# Mathematics and Statistics (Part 1) Standard XII

# 1 MATHEMATICAL LOGIC

## CHAPTER OUTLINE



## IMPORTANT FORMULAE

1. A declarative (assertive) sentence, which is either true or false, but not both simultaneously is called a **statement** (or **proposition**).
2. Sentences which are incomplete or imperative or interrogative or exclamatory or suggestive are not taken as statements.
3. The symbol ∀ stands for 'for all' or 'for every'. It is called **universal quantifier.**
4. The symbol ∃ stands for 'for some' or 'for one' or 'there exists at least one'. It is called **existential quantifier.**
5. **Logical Connectives :**

| Sr. No. | Connective | Symbolic form | Name of the compound statement | Negation |
|---|---|---|---|---|
| (1) | and | $p \wedge q$ | conjunction | $\sim p \vee \sim q$ |
| (2) | or | $p \vee q$ | disjunction | $\sim p \wedge \sim q$ |
| (3) | not | $\sim p$ | negation | $\sim(\sim p) \equiv p$ |
| (4) | If...then | $p \rightarrow q$ | conditional or implication | $p \wedge \sim q$ |
| (5) | If and only if or iff | $p \leftrightarrow q$ | biconditional or double implication or equivalence | $(p \wedge \sim q) \vee (q \wedge \sim p)$ |

6. A sentence which contains one or more variables is called an **open sentence.**
7. An open sentence with a quantifier becomes a quantified statement.
8. Negation of universal quantifier is existential quantifier and vice versa.
9. A statement pattern which is always true is called a **tautology.**
10. A statement pattern which is always false is called a **contradiction.**
11. A statement pattern which is neither a tautology nor a contradiction is called a **contingency.**
12. **Standard equivalent statements :**

**(1) Idempotent Laws :**

(i) $p \vee p \equiv p$ (ii) $p \wedge p \equiv p$

**(2) Commutative Laws :**

(i) $p \vee q \equiv q \vee p$ (ii) $p \wedge q \equiv q \wedge p$

**(3) Associative Laws :**

(i) $p \vee (q \vee r) \equiv (p \vee q) \vee r = p \vee q \vee r$

(ii) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r = p \wedge q \wedge r$

**(4) Distributive Laws :**

(i) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(ii) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

**(5) De Morgan's Laws :**

(i) $\sim(p \vee q) \equiv \sim p \wedge \sim q$ (ii) $\sim(p \wedge q) \equiv \sim p \vee \sim q$

**(6) Complement Laws :**

(i) $p \vee \sim p \equiv t$ (ii) $p \wedge \sim p \equiv c$

(iii) $\sim t \equiv c$ (iv) $\sim c \equiv t$.

**(7) Identity Laws :**

(i) $p \vee c \equiv p$ (ii) $p \wedge c \equiv c$

(iii) $p \vee t \equiv t$ (iv) $p \wedge t \equiv p$

**(8) Involution Law :**

$\sim(\sim p) \equiv p$

**(9) Absorption Laws :**

(i) $p \vee (p \wedge q) \equiv p$ (ii) $p \wedge (p \vee q) \equiv p$

**(10) Conditional Laws :**

(i) $p \to q \equiv \sim p \vee q$

(ii) $p \leftrightarrow q \equiv (\sim p \vee q) \wedge (\sim q \vee p)$

**(11) Contrapositive Law :**

$p \to q \equiv \sim q \to \sim p$

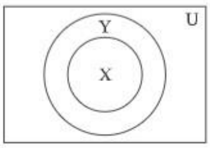
**13.** (i) $p \to q \equiv \sim q \to \sim p \equiv \sim p \vee q$

(ii) $(p \leftrightarrow q) \equiv (p \to q) \wedge (q \to p)$

$\equiv (\sim p \vee q) \wedge (\sim q \vee p)$

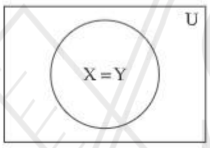
**14. Venn Diagrams :**

**(1) All X's are Y's**



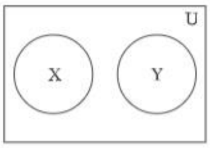

$X \cap Y = X \neq \phi$

**OR**



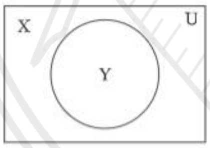

$X = Y$

**(2) No X's are Y's**



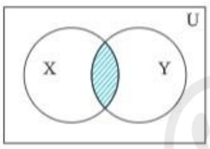

$X \cap Y = \phi$

**OR**



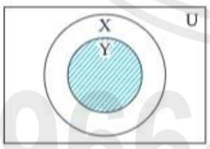

$X \cap Y = \phi$

**(3) Some X's are Y's**



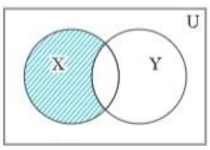

$X \cap Y \neq \phi$

**OR**



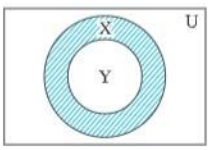

$X \cap Y \neq \phi$

**(4) Some X's are not Y's**




$X - Y \neq \phi$

**OR**




$X - Y \neq \phi$

## INTRODUCTION

Logic is a science of accurate reasoning or thinking. It consists of the rules used in drawing correct conclusions from given informations. As in the case of every subject, logic has its own vocabulary. We first introduce some of the terms and symbols which are basic to logic vocabulary. After becoming familiar with them, we shall learn the methods of analysing sentences.

## 1.1 : STATEMENT

**Definition :** A declarative (assertive) sentence, which is either true or false, but not both simultaneously is called a *Statement* (or *proposition*) in logic.

Sentences which are incomplete or imperative or interrogative or exclamatory or suggestive are not taken as statements.

A statement is either true or false or equivalently either valid or invalid. It cannot be both true and false and also neither true nor false. This fact is known as the **law of excluded middle**.

Statements are usually denoted by the letters $p, q, r, \ldots$. Consider the following sentences :

(1) Delhi is the capital of India.

(2) The sum of two odd integers is an odd integer.

(3) It is white in colour.

(4) $x + 3 = 5$.

(5) Please, come here.

(6) When is your examination going to start?

(7) What a terrible accident it is!

(8) Let us go for a walk.

The first two sentences declare something emphatically. Sentence (1) is true but not false, and sentence (2) is false but not true. Hence these are statements.

We cannot decide whether the sentence (3) is true or false. If the pronoun 'it' stands for 'chalk', then it is true. But if 'it' stands for 'blackboard', then it is false. Hence it is not a statement.

Similarly, we cannot decide whether the sentence (4) is true or false, unless we know the value of $x$. If $x = 2$, it is true and if $x \neq 2$, it is false. Such a sentence is called an *open sentence*. It is not a statement.

Sentence (5) is a request. It is an imperative sentence. Sentence (6) is a question, i.e. an interrogative sentence. Sentence (7) is an exclamatory sentence. Sentence (8) is a suggestion. Hence these sentences are not statements.

**Truth value of the statement :**

A statement (proposition) is either true or false. This truth or falsity of a statement is called the *truth value* of the statement. If a statement is true, its truth value is denoted by T and if it is false, then its truth value is denoted by F.

Consider the statement : The sum of two natural numbers is a natural number.

Let us denote this statement by $p$. Then we write :

$p$ : The sum of two natural numbers is a natural number.

We know that the statement $p$ is true. Hence its truth value is T.

The statement 'three plus four equals ten' can be written as $3+4=10$.

Let us denote this statement by $q$. Then $q : 3+4=10$.

Clearly the statement $q$ is false. Hence its truth value is F.

**Notes :**

(i) 1 and 0 can be used for T and F respectively.

(ii) An open sentence is not considered a statement in logic.

(iii) Mathematical identities are true statements.

**ACTIVITY** Textbook page 2

**Determine whether the following sentences are statements in logic and write down the truth values of the statements :**

| Sr. No. | Sentence | Whether it is a statement or not (Yes/No) | If 'No' then reason | Truth value of the statement |
|---|---|---|---|---|
| 1. | $\sqrt{-9}$ is a rational number. | Yes | – | False (F) |
| 2. | Can you speak in French? | No | Interrogative | – |
| 3. | Tokyo is in Gujarat. | Yes | – | False (F) |
| 4. | Fantastic, let's go! | No | Exclamatory | – |
| 5. | Please open the door quickly. | No | Imperative | – |
| 6. | Square of an even number is even. | Yes | – | True (T) |
| 7. | $x+5<14$. | No | Open sentence | – |
| 8. | 5 is a perfect square. | Yes | – | False (F) |
| 9. | West Bengal is capital of Kolkata. | Yes | – | False (F) |
| 10. | $i^2=-1$. | Yes | – | True (T) |

**EXERCISE 1.1** Textbook pages 2 and 3

**State which of the following sentences are statements. Justify your answer. In case of satements, write down the truth value :**

1. A triangle has '$n$' sides.
2. The sum of interior angles of a triangle is 180°.
3. You are amazing!
4. Please grant me a loan.
5. $\sqrt{-4}$ is an irrational number.
6. $x^2-6x+8=0$ implies $x=-4$ or $x=-2$.
7. He is an actor.
8. Did you eat lunch yet?
9. Have a cup of cappuccino.
10. $(x+y)^2=x^2+2xy+y^2$ for all $x, y \in R$.

11. Every real number is a complex number.
12. 1 is a prime number.
13. With the sunset the day ends.
14. $1! = 0$.
15. $3 + 5 > 11$.
16. The number $\pi$ is an irrational number.
17. $x^2 - y^2 = (x+y)(x-y)$ for all $x, y \in R$.
18. The number 2 is only even prime number.
19. Two coplanar lines are either parallel or intersecting.
20. The number of arrangements of 7 girls in a row for a photograph is 7!
21. Give me a compass box.
22. Bring the motor car here.
23. It may rain today.
24. If $a + b < 7$, where $a \geqslant 0$ and $b \geqslant 0$, then $a < 7$ and $b < 7$.
25. Can you speak in English?

**Solution :**

1. It is a statement which is **false**, hence its truth value is 'F'.
2. It is a statement which is **true,** hence its truth value is 'T'.
3. It is an exclamatory sentence, hence it is not a statement.
4. It is an imperative sentence, hence it is not a statement.
5. It is a statement which is **false**, hence its truth value is 'F'.
6. It is a statement which is **false,** hence its truth value is 'F'.
7. It is an open sentence, hence it is not a statement.
8. It is an interrogative sentence, hence it is not a statement.
9. It is an imperative sentence, hence it is not a statement.
10. It is a mathematical identity which is **true,** hence its truth value is 'T'.
11. It is a statement which is **true,** hence its truth value is 'T'.

    [**Note :** *Answer in the textbook is incorrect.*]
12. It is a statement which is **false,** hence its truth value is 'F'.
13. It is a statement which is **true**, hence its truth value is 'T'.
14. It is a statement which is **false,** hence its truth value is 'F'.
15. It is a statement which is **false,** hence its truth value is 'F'.
16. It is a statement which is **true,** hence its truth value is 'T'.
17. It is a mathematical identity which is **true,** hence its truth value is 'T'.
18. It is a statement which is **true,** hence its truth value is 'T'.
19. It is a statement which is **true,** hence its truth value is 'T'.
20. It is a statement which is **true,** hence its truth value is 'T'.
21. It is an imperative sentence, hence it is not a statement.
22. It is an imperative sentence, hence it is not a statement.
23. It is an open sentence, hence it is not a statement.
24. It is a statement which is **true,** hence its truth value is 'T'.
25. It is an interrogative sentence, hence it is not a statement.

## EXAMPLES FOR PRACTICE 1.1

**State which of the following sentences are statements. Justify your answer. In case of statements, write down the truth value :**

1. The square of any odd integers is even.
2. $5 + 4 = 11$.
3. $x + 2 = 5$.
4. Mumbai is the capital of India.
5. $x^2 - 5x + 6 = 0$, when $x = 2$.
6. I am lying.
7. The quadratic equation $x^2 - 7x + 12 = 0$ has two real roots.
8. Please get me a pen.
9. $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$ for all $x, y \in R$.
10. A quadrilateral has five sides.
11. Let us go for a walk.

12. Congruent triangles are also similar.
13. When is your examination going to start?
14. Moon revolves around the sun.
15. $x+0=x$, for all $x \in R$.
16. Today is Saturday.
17. 14 is a perfect square.
18. Give me a glass of water.
19. Two parallel lines intersect each other.
20. $x^2+7x+10=0$ implies that $x=-2$ or $x=-5$.
21. $1+2+3+\ldots+n=\frac{n(n+1)}{2}$, for all $n \in N$.
22. Zero is a complex number.
23. Close the door.
24. A cyclic trapezium has its non-parallel sides congruent.
25. Do you like Mathematics?

**Answers**

**True statements :** 5, 7, 9, 12, 14, 15, 20, 21, 22, 24.
Hence their truth values are T.

**False statements :** 1, 2, 4, 10, 17, 19.
Hence their truth values are F.

**Not statements :** 3, 6, 8, 11, 13, 16, 18, 23, 25.

## 1.2 : LOGICAL CONNECTIVES

### Compound statement :

A statement which is not a combination of two or more statements is a *simple statement*. For example :

(1) Paris is the capital of France.
(2) The set of all rational numbers is finite.

These are simple statements.

A combination of two or more simple statements is called a *compound statement*.

For example,

(1) 2 is a rational number and $\sqrt{2}$ is an irrational number.
(2) If a quadrilateral is a rectangle, then its diagonals are congruent.

These are compound statements.

The simple statements whose combination is a compound statement are called *constituents* or *components* of the compound statement.

### Logical connectives :

The words or phrases which connect two or more simple statements to form a compound statement are called *sentential connectives* or *logical connectives* or simply *connectives*.

We are going to study the five connectives along with their truth tables which are *'and'*, *'or'*, *'If ... then'*, *'if and only if'* (shortly written as *'iff'*) and *'not'*.

The truth value of a compound statement depends upon the truth values of its constituents and the connective used.

We shall discuss the logical connectives in the order given in the following table :

| Sr. No. | Connective | Symbol | Name of the compound statement |
|---|---|---|---|
| (1) | and | $\wedge$ | conjunction |
| (2) | or | $\vee$ | disjunction |
| (3) | not | $\sim$ | negation |
| (4) | If... then | $\rightarrow$ or $\Rightarrow$ | conditional or implication |
| (5) | if and only if or iff | $\leftrightarrow$ or $\Leftrightarrow$ | biconditional or double implication or equivalence |

### (A) Conjunction ($\wedge$) :

A compound statement formed by combining two given simple statements by the connective word *'and'*, is called the *conjunction* of the two given statements.

The symbol for the connective *'and'* is $\wedge$.

If $p$ and $q$ are two simple statements, their conjunction is '$p$ *and* $q$' which is written as $p \wedge q$ and read as '$p$ *and* $q$'.

**Illustrations :**

(i) Let $p : 2 > 3$, $q : 5 < 7$.
Then their conjunction is
$p \wedge q : 2 > 3$ *and* $5 < 7$

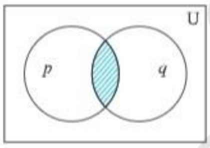
(ii) Let $p$ : Rama is tall.
$q$ : Rama is thin.
Then their conjunction is
$p \wedge q$ : Rama is tall *and* Rama is thin.
We can write this as :
$p \wedge q$ : Rama is tall *and* thin.

**The truth value of the conjunction $p \wedge q$ is T, if both $p$ and $q$ have the truth value T. In all other cases, $p \wedge q$ is false.**

This is shown in the truth table for the conjunction $p \wedge q$ given below :

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

The diagrammatic representation of conjunction is as below :




$p \wedge q$

**Notes :**

(1) Other words such as 'but', 'yet', 'though', 'still', 'moreover' are also used for the conjunction.
e.g., The statements :
(a) He is tall and handsome.
(b) He is tall but handsome.
have the same meaning.

(2) Conjunction of two statements corresponds to the 'intersection of two sets' in set theory.

**(B) Disjunction (∨) :**

A compound statement formed by combining two given simple statements by the connective word *'or'* is called the *disjunction* of the two given statements.

The symbol for the connective word *'or'* is ∨.

If $p$ and $q$ are two simple statements, their disjunction is '*p or q*' which is written as $p \vee q$ and read as '*p or q*'.

**Illustrations :**

(i) Let $p : 2 > 3$, $q : 5 < 7$
Then their disjunction is $p \vee q : 2 > 3$ or $5 < 7$.

(ii) Let $p$ : The sun shines.
$q$ : It rains.
Then their disjunction is
$p \vee q$ : The sun shines or it rains.

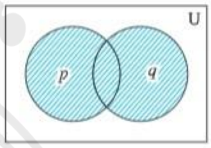
Note that in symbolic logic and in Mathematics 'or' is always taken in the **inclusive sense** unless otherwise stated. Thus $p \vee q$ means $p$ or $q$ or both $p$ and $q$.

**The truth value of the disjunction $p \vee q$ is F, if both $p$ and $q$ have the truth value F. In all other cases, $p \vee q$ has the truth value T, i.e., $p \vee q$ is true, if at least one of $p$ and $q$ is true.**

This is shown in the truth table for the disjunction $p \vee q$ given below :

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

The diagrammatic representation of disjunction is as below :




$p \vee q$

**Note :** Disjunction of two statements corresponds to 'union of two sets' in set theory.

**EXERCISE 1.2** | **Textbook page 6**

**1. Express the following statements in symbolic form :**

**(i) $e$ is a vowel or $2 + 3 = 5$.**

**(ii) Mango is a fruit but potato is a vegetable.**

**(iii) Milk is white or grass is green.**

**(iv) I like playing but not singing.**

**(v) Even though it is cloudy, it is still raining.**

**Solution :**

(i) Let $p$ : $e$ is a vowel.
$q : 2 + 3 = 5$.
Then the symbolic form of the given statement is $p \vee q$.

(ii) Let $p$ : Mango is a fruit.
$q$ : Potato is a vegetable.
Then the symbolic form of the given statement is $p \wedge q$.

(iii) Let $p$ : Milk is white.
$q$ : Grass is green.
Then the symbolic form of the given statement is $p \vee q$.

(iv) Let $p$ : I like playing.
$q$ : I am not singing.
Then the symbolic form of the given statement is $p \wedge q$.

**(v)** The given statement is equivalent to :

It is cloudy and it is still raining.

Let $p$ : It is cloudy.

$q$ : It is still raining.

Then the symbolic form of the given statement is $p \wedge q$.

**2. Write the truth values of the following statements :**

**(i) Earth is a planet and Moon is a star.**

**(ii) 16 is an even number and 8 is a perfect square.**

**(iii) A quadratic equation has two distinct roots or 6 has three prime factors.**

**(iv) The Himalayas are the highest mountains but they are part of India in the north-east.**

**Solution :**

**(i)** Let $p$ : Earth is a planet.

$q$ : Moon is a star.

Then the symbolic form of the given statement is $p \wedge q$.

The truth values of $p$ and $q$ are T and F respectively.

$\therefore$ the truth value of $p \wedge q$ is F. ... $[T \wedge F \equiv F]$

**(ii)** Let $p$ : 16 is an even number.

$q$ : 8 is a perfect square.

Then the symbolic form of the given statement is $p \wedge q$.

The truth values of $p$ and $q$ are T and F respectively.

$\therefore$ the truth value of $p \wedge q$ is F. ... $[T \wedge F \equiv F]$

**(iii)** Let $p$ : A quadratic equation has two distinct roots.

$q$ : 6 has three prime factors.

Then the symbolic form of the given statement is $p \vee q$.

The truth values of both $p$ and $q$ are F.

$\therefore$ the truth value of $p \vee q$ is F. ... $[F \vee F \equiv F]$

**(iv)** Let $p$ : Himalayas are the highest mountains.

$q$ : They are part of India in north-east.

Then the symbolic form of the given statement is $p \wedge q$.

The truth values of both $p$ and $q$ are T.

$\therefore$ the truth value of $p \wedge q$ is T. ... $[T \wedge T \equiv T]$

## EXAMPLES FOR PRACTICE 1.2

**1. Express the following statements in symbolic form :**

(i) It is raining and cold.

(ii) Sachin Tendulkar scored double century or India won the match.

(iii) The school is open or there is a holiday.

(iv) He is tall but handsome.

(v) Leela speaks Marathi or English.

(vi) The sky is blue and the rose is red.

**2. Write the truth values of the following statements :**

(i) Paris is in China or London is in Germany.

(ii) $2+3=5$ or $3<0$.

(iii) All cows are white and all flowers are red.

(iv) Taj Mahal is in Madhya Pradesh and Hyderabad is in Tamil Nadu.

(v) 2 is a prime number and 3 is a rational number.

(vi) Fixed deposit scheme gives fixed return or share market gives uncertain returns.

(vii) Prepaid expense is an asset or Accountancy is a part of book-keeping.

(viii) Goodwill is fixed asset or a Bill of exchange needs acceptance.

### Answers

**1.** (i) $p \wedge q$ (ii) $p \vee q$ (iii) $p \vee q$ (iv) $p \wedge q$ (v) $p \vee q$ (vi) $p \wedge q$.

**2.** (i) $F \vee F \equiv F$ (ii) $T \vee F \equiv T$ (iii) $F \wedge F \equiv F$ (iv) $F \wedge F \equiv F$ (v) $T \wedge T \equiv T$ (vi) $T \vee T \equiv T$ (vii) $T \vee F \equiv T$ (viii) $F \vee T \equiv T$.

### (C) Negation ($\sim$) :

The connective word '*not*' operates on only one statement. When it is used at a proper place in a given statement, it forms a new statement which has a meaning opposite to that of the given statement. This new statement is called the '*negation*' of the given statement.

The symbol for '*not*' is $\sim$. Thus if $p$ is a statement, then its negation is denoted by $\sim p$ and read as '*not p*'.

The negation of a given statement can also be obtained by using '*It is not true that*' or '*It is false that*'.

**Illustrations :**

(i) Let $p$ : 2 is a prime number.

Then its negation is

$\sim p$ : 2 is not a prime number.

or $\sim p$ : It is not true that 2 is a prime number.

or $\sim p$ : It is false that 2 is a prime number.

Observe that $p$ is true and $\sim p$ is false.

(ii) Let $p$ : 2 is a root of the equation $x^2+3x+2=0$.

Then its negation is

$\sim p$ : 2 is not a root of the equation $x^2+3x+2=0$.

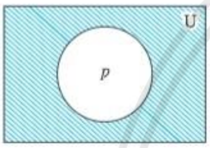
Observe that $p$ is false and $\sim p$ is true.

**If $p$ is true, then its negation $\sim p$ is false and if $p$ is false, then its negation $\sim p$ is true.**

This is expressed in the following truth table for $\sim p$ :

| $p$ | $\sim p$ |
|---|---|
| T | F |
| F | T |

The diagrammatic representation of negation is as below :




**Note :** Negation of a statement is equivalent to 'complement of the set' in set theory.

**EXERCISE 1.3** | **Textbook page 7**

**1. Write the negation of each of the following statements :**

**(i) All men are animals.**

**(ii) $-3$ is a natural number.**

**(iii) It is false that Nagpur is the capital of Maharashtra.**

**(iv) $2+3 \neq 5$.**

**Solution :** The negations of the given statements are :

**(i)** Some men are not animals.

**(ii)** $-3$ is not a natural number.

**(iii)** Nagpur is the capital of Maharashtra.

**(iv)** $2+3=5$.

**2. Write the truth value of the negation of each of the following statements :**

**(i) $\sqrt{5}$ is an irrational number.**

**(ii) London is in England.**

**(iii) For every $x \in N$, $x+3<8$.**

**Solution :**

**(i)** Let $p$ : $\sqrt{5}$ is an irrational number.

The truth value of $p$ is T.

Therefore, the truth value of $\sim p$ is F.

**(ii)** Let $p$ : London is in England.

The truth value of $p$ is T.

Therefore, the truth value of $\sim p$ is F.

**(iii)** Let $p$ : For every $x \in N$, $x+3<8$.

The truth value of $p$ is F.

Therefore, the truth value of $\sim p$ is T.

## EXAMPLES FOR PRACTICE 1.3

**1. Write the negations of the following statements :**

(i) ABCD is a quadrilateral.

(ii) Jagannath Puri is in Odisha.

(iii) 3 is a rational number.

(iv) 2 is not a root of the equation $x^2+3x+2=0$.

(v) Some teachers are sincere.

(vi) Some students don't like studying.

(vii) All prime numbers are odd numbers.

(viii) 5 is less than 7.

(ix) $\text{Im}(z) \leqslant |z|$.

(x) $7+2>5$.

(xi) It is not true that mangoes are inexpensive.

**2. Write the truth value of the negation of each of the following statements :**

(i) The square of a real number is positive.

(ii) 5 is greater than 4.

(iii) Nagpur is in Maharashtra.

(iv) Every equilateral triangle is an isosceles triangle.

(v) The sun sets in the east.

(vi) $\text{Re}(z) \leqslant |z|$, where $z$ is a complex number.

### Answers

**1.** (i) ABCD is not a quadrilateral.

(ii) Jagannath Puri is not in Odisha.

(iii) 3 is not a rational number.

(iv) 2 is a root of the equation $x^2+3x+2=0$.

(v) No teacher is sincere.

(vi) All students like studying.

(vii) Some prime numbers are not odd numbers. **OR** It is not true that all prime numbers are odd numbers. **OR** It is false that all prime numbers are odd numbers.

(viii) 5 is not less than 7.

(ix) $\text{Im}(z) \nleqslant |z|$ **OR** $\text{Im}(z) > |z|$.

(x) $7+2 \ngtr 5$ **OR** $7+2 \leqslant 5$.

(xi) It is true that mangoes are inexpensive. **OR**
The mangoes are inexpensive.

**2.** (i) T (ii) F (iii) F (iv) F (v) T (vi) F.

### (D) Conditional Statement (Implication) ($\rightarrow$) :

A compound statement formed by combining two given simple statements by the connective phrase '*If ... then*' is called the *conditional* or *implication* of the two given statements.

The symbol for the connective phrase '*If ... then*' is $\rightarrow$ or $\Rightarrow$.

If $p$ and $q$ are two simple statements, then their implication is '*If p then q*' which is written as $p \rightarrow q$ or $p \Rightarrow q$ and read as '*p implies q*'.

The components of the implication $p \rightarrow q$ are $p$ and $q$. The first component $p$ is called *antecedent* or the *hypothesis* and the second component is called *consequent* or the *conclusion*.

For the implication $p \rightarrow q$, all the following statements have the same meaning :

(1) $p$ implies $q$.
(2) If $p$ then $q$.
(3) $q$ whenever $p$.
(4) $p$ only if $q$.
(5) $p$ is sufficient for $q$.
(6) $q$ is necessary for $p$.
(7) Sufficient condition for $q$ is $p$.
(8) $q$ follows from $p$.

**Illustrations :**

(i) Let $p$ : A number $n$ is divisible by 3.
$q$ : The sum of the digits in the number $n$ is divisible by 3.

Then their implication is

$p \rightarrow q$ : *If* a number $n$ is divisible by 3, *then* the sum of the digits in the number $n$ is divisible by 3.

(ii) Let $p$ : There is no cool breeze.
$q$ : The night is hot.

Then their implication is

$p \rightarrow q$ : *If* there is no cool breeze, *then* the night is hot.

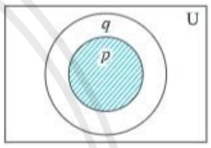
**The truth value of the implication $p \rightarrow q$ is F, if $p$ has truth value T and $q$ has the truth value F. In all other cases, $p \rightarrow q$ has the truth value T.**

This is expressed in the truth table for $p \rightarrow q$ given below :

| $p$ | $q$ | $p \rightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

Particularly, we note that even if $p$ and $q$ are both false, $p \rightarrow q$ is taken to be true.

The diagrammatic representation of the implication is as below :




$p \rightarrow q \ (p \subset q)$

**Note :** Implications of two statements is equivalent to 'subset' in set theory.

### (E) Biconditional (Double Implication) ($\leftrightarrow$) :

A compound statement formed by combining two given simple statements by the connective phrase '*if and only if*' is called the *biconditional* or *double implication* of the two given statements.

The connective phrase '*if and only if*' is shortly written as '*iff*' and the symbol for this connective phrase is $\leftrightarrow$ or $\Leftrightarrow$.

If $p$ and $q$ are two simple statements, then their biconditional is '*p if and only if q*', i.e., '*p iff q*' and is written as $p \leftrightarrow q$ or $p \Leftrightarrow q$ and is read as '*p implies and implied by q*'.

For the double implication $p \leftrightarrow q$, all the following statements have the same meaning :

(1) $p$ if and only if $q$.
(2) $p$ implies $q$ and $q$ implies $p$.
(3) $p$ implies and implied by $q$.
(4) $p$ is necessary and sufficient for $q$.
(5) $q$ is necessary and sufficient for $p$.
(6) $p$ is equivalent to $q$.

Here we note that $p \leftrightarrow q$ means $p \to q$ and $q \to p$. Hence $p \leftrightarrow q$ is the conjunction of the statements $p \to q$ and $q \to p$, i.e. $(p \to q) \wedge (q \to p)$.

Hence the name double implication.

**Illustrations :**

(i) Let $p$ : An integer $n$ is divisible by 3.

$q$ : The sum of the digits in the integer $n$ is divisible by 3.

Then their biconditional is

$p \leftrightarrow q$ : An integer $n$ is divisible by 3, *iff* the sum of the digits in the integer $n$ is divisible by 3.

(ii) Let $p$ : Three points are collinear.

$q$ : They lie on the same line.

Then their biconditional is

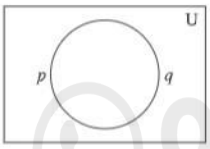
$p \leftrightarrow q$ : Three points are collinear, *if* and only *if* they lie on the same line.

**The truth value of the double implication $p \leftrightarrow q$ is T, if both $p$ and $q$ have the same truth value. In other cases, the truth value of $p \leftrightarrow q$ is F.**

This is expressed in the truth table for $p \leftrightarrow q$ given below :

| $p$ | $q$ | $p \leftrightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |

The diagrammatic representation of the biconditional is as below :




$p \leftrightarrow q$

**Note :** Biconditional of two statements is equivalent to 'equality of two sets' in set theory.

**Remark :** Since negation operates on a single logical statement, it is known as **unary** operation. All other connectives combine two statements, hence they are called **binary** connectives.

**ACTIVITY** | **Textbook page 10**

**If $p$ and $q$ are true and $r$ and $s$ are false, find the truth value of $\sim [(p \wedge \sim s) \vee (q \wedge \sim r)]$**

**Solution :**

The truth values of both $p$ and $q$ are T and truth values of both $r$ and $s$ are F.

$\therefore \sim [(p \wedge \sim s) \vee (q \wedge \sim r)]$

$\equiv \sim [(\boxed{T} \wedge \sim \boxed{F}) \vee (\boxed{T} \wedge \sim \boxed{F})]$

$\equiv \sim [(\boxed{T} \wedge T) \vee (T \wedge T)]$

$\equiv \sim (\boxed{T} \vee \boxed{T})$

$\equiv \sim \boxed{T} \equiv \boxed{F}$

Hence, the truth value of the given statement is **false.**

**EXERCISE 1.4** | **Textbook pages 10 and 11**

**1. Write the following statements in symbolic form :**

**(i) If triangle is equilateral, then it is a equiangular.**

**(ii) It is not true that '$i$' is a real number.**

**(iii) Even though it is not cloudy, it is still raining.**

**(iv) Milk is white if and only if the sky is not blue.**

**(v) Stock prices are high if and only if stocks are rising.**

**(vi) If Kutub-minar is in Delhi, then Taj Mahal is in Agra.**

**Solution :**

**(i)** Let $p$ : Triangle is equilateral.

$q$ : It is equiangular.

Then the symbolic form of the given statement is $p \to q$.

**(ii)** Let $p$ : '$i$' is a real number.

Then the symbolic form of the given statement is $\sim p$.

**(iii)** Let $p$ : It is cloudy.

$q$ : It is still raining.

Then the symbolic form of the given statement is $\sim p \wedge q$.

**(iv)** Let $p$ : Milk is white.

$q$ : Sky is blue.

Then the symbolic form of the given statement is $p \leftrightarrow (\sim q)$.

**(v)** Let $p$ : Stock prices are high.

$q$ : stocks are rising.

Then the symbolic form of the given statement is $p \leftrightarrow q$.

**(vi)** Let $p$ : Kutub-minar is in Delhi.

$q$ : Taj Mahal is in Agra.

Then the symbolic form of the given statement is $p \rightarrow q$.

**2. Find the truth value of each of the following statements :**

**(i) It is not true that $3-7i$ is a real number.**

**(ii) If a joint venture is a temporary partnership, then discount on purchase is credited to the supplier.**

**(iii) Every accountant is free to apply his own accounting rules if and only if machinery is an asset.**

**(iv) Neither 27 is a prime number nor divisible by 4.**

**(v) 3 is a prime number and an odd number.**

**Solution :**

**(i)** Let $p$ : $3-7i$ is a real number.

Then the symbolic form of the given statement is $\sim p$.

The truth value of $p$ is F.

$\therefore$ the truth value of $\sim p$ is T. ... $[\sim F \equiv T]$

**(ii)** Let $p$ : Joint venture is a temporary partnership.

$q$ : Discount on purchases is credited to supplier.

Then the symbolic form of the given statement is $p \rightarrow q$.

The truth values of $p$ and $q$ are T and F respectively.

$\therefore$ the truth value of $p \rightarrow q$ is F. ... $[T \rightarrow F \equiv F]$

**(iii)** Let $p$ : Every accountant is free to apply his own accounting rules.

$q$ : Machinery is an asset.

Then the symbolic form of the given statement is $p \leftrightarrow q$.

The truth values of $p$ and $q$ are F and T respectively.

$\therefore$ the truth value of $p \leftrightarrow q$ is F. ... $[F \leftrightarrow T \equiv F]$

**(iv)** Let $p$ : 27 is a prime number.

$q$ : 27 is divisible by 4.

Then the symbolic form of the given statement is $\sim p \wedge \sim q$.

The truth values of both $p$ and $q$ are F.

$\therefore$ the truth value of $\sim p \wedge \sim q$ is T.

... $[\sim F \wedge \sim F \equiv T \wedge T \equiv T]$

**(v)** Let $p$ : 3 is a prime number.

$q$ : 3 is an odd number.

Then the symbolic form of the given statement is $p \wedge q$.

The truth values of both $p$ and $q$ are T.

$\therefore$ the truth value of $p \wedge q$ is T. ... $[T \wedge T \equiv T]$

**3. If $p$ and $q$ are true and $r$ and $s$ are false, find the truth value of each of the following statements :**

**(i)** $p \wedge (q \wedge r)$

**(ii)** $(p \rightarrow q) \vee (r \wedge s)$

**(iii)** $\sim [(\sim p \vee s) \wedge (\sim q \wedge r)]$

**(iv)** $(p \rightarrow q) \leftrightarrow \sim (p \vee q)$

**(v)** $[(p \vee s) \rightarrow r] \vee \sim [\sim (p \rightarrow q) \vee s]$

**(vi)** $\sim [p \vee (r \wedge s)] \wedge \sim [(r \wedge \sim s) \wedge q]$

**Solution :** Truth values of $p$ and $q$ are T and truth values of $r$ and $s$ are F.

**(i)** $p \wedge (q \wedge r) \equiv T \wedge (T \wedge F)$

$\equiv T \wedge F \equiv F$

Hence, the truth value of the given statement is **false.**

**(ii)** $(p \rightarrow q) \vee (r \wedge s) \equiv (T \rightarrow T) \vee (F \wedge F)$

$\equiv T \vee F \equiv T$

Hence, the truth value of the given statement is **true.**

**(iii)** $\sim [(\sim p \vee s) \wedge (\sim q \wedge r)] \equiv \sim [(\sim T \vee F) \wedge (\sim T \wedge F)]$

$\equiv \sim [(F \vee F) \wedge (F \wedge F)]$

$\equiv \sim (F \wedge F)$

$\equiv \sim F \equiv T$

Hence, the truth value of the given statement is **true.**

**(iv)** $(p \rightarrow q) \leftrightarrow \sim (p \vee q) \equiv (T \rightarrow T) \leftrightarrow \sim (T \vee T)$

$\equiv T \leftrightarrow \sim (T)$

$\equiv T \leftrightarrow F \equiv F$

Hence, the truth value of the given statement is **false.**

**(v)** $[(p \vee s) \rightarrow r] \vee \sim [\sim (p \rightarrow q) \vee s]$

$\equiv [(T \vee F) \rightarrow F] \vee \sim [\sim (T \rightarrow T) \vee F]$

$\equiv (T \rightarrow F) \vee \sim (\sim T \vee F)$

$\equiv F \vee \sim (F \vee F)$

$\equiv F \vee \sim F \equiv F \vee T \equiv T$

Hence, the truth value of the given statement is **true.**

**(vi)** $\sim [p \vee (r \wedge s)] \wedge \sim [(r \wedge \sim s) \wedge q]$

$\equiv \sim [T \vee (F \wedge F)] \wedge \sim [(F \wedge \sim F) \wedge T]$

$\equiv \sim [T \vee F] \wedge \sim [(F \wedge T) \wedge T]$

$\equiv \sim T \wedge \sim (F \wedge T)$

$\equiv F \wedge \sim F \equiv F \wedge T \equiv F$

Hence, the truth value of the given statement is **false.**

**4. Assuming that the following statements are true :**

***p* : Sunday is holiday.**

***q* : Ram does not study on holiday.**

**Find the truth values of the following statements :**

**(i) Sunday is not holiday or Ram studies on holiday.**

**(ii) If Sunday is not holiday, then Ram studies on holiday.**

**(iii) Sunday is a holiday and Ram studies on holiday.**

**Solution :**

**(i)** The symbolic form of the statement is $\sim p \vee \sim q$.

| $p$ | $q$ | $\sim p$ | $\sim q$ | $\sim p \vee \sim q$ |
|---|---|---|---|---|
| T | T | F | F | F |

$\therefore$ truth value of the given statement is F.

**(ii)** The symbolic form of the given statement is $\sim p \rightarrow \sim q$.

| $p$ | $q$ | $\sim p$ | $\sim q$ | $\sim p \rightarrow \sim q$ |
|---|---|---|---|---|
| T | T | F | F | T |

$\therefore$ truth value of the given statement is T.

**(iii)** The symbolic form of the given statement is $p \wedge \sim q$.

| $p$ | $q$ | $\sim q$ | $p \wedge \sim q$ |
|---|---|---|---|
| T | T | F | F |

$\therefore$ truth value of the given statement is F.

**5. If *p* : He swims.**

***q* : Water is warm.**

**Give the verbal statements for the following symbolic statements :**

**(i)** $p \leftrightarrow \sim q$ **(ii)** $\sim (p \vee q)$ **(iii)** $q \rightarrow p$ **(iv)** $q \wedge \sim p$.

**Solution :**

**(i)** $p \leftrightarrow \sim q$ : He swims if and only if water is not warm.

**(ii)** $\sim (p \vee q)$ : It is not true that he swims or water is warm.

**(iii)** $q \rightarrow p$ : If water is warm, then he swims.

**(iv)** $q \wedge \sim p$ : Water is warm and he does not swim.

## EXAMPLES FOR PRACTICE 1.4

**1. Write the following statements in symbolic form :**

(i) The school is open or there is a holiday.

(ii) If it rains heavily, then the school will be closed.

(iii) An angle is a right angle if and only if it is of measure 90°.

(iv) The sky is blue and the rose is red.

(v) If $3+3=6$, then $4+4=8$.

(vi) Leela speaks Marathi or English.

(vii) It is raining and cold.

(viii) He is neither tall nor handsome.

(ix) There are clouds in the sky and it is not raining.

(x) A right-angled triangle is either isosceles or scalene.

(xi) A metal expands, when it is heated.

(xii) The sky is clear implies that the sun is shining.

(xiii) It is not true that if, '*i*' is a real number, then '2' is an even prime number.

**2. Find the truth values of the following statements :**

(i) If Rome is in Italy, then Paris is in France.

(ii) $3+3=7$ if and only if $3+5=1$.

(iii) $3+5=8$ if and only if $3+2=7$.

(iv) If Taj Mahal is in Madhya Pradesh, then Hyderabad is in Tamil Nadu.

(v) If 4 is an odd number, then 6 is divisible by 3.

**3. Let *p* : Price increases, *q* : Demand falls.**

**Express the following statements in the symbolic form :**

(i) Price increases, then demand falls.

(ii) If demand does not fall, then price does not increase.

(iii) If price does not increase, then demand does not fall.

(iv) Price increases, if and only if, the demand falls.

**4. Let *p* : The diagonals of a parallelogram are perpendicular.**

***q* : It is a rhombus.**

**Express the following statements in the symbolic form :**

(i) If the diagonals of a parallelogram are perpendicular, then it is a rhombus.

(ii) If the parallelogram is not a rhombus, then diagonals of it are not perpendicular.

(iii) If the diagonals of a parallelogram are not perpendicular, then it is not a rhombus.

**5. Using the statements $p$ : Seema is fat, $q$ : Seema is happy, write the following statements in the symbolic form :**

(i) Seema is thin but happy.

(ii) Seema is fat or unhappy.

(iii) If Seema is fat, then she is happy.

(iv) It is not true that if Seema is happy, then she is not fat.

**6. Assuming $p$ and $q$ as given, write the verbal statements for the following symbolic statements :**

(a) $p$ : Stock prices are high.

$q$ : Stocks are rising.

(i) $p \vee q$ (ii) $p \rightarrow q$ (iii) $p \wedge \sim q$.

(b) $p$ : $\triangle$ ABC is equilateral.

$q$ : $\triangle$ ABC is equiangular.

(i) $\sim p \rightarrow \sim q$ (ii) $p \leftrightarrow q$ (iii) $q \rightarrow p$.

(c) $p$ : Sachin is a good boy.

$q$ : Viraj is tall.

(i) $p \wedge q$ (ii) $\sim p \rightarrow q$.

(d) $p$ : Rahul is rich.

$q$ : Rahul is happy.

(i) $\sim p \rightarrow q$ (ii) $\sim p \wedge q$.

**7. Write the following statements in symbolic form :**

(i) Reshma is tall or she is short and beautiful.

(ii) Amitabh is tall but not handsome.

**8. If the truth value of $p$ is T and the truth value of $q$ is F, then find the truth values of :**

(i) $p \rightarrow q$ (ii) $\sim p \wedge q$.

**9. If $p$ is true statement and $q$ and $r$ are false statements, find the truth values of :**

(i) $[p \wedge (\sim q)] \rightarrow r$ (ii) $p \vee (q \vee r)$.

**10. If the statements $p$ and $q$ are true and the statement $r$ is false, find the truth values of the following :**

(i) $\sim p \leftrightarrow (\sim q \wedge r)$ (ii) $\sim q \wedge (p \rightarrow q)$

(iii) $p \vee (\sim q \leftrightarrow r)$.

**11. If the statements $p$ and $q$ are true and the statements $r$ and $s$ are false, find the truth values of the following :**

(i) $[p \wedge (q \wedge r)] \vee [(p \vee q) \wedge (\sim r \vee s)]$

(ii) $\sim [(p \wedge \sim r) \vee (\sim q \vee s)]$

(iii) $\sim [(\sim p \wedge r) \vee (s \rightarrow \sim q)] \leftrightarrow (p \wedge r)$

(iv) $[p \wedge (\sim r)] \leftrightarrow (q \vee s)$

(v) $[p \wedge (\sim r)] \rightarrow (q \wedge s)$

(vi) $[(p \rightarrow q) \rightarrow (q \rightarrow r)] \rightarrow (r \rightarrow s)$.

**12. If the statements $p$ and $q$ are false and the statements $r$ and $s$ are true, find the truth value of :**

$[(\sim p \wedge q) \wedge \sim r] \vee [(q \rightarrow p) \rightarrow (\sim s \vee r)]$.

### Answers

**1.** (i) $p \vee q$ (ii) $p \rightarrow q$ (iii) $p \leftrightarrow q$ (iv) $p \wedge q$ (v) $p \rightarrow q$ (vi) $p \vee q$ (vii) $p \wedge q$ (viii) $\sim p \wedge \sim q$ (ix) $p \wedge \sim q$ (x) $p \vee q$ (xi) $p \rightarrow q$ (xii) $p \rightarrow q$ (xiii) $\sim (p \rightarrow q)$.

**2.** (i) $T \rightarrow T \equiv T$ (ii) $F \leftrightarrow F \equiv T$ (iii) $T \leftrightarrow F \equiv F$ (iv) $F \rightarrow F \equiv T$ (v) $F \rightarrow T \equiv T$.

**3.** (i) $p \rightarrow q$ (ii) $\sim q \rightarrow \sim p$ (iii) $\sim p \rightarrow \sim q$ (iv) $p \leftrightarrow q$.

**4.** (i) $p \rightarrow q$ (ii) $\sim q \rightarrow \sim p$ (iii) $\sim p \rightarrow \sim q$.

**5.** (i) $\sim p \wedge q$ (ii) $p \vee \sim q$ (iii) $p \rightarrow q$ (iv) $\sim (q \rightarrow \sim p)$.

**6.** (a) (i) Stock prices are high or stocks are rising.

(ii) If stock prices are high, then stocks are rising.

(iii) Stock prices are high and stocks are not rising.

(b) (i) If $\triangle$ ABC is not equilateral, then it is not equiangular.

(ii) $\triangle$ ABC is equilateral if and only if it is equiangular.

(iii) If $\triangle$ ABC is equiangular, then it is equilateral.

(c) (i) Sachin is a good boy and Viraj is tall.

(ii) If Sachin is not a good boy, then Viraj is tall.

(d) (i) If Rahul is not rich, then he is happy.

(ii) Rahul is not rich but he is happy.

**7.** (i) $p \vee (\sim p \wedge q)$ (ii) $p \wedge \sim q$.

**8.** (i) F (ii) F.

**9.** (i) F (ii) T.

**10.** (i) T (ii) F (iii) T.

**11.** (i) T (ii) F (iii) T (iv) T (v) F (vi) T.

**12.** T.

## 1.2.1 : Quantifiers and Quantified Statements

**Quantifiers :**

A sentence contains one or more variables is called an **open sentence.**

An open sentence become true or false statement when we replace the variable by some specific value from a given set.

e.g. Consider an open sentence $x+5=8$.

If $x=3$, then this sentence becomes true statement and if $x \neq 3$, then this sentence becomes false statement. The phrases quantify the variable(s) in open sentences are called **quantifiers.** There are two types of quantifiers :

**(i) Universal quantifier :** The quantifier 'for all' or 'for every' is called universal quantifier and is denoted by $\forall$.

**(ii) Existential quantifier :** The quantifier 'for some' or 'for one' or 'there exists at least one' or 'there exists' is called existential quantifier and is denoted by $\exists$.

### Quantified statement :

An open sentence with a quantifier becomes a statement. Such statement is called a quantified statement.

| EXERCISE 1.5 | Textbook page 12 |
|---|---|

**1. Use qualifiers to convert each of the following open sentences defined on $N$, into a true statement :**

**(i) $x^2+3x-10=0$ (ii) $3x-4<9$**
**(iii) $n^2 \geqslant 1$ (iv) $2n-1=5$**
**(v) $y+4>6$ (vi) $3y-2 \leqslant 9$.**

**Solution :**

(i) $\exists x \in N$, such that $x^2+3x-10=0$ is a true statement
($x=2 \in N$ satisfy $x^2+3x-10=0$)

(ii) $\exists x \in N$, such that $3x-4<9$ is a true statement.
($x=1, 2, 3, 4 \in N$ satisfy $3x-4<9$)

(iii) $\forall n \in N$, $n^2 \geqslant 1$ is a true statement.
(All $n \in N$ satisfy $n^2 \geqslant 1$)

(iv) $\exists n \in N$, such that $2n-1=5$ is a true statement.
($n=3 \in N$ satisfy $2n-1=5$)

(v) $\exists y \in N$, such that $y+4>6$ is a true statement.
($y=3, 4, 5, ... \in N$ satisfy $y+4>6$)

(vi) $\exists y \in N$, such that $3y-2 \leqslant 9$ is a true statement.
($y=1, 2, 3 \in N$ satisfy $3y-2 \leqslant 9$).

**2. If $B=\{2, 3, 5, 6, 7\}$, determine the truth value of each of the following :**

**(i) $\forall x \in B$, $x$ is a prime number.**
**(ii) $\exists n \in B$, such that $n+6>12$.**
**(iii) $\exists n \in B$, such that $2n+2<4$.**
**(iv) $\forall y \in B$, $y^2$ is negative.**
**(v) $\forall y \in B$, $(y-5) \in N$.**

**Solution :**

(i) $x=6 \in B$ does not satisfy $x$ is a prime number. So, the given statement is **false,** hence its truth value is F.

(ii) Clearly $n=7 \in B$ satisfies $n+6>12$.
So, the given statement is **true**, hence its truth value is T.

(iii) No element $n \in B$ satisfy $2n+2<4$.
So, the given statement is **false,** hence its truth value is F.

(iv) No element $y \in B$ satisfy $y^2$ is negative.
So, the given statement is **false,** hence its truth value is F.

(v) $y=2 \in B$, $y=3 \in B$ and $y=5 \in B$ do not satisfy $(y-5) \in N$.
So, the given statement is **false,** hence its truth value is F.

## EXAMPLES FOR PRACTICE 1.5

**1. If $A=\{2, 3, 4, 5, 6, 7, 8\}$, determine the truth value of each of the following :**

(i) $\exists x \in A$, such that $x+5=8$.
(ii) $\forall x \in A$, $x+1 \leqslant 10$.
(iii) $\forall x \in A$, $x+5>8$.
(iv) $\exists x \in A$, such that $x^2+1$ is even.
(v) $\forall x \in A$, such that $x-2$ is a rational number.

**2. Use quantifiers to convert each of the following open sentences defined on N, into a true statement :**

(i) $x^2+1=26$ (ii) $3x+1 \leqslant 5$ (iii) $x^2>0$
(iv) $x^2-5x+6=0$ (v) $2x-3=1$ (vi) $x^2+1<10$.

### Answers

1. (i) T (ii) T (iii) F (iv) T (v) T.

2. (i) $\exists x \in N$, such that $x^2+1=26$ ($x=5$)
(ii) $\exists x \in N$, such that $3x+1 \leqslant 5$ ($x=1$)
(iii) $\forall x \in N$, $x^2>0$
(iv) $\exists x \in N$, such that $x^2-5x+6=0$ ($x=2$, $x=3$)
(v) $\exists x \in N$, such that $2x-3=1$ ($x=2$)
(vi) $\exists x \in N$, such that $x^2+1<10$ ($x=1$, $x=2$).

## 1.3 : STATEMENT PATTERNS AND LOGICAL EQUIVALENCE

### (A) Statement Pattern :

The expression constructed from logical statements $p, q, r, \ldots$, which take on the value True (T) or False (F) and logical connectives $\wedge, \vee, \sim, \rightarrow$ and $\leftrightarrow$ is called a statement pattern which is generally written as $S(p, q, r, \ldots)$.

e.g. (i) $p \wedge (\sim p \vee q)$ (ii) $p \rightarrow (q \wedge r)$

(iii) $(p \wedge \sim q) \leftrightarrow (r \rightarrow s)$ are statement patterns.

To find the truth values of the statement patterns, we must know how to prepare the truth tables.

While preparing the truth table, the following points should be taken into account :

(i) If a statement pattern contains two statements $p$ and $q$, then the truth table of the statement pattern will have $2 \times 2 = 4$ rows. This is because each of $p$ and $q$ has two possible truth value T and F.

Similarly, if a statement pattern contains three statements $p, q, r$, then the truth table of the statement pattern will have $2 \times 2 \times 2 = 8$ rows.

In general, the truth table for the statement pattern having '$n$' statements will have $2^n$ rows.

(ii) If the statement pattern contains $m$ connectives then the truth table will have $(m+n)$ columns.

(iii) Parentheses must be introduced whenever necessary.

**For example :** $\sim (p \vee q)$ and $\sim p \vee q$ are not the same.

**ACTIVITY** | **Textbook page 13**

**Complete the following truth table :**

| $p$ | $q$ | $r$ | $p \wedge r$ | $q \wedge r$ | $(p \wedge r) \vee (q \wedge r)$ |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | T | F | F | F | F |
| T | F | T | T | F | T |
| T | F | F | F | F | F |
| F | T | T | F | T | T |
| F | T | F | F | F | F |
| F | F | T | F | F | F |
| F | F | F | F | F | F |

### (B) Logical Equivalence :

Two statement patterns $S_1$ and $S_2$ are said to be *logically equivalent* (or *equivalent* or *equal*), if they have identical truth values in their truth tables for each combination of the truth values of their components. In this case, we write $S_1 \equiv S_2$ or $S_1 = S_2$.

**Note :** Two statement patterns $S_1$ and $S_2$ are logically equivalent, if their biconditional is always true.

i.e. truth value of $S_1 \leftrightarrow S_2$ is always T.

**For example :** Consider the statement patterns $S_1 \equiv p \rightarrow q$ and $S_2 \equiv \sim p \vee q$

| 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| $p$ | $q$ | $p \rightarrow q$ | $\sim p$ | $\sim p \vee q$ | $(p \rightarrow q) \leftrightarrow (\sim p \vee q)$ |
| T | T | T | F | T | T |
| T | F | F | F | F | T |
| F | T | T | T | T | T |
| F | F | T | T | T | T |

From the given truth table, the entries in columns 3 and 5 are identical. Also all the entries in the last column (i.e. column 6) are T.

$\therefore p \rightarrow q \equiv \sim p \vee q$, i.e. $S_1 \equiv S_2$.

**ACTIVITIES** | **Textbook page 14**

**1.** $p \vee (q \vee r) \equiv (p \vee q) \vee (p \vee r)$.

**Solution :**

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| $p$ | $q$ | $r$ | $q \vee r$ | $p \vee (q \vee r)$ | $p \vee q$ | $p \vee r$ | $(p \vee q) \vee (p \vee r)$ |
| T | T | T | T | T | T | T | T |
| T | T | F | T | T | T | T | T |
| T | F | T | T | T | T | T | T |
| T | F | F | F | T | T | T | T |
| F | T | T | T | T | T | T | T |
| F | T | F | T | T | T | F | T |
| F | F | T | T | T | F | T | T |
| F | F | F | F | F | F | F | F |

From the truth table, we observe that all entries in columns 5 and 8 are identical.

$\therefore p \vee (q \vee r) \equiv (p \vee q) \vee (p \vee r)$.

**2.** $\sim r \to \sim (p \wedge q) \equiv \sim (q \to r) \to \sim p$.

**Solution :**

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|
| $p$ | $q$ | $r$ | $\sim p$ | $\sim r$ | $p \wedge q$ | $\sim (p \wedge q)$ | $\sim r \to \sim (p \wedge q)$ | $q \to r$ | $\sim (q \to r)$ | $\sim (q \to r) \to \sim p$ |
| T | T | T | F | F | T | F | T | T | F | T |
| T | T | F | F | T | T | F | F | F | T | F |
| T | F | T | F | F | F | T | T | T | F | T |
| T | F | F | F | T | F | T | T | T | F | T |
| F | T | T | T | F | F | T | T | T | F | T |
| F | T | F | T | T | F | T | T | F | T | T |
| F | F | T | T | F | F | T | T | T | F | T |
| F | F | F | T | T | F | T | T | T | F | T |

The entries in columns 8 and 11 are identical.

$\therefore \sim r \to \sim (p \wedge q) \equiv \sim (q \to r) \to \sim p$.

### (C) Tautology, Contradiction and Contingency :

**(1) Tautology :** A statement pattern which has all the entries in the last column of its truth table as T is called a *tautology*.

| $p$ | $\sim p$ | $p \vee \sim p$ |
|---|---|---|
| T | F | T |
| F | T | T |

In the above truth table for the statement $p \vee \sim p$, we observe that all the entries in the last column are T. Hence the statement $p \vee \sim p$ is a tautology.

**(2) Contradiction (Fallacy) :** A statement pattern which has all the entries in the last column of its truth table as F is called a *contradiction* or *fallacy*.

| $p$ | $\sim p$ | $p \wedge \sim p$ |
|---|---|---|
| T | F | F |
| F | T | F |

In the above truth table for the statement $p \wedge \sim p$, we observe that all the entries in the last column are F. Hence the statement $p \wedge \sim p$ is a contradiction.

A statement pattern which is a tautology is true, irrespective of the truth values of its components. Similarly, a statement pattern which is a contradiction is false, irrespective of the truth values of its components.

**(3) Contingency :** A statement pattern which is neither always true nor always false is called *contingency*.

**Note :** The tautology is represented by $t$ and contradiction is represented by $c$.

**ACTIVITY** | **Textbook page 15**

**Prepare the truth table for $(\sim p \vee \sim q) \leftrightarrow \sim (p \wedge q)$.**

**Solution :**

| $p$ | $q$ | $\sim p$ | $\sim q$ | $p \wedge q$ | $\sim (p \wedge q)$ | $\sim p \vee \sim q$ | $(\sim p \vee \sim q) \leftrightarrow \sim (p \wedge q)$ |
|---|---|---|---|---|---|---|---|
| T | T | F | F | T | F | F | T |
| T | F | F | T | F | T | T | T |
| F | T | T | F | F | T | T | T |
| F | F | T | T | F | T | T | T |

**EXERCISE 1.6** | **Textbook page 16**

**1. Prepare the truth tables for the following statement patterns :**

**(i)** $p \to (\sim p \vee q)$

**Solution :** Here are two statements and three connectives.

$\therefore$ there are $2 \times 2 = 4$ rows and $2 + 3 = 5$ columns in the truth table.

| $p$ | $q$ | $\sim p$ | $\sim p \vee q$ | $p \to (\sim p \vee q)$ |
|---|---|---|---|---|
| T | T | F | T | T |
| T | F | F | F | F |
| F | T | T | T | T |
| F | F | T | T | T |

**(ii)** $(\sim p \vee q) \wedge (\sim p \vee \sim q)$

**Solution :**

| $p$ | $q$ | $\sim p$ | $\sim q$ | $\sim p \vee q$ | $\sim p \vee \sim q$ | $(\sim p \vee q) \wedge (\sim p \vee \sim q)$ |
|---|---|---|---|---|---|---|
| T | T | F | F | T | F | F |
| T | F | F | T | F | T | F |
| F | T | T | F | T | T | T |
| F | F | T | T | T | T | T |

**(iii)** $(p \wedge r) \to (p \vee \sim q)$

**Solution :** Here are three statements and 4 connectives.

$\therefore$ there are $2 \times 2 \times 2 = 8$ rows and $3 + 4 = 7$ columns in the truth table.

| $p$ | $q$ | $r$ | $\sim q$ | $p \wedge r$ | $p \vee \sim q$ | $(p \wedge r) \to (p \vee \sim q)$ |
|---|---|---|---|---|---|---|
| T | T | T | F | T | T | T |
| T | T | F | F | F | T | T |
| T | F | T | T | T | T | T |
| T | F | F | T | F | T | T |
| F | T | T | F | F | F | T |
| F | T | F | F | F | F | T |
| F | F | T | T | F | T | T |
| F | F | F | T | F | T | T |

**(iv)** $(p \wedge q) \vee \sim r$

**Solution :**

| $p$ | $q$ | $r$ | $\sim r$ | $p \wedge q$ | $(p \wedge q) \vee \sim r$ |
|---|---|---|---|---|---|
| T | T | T | F | T | T |
| T | T | F | T | T | T |
| T | F | T | F | F | F |
| T | F | F | T | F | T |
| F | T | T | F | F | F |
| F | T | F | T | F | T |
| F | F | T | F | F | F |
| F | F | F | T | F | T |

**2. Examine, whether each of the following statement patterns is a tautology or a contradiction or a contingency :**

**(i)** $q \vee [\sim (p \wedge q)]$

**Solution :**

| $p$ | $q$ | $p \wedge q$ | $\sim (p \wedge q)$ | $q \vee [\sim (p \wedge q)]$ |
|---|---|---|---|---|
| T | T | T | F | T |
| T | F | F | T | T |
| F | T | F | T | T |
| F | F | F | T | T |

All the entries in the last column of the above truth table are T.

$\therefore$ $q \vee [\sim (p \wedge q)]$ is a **tautology.**

**(ii)** $(\sim q \wedge p) \wedge (p \wedge \sim p)$

**Solution :**

| $p$ | $q$ | $\sim p$ | $\sim q$ | $\sim q \wedge p$ | $p \wedge \sim p$ | $(\sim q \wedge p) \wedge (p \wedge \sim p)$ |
|---|---|---|---|---|---|---|
| T | T | F | F | F | F | F |
| T | F | F | T | T | F | F |
| F | T | T | F | F | F | F |
| F | F | T | T | F | F | F |

All the entries in the last column of the above truth table are F.

$\therefore$ $(\sim q \wedge p) \wedge (p \wedge \sim p)$ is a **contradiction.**

**(iii)** $(p \wedge \sim q) \to (\sim p \wedge \sim q)$

**Solution :**

| $p$ | $q$ | $\sim p$ | $\sim q$ | $p \wedge \sim q$ | $\sim p \wedge \sim q$ | $(p \wedge \sim q) \to (\sim p \wedge \sim q)$ |
|---|---|---|---|---|---|---|
| T | T | F | F | F | F | T |
| T | F | F | T | T | F | F |
| F | T | T | F | F | F | T |
| F | F | T | T | F | T | T |

The entries in the last column are neither all T nor all F.

$\therefore$ $(p \wedge \sim q) \to (\sim p \wedge \sim q)$ is a **contingency.**

**(iv)** $\sim p \to (p \to \sim q)$

**Solution :**

| $p$ | $q$ | $\sim p$ | $\sim q$ | $p \to \sim q$ | $\sim p \to (p \to \sim q)$ |
|---|---|---|---|---|---|
| T | T | F | F | F | T |
| T | F | F | T | T | T |
| F | T | T | F | T | T |
| F | F | T | T | T | T |

All the entries in the last column of the truth table are T.

$\therefore$ $\sim p \to (p \to \sim q)$ is a **tautology.**

**3. Prove that each of the following statement pattern is a tautology :**

**(i) $(p \wedge q) \to q$**

**Solution :**

| $p$ | $q$ | $p \wedge q$ | $(p \wedge q) \to q$ |
|---|---|---|---|
| T | T | T | T |
| T | F | F | T |
| F | T | F | T |
| F | F | F | T |

All the entries in the last column of the above truth table are T.

$\therefore$ $(p \wedge q) \to q$ is a **tautology.**

**(ii) $(p \to q) \leftrightarrow (\sim q \to \sim p)$**

**Solution :**

| $p$ | $q$ | $\sim p$ | $\sim q$ | $p \to q$ | $\sim q \to \sim p$ | $(p \to q) \leftrightarrow (\sim q \to \sim p)$ |
|---|---|---|---|---|---|---|
| T | T | F | F | T | T | T |
| T | F | F | T | F | F | T |
| F | T | T | F | T | T | T |
| F | F | T | T | T | T | T |

All the entries in the last column of the above truth table are T.

$\therefore$ $(p \to q) \leftrightarrow (\sim q \to \sim p)$ is a **tautology.**

**(iii) $(\sim p \wedge \sim q) \to (p \to q)$**

**Solution :**

| $p$ | $q$ | $\sim p$ | $\sim q$ | $\sim p \wedge \sim q$ | $p \to q$ | $(\sim p \wedge \sim q) \to (p \wedge q)$ |
|---|---|---|---|---|---|---|
| T | T | F | F | F | T | T |
| T | F | F | T | F | F | T |
| F | T | T | F | F | T | T |
| F | F | T | T | T | T | T |

All the entries in the last column of the above truth table are T.

$\therefore$ $(\sim p \wedge \sim q) \to (p \to q)$ is a **tautology.**

**(iv) $(\sim p \vee \sim q) \leftrightarrow \sim (p \wedge q)$.**

| $p$ | $q$ | $\sim p$ | $\sim q$ | $\sim p \vee \sim q$ | $p \wedge q$ | $\sim (p \wedge q)$ | $(\sim p \vee \sim q) \leftrightarrow \sim (p \wedge q)$ |
|---|---|---|---|---|---|---|---|
| T | T | F | F | F | T | F | T |
| T | F | F | T | T | F | T | T |
| F | T | T | F | T | F | T | T |
| F | F | T | T | T | F | T | T |

All the entries in the last column of the above truth table are T.

$\therefore$ $(\sim p \vee \sim q) \leftrightarrow \sim (p \wedge q)$ is a **tautology.**

**4. Prove that each of the following statement pattern is a contradiction :**

**(i) $(p \vee q) \wedge (\sim p \wedge \sim q)$**

**Solution :**

| $p$ | $q$ | $\sim p$ | $\sim q$ | $p \vee q$ | $\sim p \wedge \sim q$ | $(p \vee q) \wedge (\sim p \wedge \sim q)$ |
|---|---|---|---|---|---|---|
| T | T | F | F | T | F | F |
| T | F | F | T | T | F | F |
| F | T | T | F | T | F | F |
| F | F | T | T | F | T | F |

All the entries in the last column of the above truth table are F.

$\therefore$ $(p \vee q) \wedge (\sim p \wedge \sim q)$ is a **contradiction.**

**(ii) $(p \wedge q) \wedge \sim p$**

**Solution :**

| $p$ | $q$ | $\sim p$ | $p \wedge q$ | $(p \wedge q) \wedge \sim p$ |
|---|---|---|---|---|
| T | T | F | T | F |
| T | F | F | F | F |
| F | T | T | F | F |
| F | F | T | F | F |

All the entries in the last column of the above truth table are T.

$\therefore$ $(p \wedge q) \wedge \sim p$ is a **contradiction.**

**(iii)** $(p \wedge q) \wedge (\sim p \vee \sim q)$

**Solution :**

| $p$ | $q$ | $\sim p$ | $\sim q$ | $p \wedge q$ | $\sim p \vee \sim q$ | $(p \wedge q) \wedge (\sim p \vee \sim q)$ |
|---|---|---|---|---|---|---|
| T | T | F | F | T | F | F |
| T | F | F | T | F | T | F |
| F | T | T | F | F | T | F |
| F | F | T | T | F | T | F |

All the entries in the last column of the above truth table are F.

$\therefore$ $(p \wedge q) \wedge (\sim p \vee \sim q)$ is a **contradiction.**

**(iv)** $(p \to q) \wedge (p \wedge \sim q)$

**Solution :**

| $p$ | $q$ | $\sim q$ | $p \to q$ | $p \wedge \sim q$ | $(p \to q) \wedge (p \wedge \sim q)$ |
|---|---|---|---|---|---|
| T | T | F | T | F | F |
| T | F | T | F | T | F |
| F | T | F | T | F | F |
| F | F | T | T | F | F |

All the entries in the last column of the above truth table are F.

$\therefore$ $(p \to q) \wedge (p \wedge \sim q)$ is a **contradiction.**

**5. Show that each of the following statement pattern is a contingency :**

**(i)** $(p \wedge \sim q) \to (\sim p \wedge \sim q)$

**Solution :**

| $p$ | $q$ | $\sim p$ | $\sim q$ | $p \wedge \sim q$ | $\sim p \wedge \sim q$ | $(p \wedge \sim q) \to (\sim p \wedge \sim q)$ |
|---|---|---|---|---|---|---|
| T | T | F | F | F | F | T |
| T | F | F | T | T | F | F |
| F | T | T | F | F | F | T |
| F | F | T | T | F | T | T |

The entries in the last column of the above truth table are neither all T nor all F.

$\therefore$ $(p \wedge \sim q) \to (\sim p \wedge \sim q)$ is a **contingency.**

**(ii)** $(p \to q) \leftrightarrow (\sim p \wedge q)$

**Solution :**

| $p$ | $q$ | $\sim p$ | $p \to q$ | $\sim p \wedge q$ | $(p \to q) \leftrightarrow (\sim p \wedge q)$ |
|---|---|---|---|---|---|
| T | T | F | T | F | F |
| T | F | F | F | F | T |
| F | T | T | T | T | T |
| F | F | T | T | F | F |

The entries in the last column of the above truth table are neither all T nor all F.

$\therefore$ $(p \to q) \leftrightarrow (\sim p \wedge q)$ is a **contingency.**

[**Note :** *Question is modified.*]

**(iii)** $p \wedge [(p \to \sim q) \to q]$

**Solution :**

| $p$ | $q$ | $\sim q$ | $p \to \sim q$ | $(p \to \sim q) \to q$ | $p \wedge [(p \to \sim q) \to q]$ |
|---|---|---|---|---|---|
| T | T | F | F | T | T |
| T | F | T | T | F | F |
| F | T | F | T | T | F |
| F | F | T | T | F | F |

The entries in the last column of the above truth table are neither all T nor all F.

$\therefore$ $p \wedge [(p \to \sim q) \to q]$ is a **contingency.**

**(iv)** $(p \to q) \wedge (p \to r)$

**Solution :**

| $p$ | $q$ | $r$ | $p \to q$ | $p \to r$ | $(p \to q) \wedge (p \to r)$ |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | T | F | T | F | F |
| T | F | T | F | T | F |
| T | F | F | F | F | F |
| F | T | T | T | T | T |
| F | T | F | T | T | T |
| F | F | T | T | T | T |
| F | F | F | T | T | T |

The entries in the last column of the above truth table are neither all T nor all F.

$\therefore$ $(p \to q) \wedge (p \to r)$ is a **contingency.**

**6. Using the truth table, verify :**

**(i)** $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| $p$ | $q$ | $r$ | $q \wedge r$ | $p \vee (q \wedge r)$ | $p \vee q$ | $p \vee r$ | $(p \vee q) \wedge (p \vee r)$ |
| T | T | T | T | T | T | T | T |
| T | T | F | F | T | T | T | T |
| T | F | T | F | T | T | T | T |
| T | F | F | F | T | T | T | T |
| F | T | T | T | T | T | T | T |
| F | T | F | F | F | T | F | F |
| F | F | T | F | F | F | T | F |
| F | F | F | F | F | F | F | F |

The entries in columns 5 and 8 are identical.

$\therefore\ p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.

**(ii)** $p \to (p \to q) \equiv \sim q \to (p \to q)$

**Solution :**

| 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| $p$ | $q$ | $\sim q$ | $p \to q$ | $p \to (p \to q)$ | $\sim q \to (p \to q)$ |
| T | T | F | T | T | T |
| T | F | T | F | F | F |
| F | T | F | T | T | T |
| F | F | T | T | T | T |

The entries in columns 5 and 6 are identical.

$\therefore\ p \to (p \to q) \equiv \sim q \to (p \to q)$.

**(iii)** $\sim (p \to \sim q) \equiv p \wedge \sim (\sim q) \equiv p \wedge q$

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| $p$ | $q$ | $\sim q$ | $p \to \sim q$ | $\sim (p \to \sim q)$ | $\sim (\sim q)$ | $p \wedge \sim (\sim q)$ | $p \wedge q$ |
| T | T | F | F | T | T | T | T |
| T | F | T | T | F | F | F | F |
| F | T | F | T | F | T | F | F |
| F | F | T | T | F | F | F | F |

The entries in columns 5, 7 and 8 are identical.

$\therefore\ \sim (p \to \sim q) \equiv p \wedge \sim (\sim q) \equiv p \wedge q$.

**(iv)** $\sim (p \vee q) \vee (\sim p \wedge q) \equiv \sim p$

**Solution :**

| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| $p$ | $q$ | $\sim p$ | $p \vee q$ | $\sim (p \vee q)$ | $\sim p \wedge q$ | $\sim (p \vee q) \vee (\sim p \wedge q)$ |
| T | T | F | T | F | F | F |
| T | F | F | T | F | F | F |
| F | T | T | T | F | T | T |
| F | F | T | F | T | F | T |

The entries in columns 3 and 7 are identical.

$\therefore\ \sim (p \vee q) \vee (\sim p \wedge q) \equiv \sim p$.

**7. Prove that the following pairs of statement patterns are equivalent :**

**(i)** $p \vee (q \wedge r)$ **and** $(p \vee q) \wedge (p \vee r)$

**Solution :** *Refer to the solution of Q. 6 (i).*

**(ii)** $p \leftrightarrow q$ **and** $(p \to q) \wedge (q \to p)$

| 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| $p$ | $q$ | $p \leftrightarrow q$ | $p \to q$ | $q \to p$ | $(p \to q) \wedge (q \to p)$ |
| T | T | T | T | T | T |
| T | F | F | F | T | F |
| F | T | F | T | F | F |
| F | F | T | T | T | T |

The entries in columns 3 and 6 are identical.

$\therefore\ p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$.

**(iii)** $p \to q$ **and** $\sim q \to \sim p$ **and** $\sim p \vee q$

**Solution :**

| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| $p$ | $q$ | $\sim p$ | $\sim q$ | $p \to q$ | $\sim q \to \sim p$ | $\sim p \vee q$ |
| T | T | F | F | T | T | T |
| T | F | F | T | F | F | F |
| F | T | T | F | T | T | T |
| F | F | T | T | T | T | T |

The entries in columns 5, 6 and 7 are identical.

$\therefore\ p \to q \equiv \sim q \to \sim p \equiv \sim p \vee q$.

(iv) $\sim (p \wedge q)$ and $\sim p \vee \sim q$

**Solution :**

| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| $p$ | $q$ | $\sim p$ | $\sim q$ | $p \wedge q$ | $\sim (p \wedge q)$ | $\sim p \vee \sim q$ |
| T | T | F | F | T | F | F |
| T | F | F | T | F | T | T |
| F | T | T | F | F | T | T |
| F | F | T | T | F | T | T |

The entries in columns 6 and 7 are identical.

$\therefore \sim (p \wedge q) \equiv \sim p \vee \sim q$.

## EXAMPLES FOR PRACTICE 1.6

1. **Prepare the truth tables of the following statement patterns :**
   (i) $(p \wedge q) \rightarrow (\sim p)$
   (ii) $\sim p \wedge [(p \vee \sim q) \wedge q]$
   (iii) $(\sim p \vee \sim q) \leftrightarrow [\sim (p \wedge q)]$
   (iv) $(\sim p \rightarrow \sim q) \wedge (\sim q \rightarrow \sim p)$
   (v) $(q \rightarrow p) \vee (\sim p \leftrightarrow q)$
   (vi) $p \vee (p \wedge q)$
   (vii) $(p \leftrightarrow q) \wedge (p \wedge \sim q)$
   (viii) $(p \vee q) \rightarrow (\sim p)$
   (ix) $(\sim p \wedge \sim q) \rightarrow (p \leftrightarrow r)$
   (x) $[p \rightarrow (q \rightarrow r)] \leftrightarrow [(p \wedge q) \rightarrow r]$.

2. **Examine whether each of the following statement patterns is a tautology or a contradiction or a contingency :**
   (i) $[(p \rightarrow q) \wedge (\sim q)] \rightarrow (\sim p)$
   (ii) $p \vee [\sim (p \wedge q)]$
   (iii) $(p \leftrightarrow q) \leftrightarrow (q \leftrightarrow p)$
   (iv) $(p \wedge q) \rightarrow (p \vee q)$
   (v) $p \rightarrow [q \rightarrow (p \wedge q)]$
   (vi) $(p \rightarrow q) \leftrightarrow (\sim p \vee q)$
   (vii) $(p \leftrightarrow q) \wedge (p \rightarrow \sim q)$
   (viii) $p \vee \sim (p \wedge q)$
   (ix) $(p \wedge q) \leftrightarrow (\sim p)$
   (x) $(\sim q \wedge p) \vee (p \vee \sim q)$
   (xi) $(p \wedge \sim q) \leftrightarrow [(p \wedge q) \vee (\sim p)]$
   (xii) $\sim p \wedge [(p \vee \sim q) \wedge q]$
   (xiii) $[p \rightarrow (\sim q \vee r)] \leftrightarrow \sim [p \rightarrow (q \rightarrow r)]$
   (xiv) $(p \wedge q) \vee (p \wedge r)$.

3. **Prove that each of the following statement patterns is a tautology :**
   (i) $(q \rightarrow p) \vee (p \rightarrow q)$
   (ii) $\sim (p \leftrightarrow q) \leftrightarrow \{[p \wedge (\sim q)] \vee [q \wedge (\sim p)]\}$
   (iii) $(\sim p \vee q) \vee (p \wedge \sim q)$ (iv) $p \vee [\sim (p \wedge q)]$.

4. **Prove that each of the following statement patterns is a contradiction :**
   (i) $(p \wedge q) \wedge (\sim q)$ (ii) $(p \vee q) \wedge [\sim (p \vee q)]$
   (iii) $(p \wedge \sim q) \wedge (p \rightarrow q)$ (iv) $(\sim p \wedge \sim q) \wedge (q \wedge r)$.

5. **Show that each of the following statement patterns is a contingency (neither) :**
   (i) $(p \vee q) \wedge (\sim p)$
   (ii) $(p \leftrightarrow q) \wedge [\sim (p \rightarrow \sim q)]$
   (iii) $(\sim p \vee q) \rightarrow [p \wedge (q \vee \sim q)]$
   (iv) $(\sim p \rightarrow q) \wedge (p \wedge r)$.

6. **Using truth tables, prove the following equivalences :**
   (i) $\sim p \wedge q \equiv (p \wedge q) \wedge (\sim p)$
   (ii) $\sim p \vee q \equiv \sim (p \wedge q) \rightarrow [\sim p \vee (\sim p \vee q)]$
   (iii) $p \leftrightarrow q \equiv \sim [(p \vee q) \wedge \sim (p \wedge q)]$
   (iv) $p \rightarrow (q \rightarrow p) \equiv \sim p \rightarrow (p \rightarrow q)$
   (v) $p \rightarrow (q \vee r) \equiv (p \rightarrow q) \vee (p \rightarrow r)$
   (vi) $[\sim (p \vee q) \vee (p \vee q)] \wedge r \equiv r$
   (vii) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$
   (viii) $(p \vee q) \rightarrow r \equiv (p \rightarrow r) \wedge (q \rightarrow r)$
   (ix) $(p \wedge q) \rightarrow r \equiv p \rightarrow (q \rightarrow r)$
   (x) $p \wedge (\sim q \vee r) \equiv \sim [p \rightarrow (q \wedge \sim r)]$.

### Answers

1. (i)

| $p$ | $q$ | $p \wedge q$ | $\sim p$ | $(p \wedge q) \rightarrow (\sim p)$ |
|---|---|---|---|---|
| T | T | T | F | F |
| T | F | F | F | T |
| F | T | F | T | T |
| F | F | F | T | T |

(ii)

| $p$ | $q$ | $\sim p$ | $\sim q$ | $p \vee \sim q$ | $(p \vee \sim q) \wedge q$ | $\sim p \wedge [(p \vee \sim q) \wedge q]$ |
|---|---|---|---|---|---|---|
| T | T | F | F | T | T | F |
| T | F | F | T | T | F | F |
| F | T | T | F | F | F | F |
| F | F | T | T | T | F | F |

(iii)

| $p$ | $q$ | $\sim p$ | $\sim q$ | $\sim p \vee \sim q$ | $p \wedge q$ | $\sim(p \wedge q)$ | $(\sim p \vee \sim q) \leftrightarrow [\sim(p \wedge q)]$ |
|---|---|---|---|---|---|---|---|
| T | T | F | F | F | T | F | T |
| T | F | F | T | T | F | T | T |
| F | T | T | F | T | F | T | T |
| F | F | T | T | T | F | T | T |

(iv)

| $p$ | $q$ | $\sim p$ | $\sim q$ | $\sim p \to \sim q$ | $\sim q \to \sim p$ | $(\sim p \to \sim q) \wedge (\sim q \to \sim p)$ |
|---|---|---|---|---|---|---|
| T | T | F | F | T | T | T |
| T | F | F | T | T | F | F |
| F | T | T | F | F | T | F |
| F | F | T | T | T | T | T |

(v)

| $p$ | $q$ | $\sim p$ | $q \to p$ | $\sim p \leftrightarrow q$ | $(q \to p) \vee (\sim p \leftrightarrow q)$ |
|---|---|---|---|---|---|
| T | T | F | T | F | T |
| T | F | F | T | T | T |
| F | T | T | F | T | T |
| F | F | T | T | F | T |

(vi)

| $p$ | $q$ | $p \wedge q$ | $p \vee (p \wedge q)$ |
|---|---|---|---|
| T | T | T | T |
| T | F | F | T |
| F | T | F | F |
| F | F | F | F |

(vii)

| $p$ | $q$ | $\sim q$ | $p \leftrightarrow q$ | $p \wedge \sim q$ | $(p \leftrightarrow q) \wedge (p \wedge \sim q)$ |
|---|---|---|---|---|---|
| T | T | F | T | F | F |
| T | F | T | F | T | F |
| F | T | F | F | F | F |
| F | F | T | T | F | F |

(viii)

| $p$ | $q$ | $\sim p$ | $p \vee q$ | $(p \vee q) \to (\sim p)$ |
|---|---|---|---|---|
| T | T | F | T | F |
| T | F | F | T | F |
| F | T | T | T | T |
| F | F | T | F | T |

(ix)

| $p$ | $q$ | $r$ | $\sim p$ | $\sim q$ | $\sim p \wedge \sim q$ | $p \leftrightarrow r$ | $(\sim p \wedge \sim q) \to (p \leftrightarrow r)$ |
|---|---|---|---|---|---|---|---|
| T | T | T | F | F | F | T | T |
| T | T | F | F | F | F | F | T |
| T | F | T | F | T | F | T | T |
| T | F | F | F | T | F | F | T |
| F | T | T | T | F | F | F | T |
| F | T | F | T | F | F | T | T |
| F | F | T | T | T | T | F | F |
| F | F | F | T | T | T | T | T |

(x)

| $p$ | $q$ | $r$ | $q \to r$ | $p \to (q \to r)$ | $p \wedge q$ | $(p \wedge q) \to r$ | $[(p \to (q \to r)] \leftrightarrow [(p \wedge q) \to r]$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T |
| T | T | F | F | F | T | F | T |
| T | F | T | T | T | F | T | T |
| T | F | F | T | T | F | T | T |
| F | T | T | T | T | F | T | T |
| F | T | F | F | T | F | T | T |
| F | F | T | T | T | F | T | T |
| F | F | F | T | T | F | T | T |

**2. Tautology :** (i), (ii), (iii), (iv), (v), (vi), (viii)

**Contradiction :** (xi), (xii), (xiii)

**Contingency :** (vii), (ix), (x), (xiv)

### (D) Duality :

Two compound statements $S_1$ and $S_2$ are said to be **duals** of each other if one can be obtained from the other by replacing $\vee$ by $\wedge$ and $\wedge$ by $\vee$.

The connectives $\wedge$ and $\vee$ are called duals of each other.

e.g.

Consider the following results :

$\sim(p \wedge q) \equiv \sim p \vee \sim q$ ... (1)

$\sim(p \vee q) \equiv \sim p \wedge \sim q$ ... (2)

From these results, we observe that a statement pattern involving a conjunction ($\wedge$) can be expressed as a disjunction ($\vee$) and vice versa.

If we replace ($\wedge$ by $\vee$) and ($\vee$ by $\wedge$) in (1) we get (2) and by the same replacement, we get (1) from (2).

This is referred to as *duality* of conjunction and disjunction. Statement patterns (1) and (2) are called *duals* of each other. Results (1) and (2) are known as *De Morgan's Laws.*

**Note :** If a compound statement S contains $t$ (tautology) or $c$ (contradiction), then the dual of S is obtained by replacing $t$ by $c$, $c$ by $t$, $\wedge$ by $\vee$ and $\vee$ by $\wedge$.

**Principle of Duality :**

If a compound statement $S_1$ contains only $\sim$, $\wedge$ and $\vee$ and statement $S_2$ is obtained from $S_1$ by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$, then $S_1$ is a tautology if and only if $S_2$ is a contradiction.

## EXERCISE 1.7 | Textbook page 17

**1. Write the dual of each of the following :**

(i) $(p \vee q) \vee r$

(ii) $\sim(p \vee q) \wedge [p \vee \sim(q \wedge \sim r)]$

(iii) $p \vee (q \vee r) \equiv (p \vee q) \vee r$

(iv) $\sim(p \wedge q) \equiv \sim p \vee \sim q$.

**Solution :** The duals are given by :

(i) $(p \wedge q) \wedge r$

(ii) $\sim(p \wedge q) \vee [p \wedge \sim(q \vee \sim r)]$

(iii) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$

(iv) $\sim(p \vee q) \equiv \sim p \wedge \sim q$.

**2. Write the dual statement of each of the following compound statements :**

(i) 13 is prime number and India is a democratic country.

(ii) Karina is very good or everybody likes her.

(iii) Radha and Sushmita can not read Urdu.

(iv) A number is real number and the square of the number is non-negative.

**Solution :** The duals are given by

(i) 13 is prime number or India is a democratic country.

(ii) Karina is very good and everybody likes her.

(iii) Radha or Sushmita can not read Urdu.

(iv) A number is real number or the square of the number is non-negative.

## ADDITIONAL SOLVED PROBLEMS – 1 (A)

**1. Write the duals of each of the following :**

(i) $t \vee (p \wedge q)$  (ii) $(p \wedge t) \vee (c \wedge \sim q)$

(iii) $p \wedge (\sim q \vee c)$.

**Solution :** The duals are given by :

(i) $c \wedge (p \vee q)$  (ii) $(p \vee c) \wedge (t \vee \sim q)$

(iii) $p \vee (\sim q \wedge t)$.

**2. State the dual of the following statement by applying the principle of duality :**

**$p \vee (q \vee r) \equiv (p \vee q) \vee r$ and**

**prove that both the sides of the dual are equivalent.**

**Solution :** The dual of the statement

$p \vee (q \vee r) \equiv (p \vee q) \vee r$ is $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| $p$ | $q$ | $r$ | $q \wedge r$ | $p \wedge (q \wedge r)$ | $p \wedge q$ | $(p \wedge q) \wedge r$ |
| T | T | T | T | T | T | T |
| T | T | F | F | F | T | F |
| T | F | T | F | F | F | F |
| T | F | F | F | F | F | F |
| F | T | T | T | F | F | F |
| F | T | F | F | F | F | F |
| F | F | T | F | F | F | F |
| F | F | F | F | F | F | F |

The entries in column 5 for $p \wedge (q \wedge r)$ and those in the column 7 for $(p \wedge q) \wedge r$ are identical. Hence, the statements $p \wedge (q \wedge r)$ and $(p \wedge q) \wedge r$ are logically equivalent.

$\therefore p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$.

**3. Prove that $(p \wedge q) \vee \sim q$ is logically equivalent to $p \vee \sim q$. Hence state its dual result.**

**Solution :**

| 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| $p$ | $q$ | $p \wedge q$ | $\sim q$ | $(p \wedge q) \vee \sim q$ | $p \vee \sim q$ |
| T | T | T | F | T | T |
| T | F | F | T | T | T |
| F | T | F | F | F | F |
| F | F | F | T | T | T |

The entries in columns 5 and 6 are identical.

$\therefore (p \wedge q) \vee \sim q \equiv p \vee \sim q$ … (1)

The dual of (1) is $(p \vee q) \wedge \sim q \equiv p \wedge \sim q$.

## EXAMPLES FOR PRACTICE 1.7

**1. Write the duals of each of the following :**

(i) $(p \wedge q) \wedge r$  (ii) $p \vee (q \wedge r)$

(iii) $(p \wedge q) \vee \sim q$  (iv) $[\sim(p \vee q)] \wedge \{p \wedge (\sim q \vee s)\}$

(v) $(p \vee q) \wedge t$  (vi) $c \vee (p \wedge q)$

**2. Write the dual of each of the following :**

(i) $p \vee (p \wedge q) \equiv p$  (ii) $t \vee c \equiv t$

(iii) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$.

**3. Write the dual statement of each of the following compound statement :**

(i) Madhu is fair and Mahesh is intelligent.

(ii) Mumbai is in Maharashtra or Patna is in Bihar.

(iii) He is tall and handsome.

(iv) Leela speaks Marathi or English.

(v) It is raining and cold.

4. State and prove the dual of the statement
$(\sim p \wedge q) \equiv (p \vee q) \wedge (\sim p)$
by constructing the truth table.

### Answers

1. (i) $(p \vee q) \vee r$ (ii) $p \wedge (q \vee r)$
(iii) $(p \vee q) \wedge \sim q$ (iv) $[\sim (p \wedge q)] \vee \{p \vee (\sim q \wedge s)\}$
(v) $(p \wedge q) \vee c$ (vi) $t \wedge (p \vee q)$.

2. (i) $p \wedge (p \vee q) \equiv p$ (ii) $c \wedge t \equiv c$
(iii) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.

3. (i) Madhu is fair or Mahesh is intelligent.
(ii) Mumbai is in Maharashtra and Patna is in Bihar.
(iii) He is tall or handsome.
(iv) Leela speaks Marathi and English.
(v) It is raining or cold.

4. $(\sim p \vee q) \equiv (p \wedge q) \vee (\sim p)$.

### (E) Negation of a Compound Statement :

Negation of a simple statement is obtained by inserting 'not' at the appropriate place in the statement.
e.g. The negation of 'Ram is a good boy' is 'Ram is not a good boy'.

Now we will study the negations of the compound statements involving conjunction, disjunction, conditional and biconditional.

**(1) Negation of Conjunction :** The conjunction of two simple statements $p$ and $q$ is $p \wedge q$ and the negation of this conjunction is denoted by $\sim (p \wedge q)$. Using truth tables, we show that $\sim (p \wedge q) \equiv \sim p \vee \sim q$.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| $p$ | $q$ | $p \wedge q$ | $\sim (p \wedge q)$ | $\sim p$ | $\sim q$ | $\sim p \vee \sim q$ |
| T | T | T | F | F | F | F |
| T | F | F | T | F | T | T |
| F | T | F | T | T | F | T |
| F | F | F | T | T | T | T |

The entries in the columns 4 and 7 are identical.
$\therefore \sim (p \wedge q) \equiv \sim p \vee \sim q$.

Thus we see that **the negation of the *conjunction* of two statements is the *disjunction* of their negations.**

**Illustration :**

Let $p$ : The sky is cloudy. $q$ : It will be raining.
Then $p \wedge q$ : The sky is cloudy *and* it will be raining.
Now $\sim p$ : The sky is not cloudy.
$\sim q$ : It will not be raining.
$\therefore \sim (p \wedge q) \equiv \sim p \vee \sim q$ gives
$\sim (p \wedge q)$ : The sky is not cloudy *or* it will not be raining.

**(2) Negation of Disjunction :** The disjunction of two simple statements $p$ and $q$ is $p \vee q$ and its negation is denoted by $\sim (p \vee q)$. Using truth tables, we show that $\sim (p \vee q) \equiv \sim p \wedge \sim q$.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| $p$ | $q$ | $p \vee q$ | $\sim (p \vee q)$ | $\sim p$ | $\sim q$ | $\sim p \wedge \sim q$ |
| T | T | T | F | F | F | F |
| T | F | T | F | F | T | F |
| F | T | T | F | T | F | F |
| F | F | F | T | T | T | T |

The entries in the columns 4 and 7 are identical.
$\therefore \sim (p \vee q) \equiv \sim p \wedge \sim q$.

Thus we see that **the negation of the *disjunction* of two statements is the *conjunction* of their negations.**

**Illustration :**

Let $p$ : The sky is cloudy.
$q$ : It will be raining.
Then $p \vee q$ : The sky is cloudy *or* it will be raining.
$\therefore \sim (p \vee q) \equiv \sim p \wedge \sim q$ gives
$\sim (p \vee q)$ : The sky is not cloudy *and* it will not be raining.

**(3) Negation of Negation :** The negation of negation of a statement is the statement itself.
i.e. $\sim (\sim p) \equiv p$

| 1 | 2 | 3 |
|---|---|---|
| $p$ | $\sim p$ | $\sim (\sim p)$ |
| T | F | T |
| F | T | F |

The entries in columns 1 and 3 are identical.
$\therefore \sim (\sim p) \equiv p$.

**Illustration :**

Let $p$ : The sky is cloudy.

The negation of $p$ is given by :

$\sim p$ : The sky is not cloudy.

$\therefore$ the negation of negation of $p$ is $\sim(\sim p)$. It is false that the sky is not cloudy.

**OR** The sky is cloudy.

**(4) Negation of Conditional (Implication) :** The negation of the conditional $p \rightarrow q$ is denoted by $\sim(p \rightarrow q)$. Using truth tables, we show that

$\sim(p \rightarrow q) \equiv p \wedge \sim q$.

| 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| $p$ | $q$ | $p \rightarrow q$ | $\sim(p \rightarrow q)$ | $\sim q$ | $p \wedge \sim q$ |
| T | T | T | F | F | F |
| T | F | F | T | T | T |
| F | T | T | F | F | F |
| F | F | T | F | T | F |

The entries in the columns 4 and 6 are identical.

$\therefore \sim(p \rightarrow q) \equiv p \wedge \sim q$.

**Illustration :**

Let $p$ : The sky is cloudy.

$q$ : It will be raining.

Then $p \rightarrow q$ : *If* the sky is cloudy, *then* it will be raining.

$\therefore \sim(p \rightarrow q) \equiv p \wedge \sim q$ gives

$\sim(p \rightarrow q)$ : The sky is cloudy *and* it will not be raining.

**(5) Negation of Biconditional (Double implication) :** The negation of the biconditional $p \leftrightarrow q$ is denoted by $\sim(p \leftrightarrow q)$. Using truth tables, we show that

$\sim(p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|
| $p$ | $q$ | $\sim p$ | $\sim q$ | $p \leftrightarrow q$ | $\sim(p \leftrightarrow q)$ | $p \wedge \sim q$ | $q \wedge \sim p$ | $(p \wedge \sim q) \vee (q \wedge \sim p)$ |
| T | T | F | F | T | F | F | F | F |
| T | F | F | T | F | T | T | F | T |
| F | T | T | F | F | T | F | T | T |
| F | F | T | T | T | F | F | F | F |

The entries in the columns 6 and 9 are identical.

$\therefore \sim(p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$.

**Illustration :**

Let $p$ : The sky is cloudy.

$q$ : It will be raining.

Then $p \leftrightarrow q$ : The sky is cloudy *iff* it will be raining.

$\therefore \sim(p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$ gives

$\sim(p \leftrightarrow q)$ : The sky is cloudy *and* it will not be raining *or* it will be raining *and* the sky is not cloudy.

**(6) Negation of quantified Statement :**

The negation of the quantified statement is obtained by replacing the word 'all' by 'some', 'for every' by 'there exists', and vice versa.

**Note :** Negation of a statement pattern involving one or more of the simple statements $p, q, r, \ldots$, and one or more of the three connectives $\wedge, \vee, \sim$ is obtained by replacing $\wedge$ by $\vee$, $\vee$ by $\wedge$ and $p, q, r, \ldots$, by $\sim p, \sim q, \sim r, \ldots$ .

e.g. The negation of the statement pattern $(\sim p \vee q) \vee (p \wedge \sim q)$ is $(p \wedge \sim q) \wedge (\sim p \vee q)$.

i.e, $\sim[(\sim p \vee q) \vee (p \wedge \sim q)] \equiv (p \wedge \sim q) \wedge (\sim p \vee q)$.

**(F) Converse, Inverse and Contrapositive :**

If $p \rightarrow q$ is a conditional statement, then

(i) $q \rightarrow p$ is called its converse

(ii) $\sim p \rightarrow \sim q$ is called its inverse and

(iii) $\sim q \rightarrow \sim p$ is called its contrapositive.

**For example :**

**Write the converse, inverse and contrapositive of the statement :**

**'If $r$ is a rational number, then $r$ is a real number.'**

**Solution :**

Let $p$ : $r$ is a rational number.

$q$ : $r$ is a real number.

Then the symbolic form of given compound statement is $p \rightarrow q$.

**Converse :** $q \rightarrow p$ is the converse of $p \rightarrow q$,

i.e. If $r$ is a real number, then $r$ is a rational number.

**Inverse :** $\sim p \rightarrow \sim q$ is the inverse of $p \rightarrow q$,

i.e. If $r$ is not a rational number, then $r$ is not a real number.

**Contrapositive :** $\sim q \rightarrow \sim p$ is the contrapositive of $p \rightarrow q$,

i.e. If $r$ is not a real number, then $r$ is not a rational number.

**EXERCISE 1.8** | **Textbook page 21**

**1. Write negation of each of the following statements :**

**(i) All the stars are shining if it is night.**

**(ii) $\forall n \in N, n + 1 > 0$.**

**(iii) $\exists n \in N$, such that $(n^2 + 2)$ is odd number.**

**(iv) Some continuous functions are differentiable.**

**Solution :**

**(i)** The given statement can be written as :

If it is night, then all the stars are shining.

Let $p$ : It is night.

$q$ : All the stars are shining.

Then the symbolic form of the given statement is $p \to q$.

Since, $\sim(p \to q) \equiv p \wedge \sim q$, the negation of given statement is :

'It is night and all the stars are not shining.'

[**Note :** *Answer in the textbook is incorrect.*]

**(ii)** The negation of the given statement is :

'$\exists\, n \in N$, such that $n+1 \leqslant 0$.'

**(iii)** The negation of the given statement is :

'$\forall\, n \in N$, $n^2+2$ is not an odd number.'

**(iv)** The negation of given statement is :

'All continuous functions are not differentiable.'

**2. Using the rules of negation, write the negations of the following :**

**(i) $(p \to r) \wedge q$ (ii) $\sim(p \vee q) \to r$**

**(iii) $(\sim p \wedge q) \wedge (\sim q \vee \sim r)$.**

**Solution :**

(i) The negation of $(p \to r) \wedge q$ is :

$\sim[(p \to r) \wedge q] \equiv \sim(p \to r) \vee (\sim q)$ ... [Negation of conjunction]

$\equiv (p \wedge \sim r) \vee (\sim q)$ ... [Negation of implication]

(ii) The negation of $\sim(p \vee q) \to r$ is :

$\sim[\sim(p \vee q) \to r] \equiv \sim(p \vee q) \wedge (\sim r)$ ... [Negation of implication]

$\equiv (\sim p \wedge \sim q) \wedge (\sim r)$ ... [Negation of disjunction]

(iii) The negation of $(\sim p \wedge q) \wedge (\sim q \vee \sim r)$ is :

$\sim[(\sim p \wedge q) \wedge (\sim q \vee \sim r)] \equiv \sim(\sim p \wedge q) \vee \sim(\sim q \vee \sim r)$ ... [Negation of conjunction]

$\equiv [\sim(\sim p) \vee \sim q] \vee [\sim(\sim q) \wedge \sim(\sim r)]$ ... [Negation of conjunction and disjunction]

$\equiv (p \vee \sim q) \vee (q \wedge r)$ ... [Negation of negation]

**3. Write the converse, inverse and contrapositive of the following statements :**

**(i) If its snows, then they do not drive the car.**

**(ii) If he studies, then he will go to college.**

**Solution :**

(i) Let $p$ : It snows.

$q$ : They do not drive the car.

Then the symbolic form of the given statement is $p \to q$.

**Converse :** $q \to p$ is the converse of $p \to q$.

i.e. If they do not drive the car, then it snows.

**Inverse :** $\sim p \to \sim q$ is the inverse of $p \to q$.

i.e. If it does not snow, then they drive the car.

**Contrapositive :** $\sim q \to \sim p$ is the contrapositive of $p \to q$. i.e. If they drive the car, then it does not snow.

**(ii)** Let $p$ : He studies.

$q$ : He will go to college.

Then two symbolic form of the given statement is $p \to q$.

**Converse :** $q \to p$ is the converse of $p \to q$.

i.e. If he will go to college, then he studies.

**Inverse :** $\sim p \to \sim q$ is the inverse of $p \to q$.

i.e. If he does not study, then he will not go to college.

**Contrapositive :** $\sim q \to \sim p$ is the contrapositive of $p \to q$. i.e. If he will not go to college, then he does not study.

**4. With proper justification, state the negation of each of the following :**

**(i) $(p \to q) \vee (p \to r)$ (ii) $(p \leftrightarrow q) \vee (\sim q \to \sim r)$**

**(iii) $(p \to q) \wedge r$.**

**Solution :**

(i) The negation of $(p \to q) \vee (p \to r)$ is :

$\sim[(p \to q) \vee (p \to r)] \equiv \sim(p \to q) \wedge \sim(p \to r)$ ... [Negation of disjunction]

$\equiv (p \wedge \sim q) \wedge (p \wedge \sim r)$ ... [Negation of implication]

**(ii)** The negation of $(p \leftrightarrow q) \vee (\sim q \to \sim r)$ is :

$\sim[(p \leftrightarrow q) \vee (\sim q \to \sim r)] \equiv \sim(p \leftrightarrow q) \wedge \sim(\sim q \to \sim r)$ ... [Negation of disjunction]

$\equiv [(p \wedge \sim q) \vee (q \wedge \sim p)] \wedge [\sim q \wedge \sim(\sim r)]$ ... [Negation of biconditional and implication]

$\equiv [(p \wedge \sim q) \vee (q \wedge \sim p)] \wedge (\sim q \wedge r)$ ... [Negation of negation]

[**Note :** *Answer in the textbook is incorrect.*]

**(iii)** The negation of $(p \to q) \wedge r$ is :

$\sim[(p \to q) \wedge r] \equiv \sim(p \to q) \vee (\sim r)$ ... [Negation of conjunction]

$\equiv (p \wedge \sim q) \vee (\sim r)$ ... [Negation of implication]

## ADDITIONAL SOLVED PROBLEMS – 1 (B)

**1. Write the following statements in symbolic form and write their negations :**

**(i) $x+3 < 5$ or $y+5=9$.**

**(ii) $7 > 2$ and $3 > 10$.**

**(iii) Either I will be busy or I will help you.**

**(iv) Madhuri will not sing and she will not dance.**

**(v) The number is an odd number if and only if it is not divisible by 2.**

**(vi) The number is neither odd nor even.**

**Solution :**

**(i)** Let $p : x+3 < 5$, $q : y+5=9$.

Then the symbolic form of given statement is : $p \vee q$.

Since, $\sim(p \vee q) \equiv \sim p \wedge \sim q$, the negation of given statement is :

'$x+3 \nless 5$ and $y+5 \neq 9$.'

**OR** '$x+3 \geqslant 5$ and $y+5 \neq 9$.'

**(ii)** Let $p : 7 > 2$, $q : 3 > 10$.

Then the symbolic form of given statement is : $p \wedge q$.

Since, $\sim(p \wedge q) \equiv \sim p \vee \sim q$, the negation of given statement is :

'$7 \ngtr 2$ or $3 \ngtr 10$.'

**OR** '$7 \leqslant 2$ or $3 \leqslant 10$.'

**(iii)** Let $p$ : I will be busy.

$q$ : I will help you.

Then the symbolic form of given statement is : $p \vee q$.

Since, $\sim(p \vee q) \equiv \sim p \wedge \sim q$, the negation of given statement is :

'I will not be busy and I will not help you.'

**OR** 'Neither I will be busy nor I will help you.'

**(iv)** Let $p$ : Madhuri will sing.

$q$ : She will dance.

Then the symbolic form of given statement is :

$\sim p \wedge \sim q$.

Since, $\sim(\sim p \wedge \sim q) \equiv \sim(\sim p) \vee \sim(\sim q) \equiv p \vee q$,

the negation of given statement is :

'Madhuri will sing or she will dance.'

**(v)** Let $p$ : The number is an odd number.

$q$ : It is not divisible by 2.

Then the symbolic form of given statement is : $p \leftrightarrow q$.

Since, $\sim(p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$, the negation of given statement is :

'The number is an odd number and it is divisible by 2 or the number is not divisible by 2 and it is an odd number.'

**(vi)** Let $p$ : The number is odd.

$q$ : The number is even.

$\therefore$ the symbolic form of given statement is :

$\sim p \wedge \sim q$

Since, $\sim(\sim p \wedge \sim q) \equiv \sim(\sim p) \vee \sim(\sim q) \equiv p \vee q$,

the negation of given statement is :

'The number is either odd or even.'

**2. State whether the following statements are negation of each other. Justify :**

**(i) A man is not mortal.**

**A man is not immortal.**

**(ii) Swapnil likes kabaddi and kho-kho.**

**Swapnil likes neither kabaddi nor kho-kho.**

**Solution :**

**(i)** Let $p$ : A man is not mortal.

The negation of $p$ is : $\sim p$ : A man is mortal.

i.e. A man is not immortal.

Hence, the given statements are negation of each other.

**(ii)** Let $p$ : Swapnil likes kabaddi.

$q$ : Swapnil likes kho-kho.

Then the symbolic form of given statement is : $p \wedge q$.

Since, $\sim(p \wedge q) \equiv \sim p \vee \sim q$, the negation of given statement is :

'Swapnil does not like kabaddi or Swapnil does not like kho-kho.'

Hence, the given statements are not the negation of each other.

## EXAMPLES FOR PRACTICE 1.8

**1. Write the negation of the following statements :**

(i) $\sqrt{2}$ is an irrational number.

(ii) Some students don't like studying.

(iii) All men are animals.

(iv) All politicians are honest.

(v) For every real number $x$, there exists a real number $y$ such that $x$ divides $y$.
(vi) I will have tea or coffee.
(vii) All men take water and meal.
(viii) Chetan has black hair and blue eyes.
(ix) $\exists x \in R$, such that $x^2 + 3 > 0$.
(x) $\forall n \in N$, $2n + 1$ is odd.
(xi) $\forall y \in N$, $y^2 + 3 \leqslant 7$.
(xii) If the lines are parallel, then their slopes are equal.
(xiii) Radha likes tea or coffee.
(xiv) $\exists x \in R$, such that $x + 3 \geqslant 10$.

**2. Write the following statements in symbolic form and write their negations :**

(i) $2 + 3 < 6$ or $\sqrt{2}$ is an irrational number.
(ii) Madhuri is a girl and Gopal is a boy.
(iii) $2 + 3 = 5$ or $4 + 8 = 12$.
(iv) If $x = 1$, then $x^2 = -1$.
(v) A triangle is an equilateral if and only if it is an equiangular triangle.
(vi) If the diagonals of a parallelogram are perpendicular, then it is a rhombus.
(vii) I like Mathematics or English.
(viii) If a quadrilateral is a rectangle, then it is a parallelogram.
(ix) $2 \times 3 = 6$ and $2 + 3 \neq 4$.
(x) $2 + 4 > 7$ or $\pi$ is rational.

**3. Using the rules of negation, write the negations of the following statements and justify :**

(i) $p \wedge \sim q$
(ii) $\sim p \vee \sim q$
(iii) $\sim q \to p$
(iv) $q \vee (p \wedge \sim q)$
(v) $(p \vee \sim q) \wedge r$
(vi) $(\sim p \wedge q) \vee (p \wedge \sim q)$
(vii) $(p \vee q) \wedge r$
(viii) $(\sim p \to q) \wedge r$
(ix) $p \to (p \vee \sim q)$
(x) $(p \vee \sim q) \to (p \wedge \sim q)$.

**4. Write the converse, inverse and contrapositive of the following statements :**

(i) The crop will be destroyed if there is a flood.
(ii) If Ravi is good in Logic, then he is good in Mathematics.
(iii) If a function is differentiable, then it is continuous.

## Answers

**1.** (i) $\sqrt{2}$ is not an irrational number.
(ii) All students like studying.
(iii) Some men are not animals.
(iv) Some politicians are not honest.
(v) There exists a real number $x$ such that for all real number $y$, $x$ does not divide $y$.
(vi) I will not have tea and coffee.
(vii) Some men take neither water nor meal.
(viii) Chetan has no black hair or no blue eyes.
(ix) $\forall x \in R$, $x^2 + 3 \leqslant 0$.
(x) $\exists n \in N$, such that $2n + 1$ is not odd.
(xi) $\exists y \in N$, such that $y^2 + 3 > 7$.
(xii) The lines are parallel and their slopes are not equal.
(xiii) Radha neither likes tea nor coffee.
(xiv) $\forall x \in R$, $x + 3 < 10$.

**2.**

| | Symbolic Form | Negations |
|---|---|---|
| (i) | $p \vee q$ | $2 + 3 \nless 6$ and $\sqrt{2}$ is not an irrational number. |
| (ii) | $p \wedge q$ | Madhuri is not a girl or Gopal is not a boy. |
| (iii) | $p \vee q$ | $2 + 3 \neq 5$ and $4 + 8 \neq 12$. |
| (iv) | $p \to q$ | $x = 1$ and $x^2 \neq -1$. |
| (v) | $p \leftrightarrow q$ | A triangle is an equilateral and it is not equiangular triangle or the triangle is an equiangular and it is not equilateral triangle. |
| (vi) | $p \to q$ | The diagonals of a parallelogram are perpendicular and it is not a rhombus. |
| (vii) | $p \vee q$ | I do not like Mathematics and English. OR Neither I like Mathematics nor English. |
| (viii) | $p \to q$ | A quadratical is a rectangle and it is not a parallelogram. |
| (ix) | $p \wedge q$ | $2 \times 3 \neq 6$ or $2 + 3 = 4$. |
| (x) | $p \vee q$ | $2 + 4 \leqslant 7$ and $\pi$ is not rational. |

**3.** (i) $\sim (p \wedge \sim q)$

$\equiv \sim p \vee \sim (\sim q)$ ... (Negation of conjunction)

$\equiv \sim p \vee q$ ... (Negation of negation)

(ii) $\sim (\sim p \vee \sim q)$

$\equiv \sim (\sim p) \wedge \sim (\sim q)$ ... (Negation of disjunction)

$\equiv p \wedge q$ ... (Negation of negation)

(iii) $\sim(\sim q \to p)$

$\equiv \sim q \wedge \sim p$ ... (Negation of implication)

$\equiv \sim(q \vee p)$ ... (Negation of disjunction)

(iv) $\sim[q \vee (p \wedge \sim q)]$

$\equiv \sim q \wedge [\sim(p \wedge \sim q)]$ ... (Negation of disjunction)

$\equiv \sim q \wedge [\sim p \vee \sim(\sim q)]$ ... (Negation of conjunction)

$\equiv \sim q \wedge (\sim p \vee q)$ ... (Negation of negation)

(v) $\sim[(p \vee \sim q) \wedge r]$

$\equiv \sim(p \vee \sim q) \vee (\sim r)$ ... (Negation of conjunction)

$\equiv [\sim p \wedge \sim(\sim q)] \vee (\sim r)$

... (Negation of disjunction)

$\equiv (\sim p \wedge q) \vee (\sim r)$ ... (Negation of negation)

(vi) We know that

$\sim(p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$

$\equiv (\sim p \wedge q) \vee (p \wedge \sim q)$

$\therefore \sim[(\sim p \wedge q) \vee (p \wedge \sim q)] = p \leftrightarrow q.$

(vii) $\sim[(p \vee q) \wedge r]$

$\equiv \sim(p \vee q) \vee (\sim r)$ ... (Negation of conjunction)

$\equiv (\sim p \wedge \sim q) \vee (\sim r)$ ... (Negation of disjunction)

(viii) $\sim[(\sim p \to q) \wedge r]$

$\equiv \sim(\sim p \to q) \vee (\sim r)$ ... (Negation of conjunction)

$\equiv (\sim p \wedge \sim q) \vee (\sim r)$ ... (Negation of implication)

(ix) $\sim[p \to (p \vee \sim q)]$

$\equiv p \wedge \sim(p \vee \sim q)$ ... (Negation of implication)

$\equiv p \wedge [\sim p \wedge \sim(\sim q)]$ ... (Negation of disjunction)

$\equiv p \wedge (\sim p \wedge q)$ ... (Negation of negation)

(x) $\sim[(p \vee \sim q) \to (p \wedge \sim q)]$

$\equiv (p \vee \sim q) \wedge \sim(p \wedge \sim q)$

... (Negation of implication)

$\equiv (p \vee \sim q) \wedge [\sim p \vee \sim(\sim q)]$

... (Negation of conjunction)

$\equiv (p \vee \sim q) \wedge (\sim p \vee q)$ ... (Negation of negation)

4. (i) **Converse :** If the crop will be destroyed, then there is a flood.

**Inverse :** If there is no flood, then the crop will not be destroyed.

**Contrapositive :** If the crop will not be destroyed, then there is no flood.

(ii) **Converse :** If Ravi is good in Mathematics, then he is good in Logic.

**Inverse :** If Ravi is not good in Logic, then he is not good in Mathematics.

**Contrapositive :** If Ravi is not good in Mathematics, then he is not good in Logic.

(iii) **Converse :** If a function is continuous, then it is differentiable.

**Inverse :** If a function is not differentiable, then it is not continuous.

**Contrapositive :** If a function is not continuous, then it is not differentiable.

## 1.4 : ALGEBRA OF STATEMENTS

Some standard equivalent statements in logic are listed below which can be proved by using truth tables.

1. **Idempotent Laws :**

(i) $p \vee p \equiv p$ (ii) $p \wedge p \equiv p$

2. **Commutative Laws :**

(i) $p \vee q \equiv q \vee p$ (ii) $p \wedge q \equiv q \wedge p$

3. **Associative Laws :**

(i) $p \vee (q \vee r) \equiv (p \vee q) \vee r \equiv p \vee q \vee r$

(ii) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r \equiv p \wedge q \wedge r$

4. **Distributive Laws :**

(i) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(ii) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

5. **Absorption Laws :**

(i) $p \vee (p \wedge q) \equiv p$ (ii) $p \wedge (p \vee q) \equiv p$

6. **De Morgan's Laws :**

(i) $\sim(p \vee q) \equiv \sim p \wedge \sim q$

(ii) $\sim(p \wedge q) \equiv \sim p \vee \sim q$

7. **Complement Laws :**

(i) $p \vee \sim p \equiv t$ (ii) $p \wedge \sim p \equiv c$

(iii) $\sim t \equiv c$ (iv) $\sim c \equiv t.$

8. **Involution Law :** $\sim(\sim p) \equiv p$

9. **Identity Laws :**

(i) $p \vee c \equiv p$ (ii) $p \wedge c \equiv c$

(iii) $p \vee t \equiv t$ (iv) $p \wedge t \equiv p$

10. **Conditional Laws :**

(i) $p \to q \equiv \sim q \to \sim p \equiv \sim p \vee q$

(ii) $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p) \equiv (\sim p \vee q) \wedge (\sim q \vee p).$

**Notes :**

(i) $p \wedge q \wedge r$ is true if and only if $p$, $q$, $r$ are all true and $p \wedge q \wedge r$ is false if at least any one of $p$, $q$, $r$ is false.

(ii) $p \vee q \vee r$ is false if and only if $p$, $q$, $r$ are all false and $p \vee q \vee r$ is true if at least any one of $p$, $q$, $r$ is true.

(iii) $p \to q \equiv \sim q \to \sim p$ is also called *contrapositive Law.*

**EXERCISE 1.9** | **Textbook page 22**

**1. Without using truth table, show that**

(i) $p \leftrightarrow q \equiv (p \wedge q) \vee (\sim p \wedge \sim q)$

(ii) $p \wedge [\sim p \vee q) \vee (\sim q)] \equiv p$

(iii) $\sim [(p \wedge q) \rightarrow (\sim q)] \equiv p \wedge q$

(iv) $\sim r \rightarrow \sim (p \wedge q) \equiv [\sim (q \rightarrow r)] \rightarrow (\sim p)$

(v) $(p \vee q) \rightarrow r \equiv (p \rightarrow r) \wedge (q \rightarrow r)$.

**Solution :**

(i) LHS $= p \leftrightarrow q$

$\equiv (p \rightarrow q) \wedge (q \rightarrow p)$

$\equiv (\sim p \vee q) \wedge (\sim q \vee p)$ ... (Conditional Law)

$\equiv [\sim p \wedge (\sim q \vee p)] \vee [q \wedge (\sim q \vee p]$ ... (Distributive Law)

$\equiv [(\sim p \wedge \sim q) \vee (\sim p \wedge p)] \vee [(q \wedge \sim q) \vee (q \wedge p)]$ ... (Distributive Law)

$\equiv [(\sim p \wedge \sim q) \vee c] \vee [c \vee (q \wedge p)]$ ... (Complement Law)

$\equiv (\sim p \wedge \sim q) \vee (q \wedge p)$ ... (Identity Law)

$\equiv (\sim p \wedge \sim q) \vee (p \wedge q)$ ... (Commutative Law)

$\equiv (p \wedge q) \vee (\sim p \wedge \sim q)$ ... (Commutative Law)

$=$ RHS.

(ii) LHS $= p \wedge [(\sim p \vee q) \vee (\sim q)]$

$\equiv p \wedge [\sim p \vee (q \vee \sim q)]$ ... (Associative Law)

$\equiv p \wedge [\sim p \vee t]$ ... (Complement Law)

$\equiv p \wedge t$ ... (Identity Law)

$\equiv p$ ... (Identity Law)

$=$ RHS.

(iii) LHS $= \sim [(p \wedge q) \rightarrow (\sim q)]$

$\equiv (p \wedge q) \wedge \sim (\sim q)$ ... (Negation of implication)

$\equiv (p \wedge q) \wedge q$ ... (Negation of negation)

$\equiv p \wedge (q \wedge q)$ ... (Associative Law)

$\equiv p \wedge q$ ... (Idempotent Law)

$=$ RHS

(iv) LHS $= \sim r \rightarrow \sim (p \wedge q)$

$\equiv \sim r \rightarrow (\sim p \vee \sim q)$ ... (De Morgan's Law)

$\equiv \sim (\sim r) \vee (\sim p \vee \sim q)$ ... (Conditional Law)

$\equiv r \vee (\sim p \vee \sim q)$ ... (Involution Law)

$\equiv r \vee \sim q \vee \sim p$ ... (Commutative Law)

$\equiv (\sim q \vee r) \vee (\sim p)$ ... (Commutative Law)

$\equiv (q \rightarrow r) \vee (\sim p)$ ... (Conditional Law)

$\equiv \sim (q \rightarrow r) \rightarrow (\sim p)$ ... (Conditional Law)

$=$ RHS.

(v) LHS $= (p \vee q) \rightarrow r$

$\equiv \sim (p \vee q) \vee r$ ... (Conditional Law)

$\equiv (\sim p \wedge \sim q) \vee r$ ... (De Morgan's Law)

$\equiv (\sim p \vee r) \wedge (\sim q \vee r)$ ... (Distributive Law)

$\equiv (p \rightarrow r) \wedge (q \rightarrow r)$ ... (Conditional Law)

$=$ RHS.

**2. Using the algebra of statement, prove that :**

(i) $[p \wedge (q \vee r)] \vee [\sim r \wedge \sim q \wedge p] \equiv p$

(ii) $(p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge \sim q) \equiv p \vee \sim q$

(iii) $(p \vee q) \wedge (\sim p \vee \sim q) \equiv (p \wedge \sim q) \vee (\sim p \wedge q)$.

**Solution :**

(i) LHS $= [p \wedge (q \vee r)] \vee [\sim r \wedge \sim q \wedge p]$

$\equiv [p \wedge (q \vee r)] \vee [(\sim r \wedge \sim q) \wedge p]$ ... (Associative Law)

$\equiv [p \wedge (q \vee r)] \vee [(\sim q \wedge \sim r) \wedge p]$ ... (Commutative Law)

$\equiv [p \wedge (q \vee r)] \vee [\sim (q \vee r) \wedge p]$ ... (De Morgan's Law)

$\equiv [p \wedge (q \vee r)] \vee [p \wedge \sim (q \vee r)]$ ... (Commutative Law)

$\equiv p \wedge [(q \vee r) \vee \sim (q \vee r)]$ ... (Distributive Law)

$\equiv p \wedge t$ ... (Complement Law)

$\equiv p$ ... (Identity Law)

$=$ RHS.

(ii) LHS $= (p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge \sim q)$

$\equiv (p \wedge q) \vee [(p \wedge \sim q) \vee (\sim p \wedge \sim q)]$ ... (Associative Law)

$\equiv (p \wedge q) \vee [(\sim q \wedge p) \vee (\sim q \wedge \sim p)]$ ... (Commutative Law)

$\equiv (p \wedge q) \vee [\sim q \wedge (p \vee \sim p)]$ ... (Distributive Law)

$\equiv (p \wedge q) \vee (\sim q \wedge t)$ ... (Complement Law)

$\equiv (p \wedge q) \vee (\sim q)$ ... (Identity Law)

$\equiv (p \vee \sim q) \wedge (q \vee \sim q)$ ... (Distributive Law)

$\equiv (p \vee \sim q) \wedge t$ ... (Complement Law)

$\equiv p \vee \sim q$ ... (Identity Law)

$=$ RHS.

(iii) LHS $= (p \vee q) \wedge (\sim p \vee \sim q)$

$\equiv [p \wedge (\sim p \vee \sim q)] \vee [q \wedge (\sim p \vee \sim q)]$ ... (Distributive Law)

$\equiv [(p \wedge \sim p) \vee (p \wedge \sim q)] \vee [q \wedge \sim p) \vee (q \wedge \sim q)]$ ... (Distributive Law)

$\equiv [c \vee (p \wedge \sim q)] \vee [(q \wedge \sim p) \vee c]$

... (Complement Law)

$\equiv (p \wedge \sim q) \vee (q \wedge \sim p)$ ... (Identity Law)

$\equiv (p \wedge \sim q) \vee (\sim p \wedge q)$ ... (Commutative Law)

$=$ RHS.

[**Note :** *Question is modified.*]

## EXAMPLES FOR PRACTICE 1.9

**1. Without using truth tables, show that**

(i) $p \wedge (q \vee \sim p) \equiv p \wedge q$

(ii) $(p \wedge q) \vee (\sim p \wedge q) \vee (\sim q \wedge r) \equiv q \vee r$.

**2. Using algebra of statements, prove that**

(i) $(p \wedge q) \vee (\sim p \wedge q) \vee (p \wedge \sim q) \equiv p \vee q$

(ii) $p \vee \{[\sim p \wedge (p \vee q)] \vee (q \wedge p)\} = p \vee q$

(iii) $(p \vee q) \wedge (p \vee \sim q) \equiv p$.

**3. Without using truth table, show that**

(i) $\sim (p \vee q) \vee (\sim p \wedge q) \equiv \sim p$

(ii) $\sim p \wedge q \equiv (p \vee q) \wedge (\sim p)$

(iii) $p \leftrightarrow q \equiv \sim (p \wedge \sim q) \wedge \sim (q \wedge \sim p)$

(iv) $\sim [(p \vee \sim q) \rightarrow (p \wedge \sim q)] \equiv (p \vee \sim q) \wedge (\sim p \vee q)$.

## 1.5 : VENN DIAGRAMS

We know that Venn diagrams are used to represent sets and to study the interrelations between sets. In Logic also, we can use Venn diagrams to represent the truth of certain statements.

We shall use the following notations :

U : universal set, X and Y : subsets of U. Any element of X will be denoted by $x$ and any element of Y will be denoted by $y$.

We consider the Venn diagrams representing the truth of the following statements :

**(1) All $x$'s are $y$'s :** There are two possibilities :

(i) $X \subseteq Y$, then all the elements of the set X, i.e. all $x$'s are in the set Y, i.e. they are $y$'s.

(ii) $X = Y$, i.e. $x$'s are precisely $y$'s.

**For example :**

(i) Consider the statement :

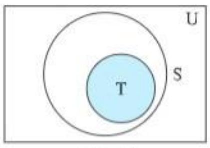
'All film stars are smart.'

Let U : set of all human beings

S : set of all smart persons

T : set of all film stars.

Then the Venn diagram represents the truth of the given statement is as below :




$T \subseteq S$

(ii) Consider the statement :

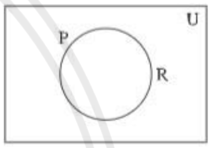
'All cyclic parallelograms are rectangles.'

Let U : set of all quadrilaterals

P : set of all cyclic parallelograms

R : set of all rectangle.

Then the Venn diagram represents the truth of the given statement is as below :




$P = R$

**(2) No $x$'s are $y$'s :** If X and Y are disjoint sets, i.e. $X \cap Y = \phi$, then no element of X can be an element of Y, i.e. no $x$'s are $y$'s.

The Venn diagram for disjoint sets represents this situation.

**For example :**

(i) Consider the statement :

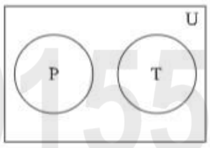
'No policeman is a thief.'

Let U : set of all human beings

P : set of policemen

T : set of thieves.

Then the Venn diagram represents the truth of the given statement is as below :




$P \cap T = \phi$

(ii) Consider the statement :

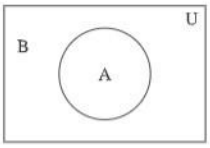
'No odd integer is an even integer.'

Let U : set of all integers

A : set of all odd integers

B : set of all even integers.

Then the Venn diagram represents the truth of the given statement is as below :




$A \cap B = \phi$

**(3) Some $x$'s are $y$'s :** 'Some' means 'at least one'. If $X \neq Y$ and $X \cap Y \neq \phi$, i.e. X and Y are not disjoint, then some $x$'s are in Y, i.e. they are $y$'s, which appear in the intersection of X and Y.

The Venn diagram for non-empty intersection of the sets X and Y represents this situation.

**For example :**

(i) Consider the statement :

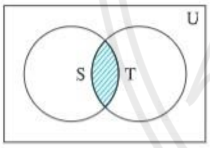
'Some students are scholars.'

Let U : set of all human beings

S : set of scholars

T : set of all students.

Then the Venn diagram represents the truth of the given statement is as follow :




$S \cap T \neq \phi$

(ii) Consider the statement :

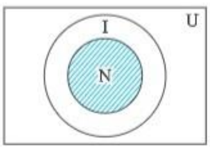
'Some integers are natural numbers'

Let U : set of all real numbers

I : set of all integers

N : set of all natural numbers.

Then the Venn diagram represents the truth of the given statement is as below :




$I \cap N \neq \phi$

**(4) Some $x$'s are not $y$'s :** If X and Y are not disjoint sets, then some $x$'s are not in Y, i.e. they are not $y$'s, which appear in the complement of the set X.

The Venn diagram for complement of set X represents this situation.

**For example :**

(i) Consider the statement :

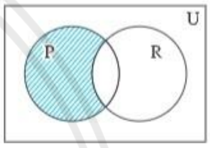
'Some parallelograms are not rhombus.'

Let U : set of all quadrilaterals

P : set of all parallelograms

R : set of all rhombus.

Then the Venn diagram represents the truth of the given statement is as below :




$P - R \neq \phi$

(ii) Consider the statement :

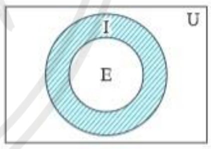
'Some isosceles triangles are not equilateral triangles.'

Let U : set of all triangles

I : set of all isosceles triangles

E : set of all equilateral triangles.

Then the Venn diagram represents the truth of the given statement is as below :




$I - E \neq \phi$

**(5) All $y$'s are $y$'s :** If the sets X and Y are equal, i.e. $X = Y$, then every $x$ is $y$ and every $y$ is $x$, i.e. all $y$'s are $y$'s.

The Venn diagram for equal sets represents this situation.

**For example :**

Consider the statement :

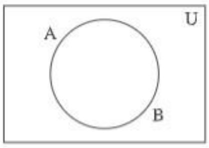
'Equilateral triangles are equiangular.'

Let U : set of all triangles

A : set of all equilateral triangles

B : set of all equiangular triangles.

Then the Venn diagram represents the truth of the given statement is as below :



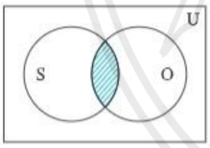

A = B

| EXERCISE 1.10 | Textbook page 27 |
|---|---|

1. **Express the truth of each of the following statements by Venn diagrams :**
   **(i) Some hardworking students are obedient.**
   **(ii) No circles are polygons.**
   **(iii) All teachers are scholars and scholars are teachers.**
   **(iv) If a quadrilateral is a rhombus, then it is a parallelogram.**

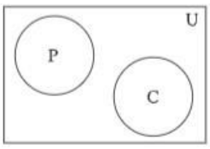
**Solution :**

**(i)** Let U : set of all students
S : set of all hardworking students
O : set of all obedient students.
Then the Venn diagram represents the truth of the given statement is as below :



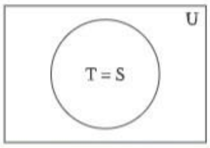

$S \cap O \neq \phi$

**(ii)** Let U : set of closed geometrical figures in plane
P : set of all polygons
C : set of all circles.
Then the Venn diagram represents the truth of the given statement is as below :

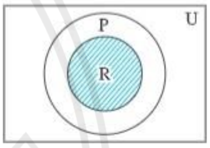



$P \cap C = \phi$

**(iii)** Let U : set of all human beings
T : set of all teachers
S : set of all scholars.
Then the Venn diagram represents the truth of the given statement is as below :




**(iv)** Let U : set of all quadrilaterals
R : set of all rhombus
P : set of all parallelograms.
Then the Venn diagram represents the truth of the given statement is as below :

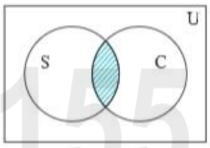



$R \subset P$

2. **Draw the Venn diagrams for the truth of the following statements :**
   **(i) Some share brokers are chartered accountants.**
   **(ii) No wicket keeper is bowler in a cricket team.**

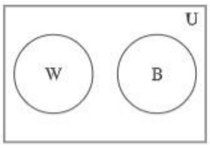
**Solution :**

**(i)** Let U : set of all human beings
S : set of all share brokers
C : set of all chartered accountants.
Then the Venn diagram represents the truth of the given statement is as below :




$S \cap C \neq \phi$

**(ii)** Let U : set of all human beings
W : set of all wicket keepers
B : set of all bowlers.
Then the Venn diagram represents the truth of the given statement is as follows :




$W \cap B = \phi$

**3. Represent the following statements by Venn diagrams:**

**(i) Some non-resident Indians are not rich.**

**(ii) No circle is a rectangle.**

**(iii) If $n$ is a prime number and $n \neq 2$, then it is odd.**

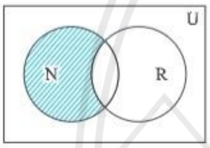
**Solution :**

**(i)** Let U : set of all human beings

N : set of all non-resident Indians

R : set of all rich people.

Then the Venn diagram represents the truth of the given statement is as below :



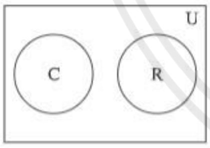

$N - R \neq \phi$

**(ii)** Let U : set of all geometrical figures

C : set of all circles

R : set of all rectangles.

Then the Venn diagram represents the truth of the given statement is as below :



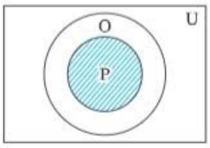

$C \cap R = \phi$

**(iii)** Let U : set of all real numbers

P : set of all prime numbers $n$, where $n \neq 2$

O : set of all odd numbers.

Then the Venn diagram represents the truth of the given statement is as below :




$P \subset O$

## EXAMPLES FOR PRACTICE 1.10

**1. Express the truth of each of the following statements by Venn diagrams :**

(i) Some children are naughty children.

(ii) Every differentiable function is continuous.

(iii) All rational numbers are real numbers.

(iv) All students are hardworking.

(v) Some rectangles are squares.

(vi) No filmstar is a director.

(vii) No obtuse angled triangle is an equilateral triangle.

(viii) Some teachers are not sincere.

(ix) Some multiples of 5 are also multiple of 2.

(x) A number is divisible by 3, if the sum of its digits is divisible by 3.

(xi) Some quadratic equations have equal roots.

**2. Draw Venn diagram to represent the following statements, assuming them to be true :**

(i) All doctors are honest.

(ii) Some doctors are honest.

**3. Draw Venn diagrams to represent the truth of the following statements :**

(i) No teacher is rich.

(ii) All engineers are intelligent.

**4. Express the truth of the following statements by Venn diagrams :**

(i) No policemen are thieves.

(ii) Some students are hardworking.

**5. Represent the following statements by Venn diagrams :**

(i) Some politicians are honest.

(ii) No straight line is a circle.

**6. Draw Venn diagrams to represent the following statements assuming them to be true :**

(i) Every rectangle is a parallelogram.

(ii) No trapezium is a parallelogram.

**7. Draw the Venn diagrams for the truth of the following statements :**

(i) Some teachers are scholars.

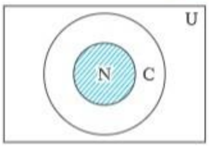
(ii) There are teachers who are scholars.

(iii) There are scholars who are not teachers.

8. **Using the Venn diagrams, examine the logical equivalence of the following statements :**
   (i) Some politicians are actors.
   (ii) There are politicians who are actors.
   (iii) There are politicians who are not actors.

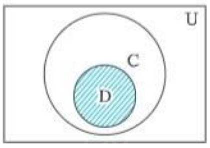
## Answers

1. (i)



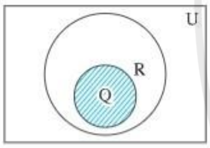

U : set of all human beings
C : set of all children
N : set of all naughty children.

(ii)



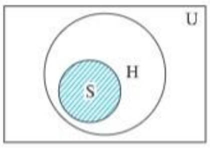

U : set of all functions
C : set of all continuous functions
D : set of all differentiable functions.

(iii)



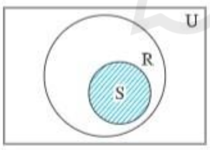

U : set of all complex numbers
Q : set of all rational numbers
R : set of all real numbers.

(iv)



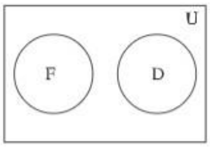

U : set of all human beings
H : set of all hardworking persons
S : set of all students.

(v)



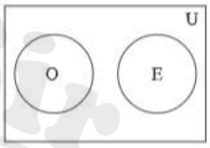

U : set of all quadrilaterals
R : set of all rectangles
S : set of all squares.

(vi)



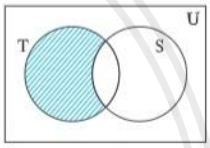

U : set of all human beings
F : set of all filmstars
D : set of all directors.

(vii)



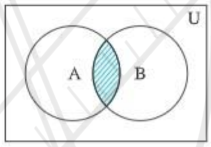

U : set of all triangles
O : set of all obtuse angled triangles
E : set of all equilateral triangles.

(viii)



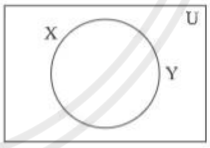

U : set of all human beings
T : set of all teachers
S : set of all sincere persons.

(ix)



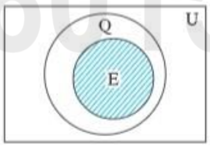

U : set of all numbers
A : set of all multiples of 5
B : set of all multiples of 2.

(x)



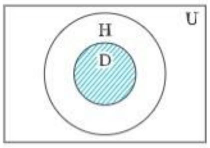

U : set of all integers
X : set of all numbers divisible by 3
Y : set of those numbers the sum of whose digits is divisible by 3.

(xi)

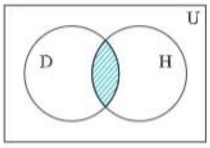



U : set of all equations
Q : set of all quadratic equations
E : set of all quadratic equations having equal roots.

2. (i)

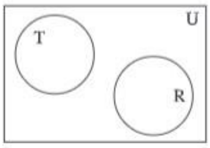



(ii)



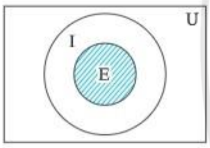

U : set of all human beings
D : set of all doctors
H : set of all honest persons.

3. (i)



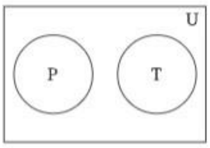

U : set of all human beings
T : set of all teachers
R : set of all rich persons.

(ii)



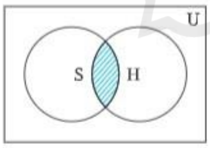

U : set of all human beings
E : set of all engineers
I : set of all intelligent persons.

4. (i)



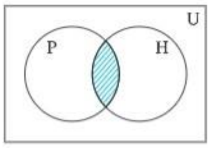

U : set of all human beings
P : set of all policemen
T : set of all thieves.

(ii)



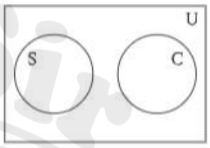

U : set of all human beings
S : set of all students
H : set of all hardworking persons.

5. (i)



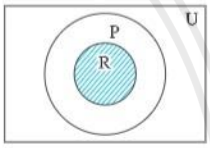

U : set of all human beings
P : set of all politicians
H : set of all honest persons.

(ii)

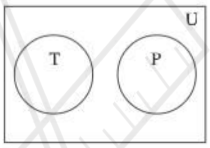



U : set of all geometrical figures
S : set of all straight lines
C : set of all circles.

6. (i)

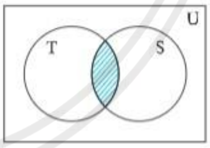



(ii)

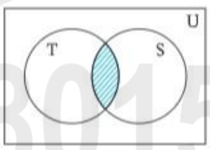



U : set of all quadrilaterals
R : set of all rectangles
P : set of all parallelograms
T : set of all trapeziums.

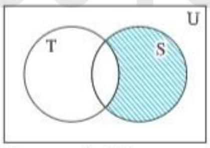
7. (i)




(ii)

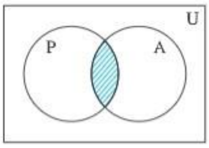



(iii)

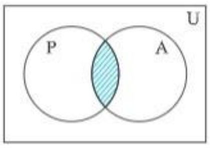



U : set of all human beings
T : set of all teachers
S : set of all scholars.

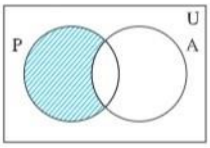
8. (i)




(ii)




(iii)




U : set of all human beings

P : set of all politicians

A : set of all actors.

By the Venn diagrams, we observe that the Venn diagrams of statements (i) and (ii) are same. Hence, statements (i) and (ii) are logically equivalent.

## MISCELLANEOUS EXERCISE – 1

**(Textbook pages 29 to 34)**

**(I) Choose the correct alternative :**

1. Which of the following is not a statement?
   (a) Smoking is injurious to health.
   (b) $2+2=4$.
   (c) 2 is only even prime number.
   (d) Come here.

2. Which of the following is an open statement?
   (a) $x$ is a natural number.
   (b) Give me a glass of water.
   (c) Wish you best of luck.
   (d) Good morning to all.

3. Let $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$. Then this law is known as
   (a) Commutative law (b) Associative law
   (c) De Morgan's law (d) Distributive law

4. The false statement in the following is :
   (a) $p \wedge (\sim p)$ is contradiction
   (b) $(p \to q) \leftrightarrow (\sim q \to \sim p)$ is a contradiction
   (c) $\sim(\sim p) \leftrightarrow p$ is a tautology
   (d) $p \vee (\sim p) \leftrightarrow p$ is a tautology.

5. Consider the following three statements
   $p$ : 2 is an even number.
   $q$ : 2 is a prime number.
   $r$ : Sum of two prime numbers is always even.
   Then, the symbolic statement $(p \wedge q) \to \sim r$ means :
   (a) 2 is an even and prime number and the sum of two prime numbers is always even.
   (b) 2 is an even and prime number and the sum of two prime numbers is not always even.
   (c) If 2 is an even and prime number, then the sum of two prime numbers is not always even.
   (d) If 2 is an even and prime number, then the sum of two prime numbers is also even.

6. If $p$ : He is intelligent.
   $q$ : He is strong.
   Then, symbolic form of statement : 'It is wrong that, he is intelligent or strong' is
   (a) $\sim p \vee \sim p$ (b) $\sim(p \wedge q)$
   (c) $\sim(p \vee q)$ (d) $p \vee \sim q$

7. The negation of the proposition 'If 2 is prime, then 3 is odd', is
   (a) If 2 is not prime, then 3 is not odd.
   (b) 2 is prime and 3 is not odd.
   (c) 2 is not prime and 3 is odd.
   (d) If 2 is not prime, then 3 is odd.

8. The statement $(\sim p \wedge q) \vee \sim q$ is
   (a) $p \vee q$ (b) $p \wedge q$
   (c) $\sim(p \vee q)$ (d) $\sim(p \wedge q)$

9. Which of the following is always true?
   (a) $\sim(p \to q) \equiv \sim q \to \sim p$
   (b) $\sim(p \vee q) \equiv \sim p \vee \sim q$
   (c) $\sim(p \to q) \equiv p \wedge \sim q$
   (d) $\sim(p \wedge q) \equiv \sim p \wedge \sim q$
   [Note : *Question is modified.*]

10. $\sim(p \vee q) \vee (\sim p \wedge q)$ is logically equivalent to
   (a) $\sim p$ (b) $p$ (c) $q$ (d) $\sim q$

11. If $p$ and $q$ are two statements, then $(p \to q) \leftrightarrow (\sim q \to \sim p)$ is
   (a) contradiction (b) tautology
   (c) neither (i) nor (ii) (d) none of these

12. If $p$ is the sentence 'This statement is false', then
   (a) truth value of $p$ is T
   (b) truth value of $p$ is F
   (c) $p$ is both true and false
   (d) $p$ is neither true nor false.

13. Conditional $p \to q$ is equivalent to
   (a) $p \to \sim q$ (b) $\sim p \vee q$
   (c) $\sim p \to \sim q$ (d) $p \vee \sim q$

14. Negation of the statement 'This is false or That is true' is
   (a) That is true or This is false
   (b) That is true and This is false
   (c) This is true and That is false
   (d) That is false and That is true.
   [**Note :** *Option (c) is modified.*]

15. If $p$ is any statement, then $(p \vee \sim p)$ is a
   (a) contingency (b) contradiction
   (c) tautology (d) none of them.

### Answers

1. (d) Come here.
2. (a) $x$ is a natural number.
3. (d) Distributive law.
4. (b) $(p \to q) \leftrightarrow (\sim q \to \sim p)$ is a contradiction.
5. (c) If 2 is an even and prime number, then the sum of two prime numbers is not always even.
6. (c) $\sim (p \vee q)$
7. (b) 2 is prime and 3 is not odd.
8. (d) $\sim (p \wedge q)$
   [**Hint :** $(\sim p \wedge q) \vee \sim q \equiv (\sim p \vee \sim q) \wedge (q \vee \sim q)$
   $\equiv (\sim p \vee \sim q) \wedge t$
   $\equiv \sim p \vee \sim q \equiv \sim (p \wedge q)$]
9. (c) $\sim (p \to q) \equiv p \wedge \sim q$
10. (a) $\sim p$
   [**Hint :** $\sim (p \vee q) \vee (\sim p \wedge q) \equiv (\sim p \wedge \sim q) \vee (\sim p \wedge q)$
   $\equiv \sim p \wedge (\sim q \vee q) \equiv \sim p \wedge t \equiv \sim p$]
11. (b) tautology
12. (d) $p$ is neither true nor false
13. (b) $\sim p \vee q$
14. (c) This is true and That is false.
15. (c) tautology

**(II) Fill in the blanks :**

1. The statement $q \to p$ is called as the ......... of the statement $p \to q$.
2. Conjuction of two statements $p$ and $q$ is symbolically written as ......... .
3. If $p \vee q$ is true, then truth value of $\sim p \vee \sim q$ is ......... .
4. Negation of 'some men are animal' is ......... .
5. Truth value of : If $x = 2$, then $x^2 = -4$ is ......... .
6. Inverse of statement pattern $p \to q$ is given by ......... .
   [**Note :** *Question is modified.*]
7. $p \leftrightarrow q$ is false when $p$ and $q$ have ......... truth values.
8. Let $p$ : The problem is easy.
   $r$ : It is not challenging.
   Then verbal form of $\sim p \to r$ is ......... .
9. Truth value of $2 + 3 = 5$ if and only if $-3 > -9$ is ......... .

### Answers

1. Converse 2. $p \wedge q$ 3. F
4. All men are not animal. *OR* No men are animals.
5. F 6. $\sim p \to \sim q$ 7. different
8. If the problem is not easy, then it is not challenging.
9. T [**Hint :** $T \leftrightarrow T \equiv T$.]

**(III) State whether each of the following is *True* or *False* :**

1. Truth value of $2 + 3 < 6$ is F.
2. There are 24 months in year is a statement.
3. $p \wedge q$ has truth value F if both $p$ and $q$ has truth value F.
   [**Note :** *Question is modified.*]
4. The negation of $10 + 20 = 30$ is, it is false that $10 + 20 \neq 30$.
5. Dual of $(p \wedge \sim q) \vee t$ is $(p \vee \sim q) \vee c$.
6. Dual of 'John and Ayub went to the forest' is 'John or Ayub went to the forest.'
   [**Note :** *Question is modified.*]
7. 'His birthday is on $29^{th}$ February' is not a statement.
8. $x^2 = 25$ is true statement.
9. Truth value of '$\sqrt{5}$ is not an irrational number' is T.
10. $p \wedge t = p$.

### Answers

1. False 2. True 3. False 4. False 5. False
6. True 7. True 8. False 9. False 10. True.

(IV) Solve the following :

1. State which of the following sentences are statement in logic :
   (i) Icecream Sundaes are my favourite.
   (ii) $x+3=8$, $x$ is variable.
   (iii) Read a lot to improve your writing skill.
   (iv) $z$ is a positive number.
   (v) $(a+b)^2=a^2+2ab+b^2$ for all $a, b \in R$.
   (vi) $(2+1)^2=9$.
   (vii) Why are you sad?
   (viii) How beautiful the flower is!
   (ix) The square of any odd number is even.
   (x) All integers are natural numbers.
   (xi) If $x$ is real number, then $x^2 \geqslant 0$.
   (xii) Do not come inside the room.
   (xiii) What a horrible sight it was!

Solution :

(i) It is a statement.
(ii) It is a statement.
(iii) It is an imperative sentence, hence it is not a statement.
(iv) It is an open sentence, hence it is not a statement.
[Note : *Answer in the textbook is incorrect.*]
(v) It is a statement.
(vi) It is a statement.
(vii) It is an interrogative sentence, hence it is not a statement.
(viii) It is an exclamatory sentence, hence it is not a statement.
(ix) It is a statement.
(x) It is a statement.
(xi) It is a statement.
(xii) It is an imperative sentence, hence it is not a statement.
(xiii) It is an exclamatory sentence, hence it is not a statement.

2. Which of the following sentences are statements? In case of a statement, write down the truth value :
   (i) What is happy ending?
   (ii) The square of every real number is positive.
   (iii) Every parallelogram is a rhombus.
   (iv) $a^2-b^2=(a+b)(a-b)$ for all $a, b \in R$.
   (v) Please carry out my instruction.
   (vi) The Himalayas is the highest mountain range.
   (vii) $(x-2)(x-3)=x^2-5x+6$ for all $x \in R$.
   (viii) What are the causes of rural unemployment?
   (ix) $0!=1$.
   (x) The quadratic equation $ax^2+bx+c=0$ $(a \neq 0)$ always has two real roots.

Solution :

(i) It is an interrogative sentence, hence it is not a statement.
(ii) It is a statement which is **false**, hence its truth value is F.
[Note : *Answer in the textbook is incorrect.*]
(iii) It is a statement which is **true**, hence its truth value is T.
(iv) It is a mathematical identity which is **true,** hence its truth value is T.
(v) It is an imperative sentence, hence it is not a statement.
(vi) It is a statement which is **true**, hence its truth value is T.
(vii) It is a mathematical identity which is **true,** hence its truth value is T.
(viii) It is an interrogative sentence, hence it is not a statement.
(ix) It is a statement which is **true**, hence its truth value is T.
(x) It is a statement which is **false,** hence its truth value is F.

3. Assuming the first statement as $p$ and second as $q$, write the following statements in symbolic form :
   (i) The Sun has set and Moon has risen.
   (ii) Mona likes Mathematics and Physics.
   (iii) 3 is prime number iff 3 is perfect square number.
   (iv) Kavita is brilliant and brave.
   (v) If Kiran drives a car, then Sameer will walk.
   (vi) The necessary condition for existence of a tangent to the curve of the function is continuity.

**(vii)** **To be brave is necessary and sufficient condtion to climb the Mount Everest.**

**(viii)** $x^3 + y^3 = (x+y)^3$, **iff** $xy = 0$.

**(ix)** **The drug is effective though it has side effects.**

**(x)** **If a real number is not rational, then it must be irrational.**

**(xi)** **It is not true that Ram is tall and handsome.**

**(xii)** **Even though it is not cloudy, it is still raining.**

**(xiii)** **It is not true that intelligent persons are neither polite nor helpful.**

**(xiv)** **If the question paper is not easy, then we shall not pass.**

**Solution :**

**(i)** Let $p$ : The Sun has set.

$q$ : Moon has risen.

Then the symbolic form of the given statement is $p \wedge q$.

**(ii)** Let $p$ : Mona likes Mathematics.

$q$ : Mona likes Physics.

Then the symbolic form of the given statement is $p \wedge q$.

**(iii)** Let $p$ : 3 is prime number.

$q$ : 3 is perfect square number.

Then the symbolic form of the given statement is $p \leftrightarrow q$.

**(iv)** Let $p$ : Kavita is brilliant.

$q$ : Kavita is brave.

Then the symbolic form of the given statement is $p \wedge q$.

**(v)** Let $p$ : Kiran drives a car.

$q$ : Sameet will walk.

Then the symbolic form of the given statement is $p \to q$.

**(vi)** The given statement can be written as :

'If the function is continuous, then the tangent to the curve exists.'

Let $p$ : The function is continuous.

$q$ : Tangent to the curve exists.

Then the symbolic form of the given statement is $p \to q$.

[**Note :** *Answer in the textbook is incorrect.*]

**(vii)** Let $p$ : To be brave.

$q$ : Climb the Mount Everest.

Then the symbolic form of the given statement is $p \leftrightarrow q$.

**(viii)** Let $p : x^3 + y^3 = (x+y)^3$.

$q : xy = 0$.

Then the symbolic form of the given statement is $p \leftrightarrow q$.

**(ix)** Let $p$ : The drug is effective.

$q$ : It has side effects.

Then the symbolic form of the given statement is $p \wedge q$.

[**Note :** *Answer in the textbook is incorrect.*]

**(x)** Let $p$ : A real number is not rational.

$q$ : It must be irrational.

Then the symbolic form of the given statement is $p \to q$.

**(xi)** Let $p$ : Ram is tall.

$q$ : Ram is handsome.

Then the symbolic form of the given statement is $\sim (p \wedge q)$.

**(xii)** The given statement is equivalent to :

It is not cloudy and it is still raining.

Let $p$ : It is not cloudy.

$q$ : It is still raining.

Then the symbolic form of the given statement is $p \wedge q$.

[**Note :** *Answer in the textbook is incorrect.*]

**(xiii)** Let $p$ : Intelligent persons are neither polite nor helpful.

Then the symbolic form of the given statement is $\sim p$.

**(xiv)** Let $p$ : The question paper is not easy.

$q$ : We shall not pass.

Then the symbolic form of the given statement is $p \to q$.

**4.** **If $p$ : Proof is lengthy.**

**$q$ : It is interesting.**

**Express the following statements in symbolic form :**

**(i) Proof is lengthy and it is not interesting.**

**(ii) If proof is lengthy, then it is interesting.**

**(iii) It is not true that the proof is lengthy but it is interesting.**

**(iv) It is interesting iff the proof is lengthy.**

**Solution :** The symbolic form of the given statements are :

(i) $p \wedge \sim q$ (ii) $p \rightarrow q$ (iii) $\sim(p \wedge q)$ (iv) $q \leftrightarrow p$.

**5.** **Let $p$ : Sachin wins the match.**

**$q$ : Sachin is a member of Rajya Sabha.**

**$r$ : Sachin is happy.**

**Write the verbal statement for each of the following :**

**(i) $(p \wedge q) \vee r$** **(ii) $p \rightarrow r$**

**(iii) $\sim p \vee q$** **(iv) $p \rightarrow (q \vee r)$**

**(v) $p \rightarrow q$** **(vi) $(p \wedge q) \wedge \sim r$**

**(vii) $\sim(p \vee q) \wedge r$.**

**Solution :**

(i) $(p \wedge q) \vee r$ : Sachin wins the match and he is a member of Rajya Sabha or Sachin is happy.

[**Note :** *Answer in the textbook is incorrect.*]

(ii) $p \rightarrow r$ : If Sachin wins the match, then he is happy.

(iii) $\sim p \vee q$ : Sachin does not win the match or he is a member of Rajya Sabha.

(iv) $p \rightarrow (q \vee r)$ : If Sachin wins the match, then he is a member of Rajya Sabha or he is happy.

[**Note :** *Question is modified.*]

(v) $p \rightarrow q$ : If Sachin wins the match, then he is a member of Rajya Sabha.

[**Note :** *Answer in the textbook is incorrect.*]

(vi) $(p \wedge q) \wedge \sim r$ : Sachin wins the match and he is a member of Rajya Sabha but he is not happy.

(vii) $\sim(p \vee q) \wedge r$ : It is false that Sachin wins the match or he is a member of Rajya Sabha but he is happy.

**6.** **Determine the truth values of the following statements :**

**(i) $4+5=7$ or $9-2=5$.**

**(ii) If $9>1$, then $x^2-2x+1=0$ for $x=1$.**

**(iii) $x+y=0$ is the equation of a straight line if and only if $y^2=4x$ is the equation of the parabola.**

**(iv) It is not true that $2+3=6$ or $12+3=5$.**

**Solution :**

(i) Let $p : 4+5=7$.

$q : 9-2=5$.

Then the symbolic form of the given statement is $p \vee q$.

The truth values of both $p$ and $q$ are F.

$\therefore$ the truth value of $p \vee q$ is F. ... $[\mathrm{F} \vee \mathrm{F} \equiv \mathrm{F}]$

(ii) Let $p : 9>1$.

$q : x^2-2x+1=0$ for $x=1$.

Then the symbolic form of the given statement is $p \rightarrow q$.

The truth values of both $p$ and $q$ are T.

$\therefore$ the truth value of $p \rightarrow q$ is T. ... $[\mathrm{T} \rightarrow \mathrm{T} \equiv \mathrm{T}]$

(iii) Let $p : x+y=0$ is the equation of a straight line.

$q : y^2=4x$ is the equation of the parabola.

Then the symbolic form of the given statement is $p \leftrightarrow q$.

The truth values of both $p$ and $q$ are T.

$\therefore$ the truth value of $p \leftrightarrow q$ is T. ... $[\mathrm{T} \leftrightarrow \mathrm{T} \equiv \mathrm{T}]$

[**Note :** *Answer in the textbook is incorrect.*]

(iv) Let $p : 2+3=6$.

$q : 12+3=5$.

Then the symbolic form of the given statement is $\sim(p \vee q)$.

The truth values of both $p$ and $q$ are F.

$\therefore$ the truth value of $\sim(p \vee q)$ is T.

... $[\sim(\mathrm{F} \vee \mathrm{F}) \equiv \sim \mathrm{F} \equiv \mathrm{T}.]$

**7.** **Assuming the following statements**

**$p$ : Stock prices are high.**

**$q$ : Stocks are rising.**

**to be true, find the truth values of the following :**

**(i) Stock prices are not high or stocks are rising.**

**(ii) Stock prices are high and stocks are rising if and only if stock prices are high.**

**(iii) If stock prices are high, then stocks are not rising.**

**(iv) It is false that stocks are rising and stock prices are high.**

**(v) Stock prices are high or stocks are not rising iff stocks are rising.**

**Solution :** $p$ and $q$ are true, i.e. T.

$\therefore$ $\sim p$ and $\sim q$ are false, i.e. F.

**(i)** The given statement in symbolic form is $\sim p \vee q$.
Since, $\sim T \vee T \equiv F \vee T \equiv T$, the given statement is **true.** Hence, its truth value is 'T'.

**(ii)** The given statement in symbolic form is $(p \wedge q) \leftrightarrow p$.
Since, $(T \wedge T) \leftrightarrow T \equiv T \leftrightarrow T \equiv T$, the given statement is **true**.
Hence, its truth value is 'T'.

**(iii)** The given statement in symbolic form is $p \to \sim q$.
Since, $T \to \sim T \equiv T \to F \equiv F$, the given statement is **false.** Hence, its truth value is 'F'.

**(iv)** The given statement in symbolic form is $\sim (q \wedge p)$.
Since, $\sim (T \wedge T) \equiv \sim T \equiv F$, the given statement is **false.**
Hence, its truth value is 'F'.
[**Note :** *Answer in the textbook is incorrect.*]

**(v)** The given statement in symbolic form is $(p \vee \sim q) \leftrightarrow q$.
Since, $(T \vee \sim T) \leftrightarrow T \equiv (T \vee F) \leftrightarrow T$
$\equiv T \leftrightarrow T \equiv T$,
the given statement is **true.**
Hence, its truth value is 'T'.

**8. Rewrite the following statements without using conditional :**
[**Hint :** $p \to q \equiv \sim p \vee q$)]
**(i) If price increases, then demand falls.**
**(ii) If demand falls, then price does not increase.**

**Solution :** Since, $p \to q \equiv \sim p \vee q$, the given statements can be written as :
(i) Price does not increase or demand falls.
(ii) Demand does not fall or price does not increase.
[**Note :** *Answer in the textbook is incorrect.*]

**9. If $p$, $q$, $r$ are statements with truth values T, T, F respectively, determine the truth values of the following :**
**(i)** $(p \wedge q) \to \sim p$
**(ii)** $p \leftrightarrow (q \to \sim p)$
**(iii)** $(p \wedge \sim q) \vee (\sim p \wedge q)$
**(iv)** $\sim (p \wedge q) \to \sim (q \wedge p)$
**(v)** $\sim [(p \to q) \leftrightarrow (p \wedge \sim q)]$

**Solution :** Truth values of $p$, $q$, $r$ are T, T, F respectively.

**(i)** $(p \wedge q) \to \sim p \equiv (T \wedge T) \to \sim T$
$\equiv T \to F \equiv F$
Hence, the truth value of the given statement is false, i.e. **F.**

**(ii)** $p \leftrightarrow (q \to \sim p) \equiv T \leftrightarrow (T \to \sim T)$
$\equiv T \leftrightarrow (T \to F)$
$\equiv T \leftrightarrow F \equiv F$
Hence, the truth value of the given statement is false, i.e. **F.**

**(iii)** $(p \wedge \sim q) \vee (\sim p \wedge q) \equiv (T \wedge \sim T) \vee (\sim T \wedge T)$
$\equiv (T \wedge F) \vee (F \wedge T)$
$\equiv F \vee F \equiv F$
Hence, the truth value of the given statement is false, i.e. **F.**

**(iv)** $\sim (p \wedge q) \to \sim (q \wedge p) \equiv \sim (T \wedge T) \to \sim (T \wedge T)$
$\equiv \sim T \to \sim T$
$\equiv F \to F \equiv T$
Hence, the truth value of the given statement is true, i.e. **T.**

**(v)** $\sim [(p \to q) \leftrightarrow (p \wedge \sim q)]$
$\equiv \sim [(T \to T) \leftrightarrow (T \wedge \sim T)]$
$\equiv \sim [T \leftrightarrow (T \wedge F)]$
$\equiv \sim [T \leftrightarrow F] \equiv \sim F \equiv T$.
Hence, the truth value of the given statement is true, i.e. **T.**

**10. Write the negations of the following :**
**(i) If $\triangle$ABC is not equilateral, then it is not equiangular.**
**(ii) Ramesh is intelligent and he is hard working.**
**(iii) A angle is a right angle if and only if it is of measure 90°.**
**(iv) Kanchanjunga is in India and Everest is in Nepal.**
**(v) If $x \in A \cap B$, then $x \in A$ and $x \in B$.**

**Solution :**
**(i)** Let $p$ : $\triangle$ABC is not equilateral.
$q$ : It is not equiangular.
Then the symbolic form of the given statement is $p \to q$.
Since, $\sim (p \to q) \equiv p \wedge \sim q$, the negation of given statement is :
'$\triangle$ABC is not equilateral and it is equiangular.'

**(ii)** Let $p$ : Ramesh is intelligent.

$q$ : He is hard working.

Then the symbolic form of the given statement is $p \wedge q$.

Since, $\sim(p \wedge q) \equiv \sim p \vee \sim q$, the negation of the given statement is :

'Ramesh is not intelligent or he is not hard working.'

**(iii)** Let $p$ : An angle is a right angle.

$q$ : It is of measure 90°.

Then the symbolic form of the given statement is $p \leftrightarrow q$.

Since, $\sim(p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$, the negation of given statement is :

'An angle is a righ angle and it is not of measure 90° or an angle is of measure 90° and it is not a right angle.'

[**Note :** *Answer in the textbook is incorrect.*]

**(iv)** Let $p$ : Kanchenjunga is in India.

$q$ : Everest is in Nepal.

Then the symbolic form of the given statement is $p \wedge q$.

Since, $\sim(p \wedge q) \equiv \sim p \vee \sim q$, the negation of the given statement is :

'Kanchenjunga is not in India or Everest is not in Nepal.'

**(v)** Let $p : x \in A \cap B$, $q : x \in A$, $r : x \in B$.

Then the symbolic form of the given statement is : $p \rightarrow (q \wedge r)$

Since, $\sim(p \rightarrow q) \equiv p \wedge \sim q$ and $\sim(p \wedge q) \equiv \sim p \vee \sim q$, the negation of the given statement is :

'$x \in A \cap B$ and $x \notin A$ or $x \notin B$.'

**11. Construct the truth table for each of the following statement patterns :**

**(i)** $(p \wedge \sim q) \leftrightarrow (q \rightarrow p)$

**(ii)** $(\sim p \vee q) \wedge (\sim p \wedge \sim q)$

**(iii)** $(p \wedge r) \rightarrow (p \vee \sim q)$

**(iv)** $(p \vee r) \rightarrow \sim(q \wedge r)$

**(v)** $(p \vee \sim q) \rightarrow (r \wedge p)$.

**Solution :**

**(i)**

| $p$ | $q$ | $\sim q$ | $p \wedge \sim q$ | $q \rightarrow p$ | $(p \wedge \sim q) \leftrightarrow (q \rightarrow p)$ |
|---|---|---|---|---|---|
| T | T | F | F | T | F |
| T | F | T | T | T | T |
| F | T | F | F | F | T |
| F | F | T | F | T | F |

**(ii)**

| $p$ | $q$ | $\sim p$ | $\sim q$ | $\sim p \vee q$ | $\sim p \wedge \sim q$ | $(\sim p \vee q) \wedge (\sim p \wedge \sim q)$ |
|---|---|---|---|---|---|---|
| T | T | F | F | T | F | F |
| T | F | F | T | F | F | F |
| F | T | T | F | T | F | F |
| F | F | T | T | T | T | T |

**(iii)** *Refer to the solution of Q. 1 (iii) of Exercise 1.6.*

**(iv)**

| $p$ | $q$ | $r$ | $p \vee r$ | $q \wedge r$ | $\sim(q \wedge r)$ | $(p \vee r) \rightarrow \sim(q \wedge r)$ |
|---|---|---|---|---|---|---|
| T | T | T | T | T | F | F |
| T | T | F | T | F | T | T |
| T | F | T | T | F | T | T |
| T | F | F | T | F | T | T |
| F | T | T | T | T | F | F |
| F | T | F | F | F | T | T |
| F | F | T | T | F | T | T |
| F | F | F | F | F | T | T |

**(v)**

| $p$ | $q$ | $r$ | $\sim q$ | $p \vee \sim q$ | $r \wedge p$ | $(p \vee \sim q) \rightarrow (r \wedge p)$ |
|---|---|---|---|---|---|---|
| T | T | T | F | T | T | T |
| T | T | F | F | T | F | F |
| T | F | T | T | T | T | T |
| T | F | F | T | T | F | F |
| F | T | T | F | F | F | T |
| F | T | F | F | F | F | T |
| F | F | T | T | T | F | F |
| F | F | F | T | T | F | F |

**12. What is tautology? What is contradiction? Show that the negation of a tautology is a contradiction and the negation of a contradiction is a tautology.**

**Solution :**

**Tautology :** A statement pattern which has all the entries in the last column of its truth table as T is called a tautology.

**For example :**

| $p$ | $\sim p$ | $p \vee \sim p$ |
|---|---|---|
| T | F | T |
| F | T | T |

In the above truth table for the statement $p \vee \sim p$, we observe that all the entries in the last column are T. Hence, the statement $p \vee \sim p$ is a tautology.

**Contradiction :** A statement pattern which has all the entries in the last column of its truth table as F is called a contradiction.

**For example :**

| $p$ | $\sim p$ | $p \wedge \sim p$ |
|---|---|---|
| T | F | F |
| F | T | F |

In the above truth table for the statement $p \wedge \sim p$, we observe that all the entries in the last column are F. Hence, the statement $p \wedge \sim p$ is a contradiction.

**To show that the negation of a tautology is a contradiction and vice versa :**

A tautology is true on every row of its truth table. Since, $\sim T = F$ and $\sim F = T$, when we negate a tautology, the resulting statement is false on every row of its table. i.e. the negation of tautology is a contradiction.

Similarly, the negation of a contradiction is a tautology.

**13. Determine whether the following statement patterns is a tautology or a contradiction or a contingency :**

(i) $[(p \wedge q) \vee (\sim p)] \vee [p \wedge (\sim q)]$

(ii) $[(\sim p \wedge q) \wedge (q \wedge r)] \vee (\sim q)$

(iii) $[\sim (p \vee q) \rightarrow p] \leftrightarrow [(\sim p) \wedge (\sim q)]$

(iv) $[\sim (p \wedge q) \rightarrow p] \leftrightarrow [(\sim p) \wedge (\sim q)]$

(v) $[p \rightarrow (\sim q \vee r)] \leftrightarrow \sim [p \rightarrow (q \rightarrow r)]$

**Solution :**

**(i)**

| $p$ | $q$ | $\sim p$ | $\sim q$ | $p \wedge q$ | $(p \wedge q) \vee (\sim p)$ | $p \wedge (\sim q)$ | $[(p \wedge q) \vee (\sim p)] \vee [(p \wedge \sim q)]$ |
|---|---|---|---|---|---|---|---|
| T | T | F | F | T | T | F | T |
| T | F | F | T | F | F | T | T |
| F | T | T | F | F | T | F | T |
| F | F | T | T | F | T | F | T |

All the entries in the last column of the above truth table are T.

$\therefore$ $[(p \wedge q) \vee (\sim p)] \vee [p \wedge (\sim q)]$ is a **tautology.**

**(ii)**

| $p$ | $q$ | $r$ | $\sim p$ | $\sim q$ | $\sim p \wedge q$ | $q \wedge r$ | $(\sim p \wedge q) \wedge (q \wedge r)$ | $[(\sim p \wedge q) \wedge (q \wedge r)] \vee (\sim q)$ |
|---|---|---|---|---|---|---|---|---|
| T | T | T | F | F | F | T | F | F |
| T | T | F | F | F | F | F | F | F |
| T | F | T | F | T | F | F | F | T |
| T | F | F | F | T | F | F | F | T |
| F | T | T | T | F | T | T | T | T |
| F | T | F | T | F | T | F | F | F |
| F | F | T | T | T | F | F | F | T |
| F | F | F | T | T | F | F | F | T |

The entries in the last column of the above truth table are neither all T nor all F.

$\therefore$ $[(\sim p \wedge q) \wedge (q \wedge r)] \vee (\sim q)$ is a **contingency.**

[**Note :** *Answer in the textbook is incorrect.*]

**(iii)**

| $p$ | $q$ | $\sim p$ | $\sim q$ | $p \vee q$ | $\sim (p \vee q)$ | $\sim (p \vee q) \rightarrow p$ | $\sim p \wedge \sim q$ | $[\sim (p \vee q) \rightarrow p] \leftrightarrow [(\sim p) \wedge (\sim q)]$ |
|---|---|---|---|---|---|---|---|---|
| T | T | F | F | T | F | T | F | F |
| T | F | F | T | T | F | T | F | F |
| F | T | T | F | T | F | T | F | F |
| F | F | T | T | F | T | F | T | F |

All the entries in the last column of the above truth table are F.

$\therefore$ $[\sim (p \vee q) \rightarrow p] \leftrightarrow [(\sim p) \wedge (\sim q)]$ is a **contradiction.**

(iv)

| $p$ | $q$ | $\sim p$ | $\sim q$ | $p \wedge q$ | $\sim(p \wedge q)$ | $\sim(p \wedge q) \to p$ | $\sim p \wedge \sim q$ | $[\sim(p \wedge q) \to p] \leftrightarrow [(\sim p) \wedge (\sim q)]$ |
|---|---|---|---|---|---|---|---|---|
| T | T | F | F | T | F | T | F | F |
| T | F | F | T | F | T | T | F | F |
| F | T | T | F | F | T | F | F | T |
| F | F | T | T | F | T | F | T | F |

The entries in the last column of the above truth table are neither all T nor all F.

$\therefore$ $[\sim(p \wedge q) \to p] \leftrightarrow [(\sim p) \wedge (\sim q)]$ is a **contingency.**

(v)

| $p$ | $q$ | $r$ | $\sim q$ | $\sim q \vee r$ | $p \to (\sim q \vee r)$ (I) | $q \to r$ | $p \to (q \to r)$ | $\sim[p \to (q \to r)]$ (II) | (I) $\leftrightarrow$ (II) |
|---|---|---|---|---|---|---|---|---|---|
| T | T | T | F | T | T | T | T | F | F |
| T | T | F | F | F | F | F | F | T | F |
| T | F | T | T | T | T | T | T | F | F |
| T | F | F | T | T | T | T | T | F | F |
| F | T | T | F | T | T | T | T | F | F |
| F | T | F | F | F | T | F | T | F | F |
| F | F | T | T | T | T | T | T | F | F |
| F | F | F | T | T | T | T | T | F | F |

All the entries in the last column of the above truth table are F.

$\therefore$ $[p \to (\sim q \vee r)] \leftrightarrow \sim[p \to (q \to r)]$ is a **contradiction.**

**14. Using the truth table, prove the following logical equivalences :**

**(i)** $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

**(ii)** $[\sim(p \vee q) \vee (p \vee q)] \wedge r \equiv r$

**(iii)** $p \wedge (\sim p \vee q) \equiv p \wedge q$

**(iv)** $p \leftrightarrow q \equiv \sim(p \wedge \sim q) \wedge \sim(q \wedge \sim p)$

**(v)** $\sim p \wedge q \equiv (p \vee q) \wedge \sim p$.

**Solution :**

(i)

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| $p$ | $q$ | $r$ | $q \vee r$ | $p \wedge (q \vee r)$ | $p \wedge q$ | $p \wedge r$ | $(p \wedge q) \vee (p \wedge r)$ |
| T | T | T | T | T | T | T | T |
| T | T | F | T | T | T | F | T |
| T | F | T | T | T | F | T | T |
| T | F | F | F | F | F | F | F |
| F | T | T | T | F | F | F | F |
| F | T | F | T | F | F | F | F |
| F | F | T | T | F | F | F | F |
| F | F | F | F | F | F | F | F |

The entries in columns 5 and 8 are identical.

$\therefore$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$.

(ii)

| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| $p$ | $q$ | $r$ | $p \vee q$ | $\sim(p \vee q)$ | $\sim(p \vee q) \vee (p \vee q)$ | $[\sim(p \vee q) \vee (p \vee q)] \wedge r$ |
| T | T | T | T | F | T | T |
| T | T | F | T | F | T | F |
| T | F | T | T | F | T | T |
| T | F | F | T | F | T | F |
| F | T | T | T | F | T | T |
| F | T | F | T | F | T | F |
| F | F | T | F | T | T | T |
| F | F | F | F | T | T | F |

The entries in columns 3 and 7 are identical.

$\therefore$ $[\sim(p \vee q) \vee (p \vee q)] \wedge r \equiv r$.

(iii)

| 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| $p$ | $q$ | $\sim p$ | $\sim p \vee q$ | $p \wedge (\sim p \vee q)$ | $p \wedge q$ |
| T | T | F | T | T | T |
| T | F | F | F | F | F |
| F | T | T | T | F | F |
| F | F | T | T | F | F |

The entries in columns 5 and 6 are identical.

$\therefore$ $p \wedge (\sim p \vee q) \equiv p \wedge q$.

(iv)

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| $p$ | $q$ | $p \leftrightarrow q$ | $\sim p$ | $\sim q$ | $p \wedge \sim q$ | $\sim(p \wedge \sim q)$ | $q \wedge \sim p$ | $\sim(q \wedge \sim p)$ | $\sim(p \wedge \sim q) \wedge \sim(q \wedge \sim p)$ |
| T | T | T | F | F | F | T | F | T | T |
| T | F | F | F | T | T | F | F | T | F |
| F | T | F | T | F | F | T | T | F | F |
| F | F | T | T | T | F | T | F | T | T |

The entries in columns 3 and 10 are identical

$\therefore$ $p \leftrightarrow q \equiv \sim(p \wedge \sim q) \wedge \sim(q \wedge \sim p)$.

(v)

| 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| $p$ | $q$ | $\sim p$ | $\sim p \wedge q$ | $p \vee q$ | $(p \vee q) \wedge \sim p$ |
| T | T | F | F | T | F |
| T | F | F | F | T | F |
| F | T | T | T | T | T |
| F | F | T | F | F | F |

The entries in columns 4 and 6 are identical.

$\therefore$ $\sim p \wedge q \equiv (p \vee q) \wedge \sim p$.

(i) **If $2+5=10$, then $4+10=20$.**

(ii) **If a man is bachelor, then he is happy.**

(iii) **If I do not work hard, then I do not prosper.**

**Solution :**

(i) Let $p : 2+5=10$.

$q : 4+10=20$.

Then the symbolic form of the given statement is $p \to q$.

**Converse :** $q \to p$ is the converse of $p \to q$

i.e. If $4+10=20$, then $2+5=10$.

**Inverse :** $\sim p \to \sim q$ is the inverse of $p \to q$

i.e. If $2+5 \neq 10$, then $4+10 \neq 20$.

**Cotrapositive :** $\sim q \to \sim p$ is the contrapositive of $p \to q$, i.e. If $4+10 \neq 20$, then $2+5 \neq 10$.

(ii) Let $p$ : A man is bachelor.

$q$ : He is happy.

Then the symbolic form of given statement is $p \to q$.

**Converse :** $q \to p$ is the converse of $p \to q$

i.e. If a man is happy, then he is a bachelor.

**Inverse :** $\sim p \to \sim q$ is the inverse of $p \to q$

i.e. If a man is not bachelor, then he is not happy.

**Contrapositive :** $\sim q \to \sim p$ is the contrapositive of $p \to q$

i.e., If a man is not happy, then he is not bachelor.

(iii) Let $p$ : I do not work hard.

$q$ : I do not prosper.

Then the symbolic form of the given statement is $p \to q$.

**Converse :** $q \to p$ is the converse of $p \to q$

i.e. If I do not prosper, then I do not work hard.

**Inverse :** $\sim p \to \sim q$ is the inverse of $p \to q$

i.e. If I work hard, then I prosper.

**Contrapositive :** $\sim q \to \sim p$ is the contrapositive of $p \to q$ i.e. If I prosper, then I work hard.

(i) $(p \wedge \sim q) \vee (\sim p \wedge q) \equiv (p \vee q) \wedge \sim (p \wedge q)$

(ii) $p \vee (q \vee r) \equiv \sim [(p \wedge q) \vee (r \vee s)]$

(iii) **2 is even number or 9 is a perfect square.**

**Solution :** The duals are given by :

(i) $(p \vee \sim q) \wedge (\sim p \vee q) \equiv (p \wedge q) \vee \sim (p \vee q)$

(ii) $p \wedge (q \wedge r) \equiv \sim [(p \vee q) \wedge (r \wedge s)]$

(iii) 2 is even number and 9 is a perfect square.

**17. Rewrite the following statements without using the connective 'If ... then' :**

(i) **If a quadrilateral is rhombus, then it is not a square.**

(ii) **If $10-3=7$, then $10 \times 3 \neq 30$.**

(iii) **If it rains, then the principal declares a holiday.**

**Solution :** Since, $p \to q \equiv \sim p \vee q$ the given statements can be written as :

(i) A quadrilateral is not a rhombus or it is not a square.

(ii) $10-3 \neq 7$ or $10 \times 3 \neq 30$.

(iii) It does not rain or the principal declares a holiday.

**18. Write the dual of each of the following :**

(i) $(\sim p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge \sim q)$

(ii) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

(iii) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(iv) $\sim (p \vee q) \equiv \sim p \wedge \sim q$.

[**Note :** *Question 18 (iii) is modified.*]

**Solution :** The duals are given by :

(i) $(\sim p \vee q) \wedge (p \vee \sim q) \wedge (\sim p \vee \sim q)$

(ii) $(p \vee q) \vee r \equiv p \vee (q \vee r)$

(iii) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

(iv) $\sim (p \wedge q) \equiv \sim p \wedge \sim q$.

**19. Consider the following statements :**

(i) **If D is dog, then D is very good.**

(ii) **If D is very good, then D is dog.**

(iii) **If D is not very good, then D is not a dog.**

(iv) **If D is not a dog, then D is not very good.**

**Identify the pairs of statements having the same meaning. Justify.**

**Solution :** Let $p$ : D is dog.

and $q$ : D is very good.

Then the given statements in the symbolic form are :

(i) $p \to q$ (ii) $q \to p$ (iii) $\sim q \to \sim p$ (iv) $\sim p \to \sim q$.

| | | | | (i) | (ii) | (iii) | (iv) |
|---|---|---|---|---|---|---|---|
| $p$ | $q$ | $\sim p$ | $\sim q$ | $p \to q$ | $q \to p$ | $\sim q \to \sim p$ | $\sim p \to \sim q$ |
| T | T | F | F | T | T | T | T |
| T | F | F | T | F | T | F | T |
| F | T | T | F | T | F | T | F |
| F | F | T | T | T | T | T | T |

**The entries in columns (i) and (iii) are identical. Hence, these statements are equivalent.**

**$\therefore$ the statements (i) and (iii) have the same meaning.**

**Similarly, the entries in columns (ii) and (iv) are identical. Hence, these statements are equivalent.**

**$\therefore$ the statements (ii) and (iv) have the same meaning.**

**20. Express the truth of each of the following statements by Venn diagrams :**

**(i) All men are mortal.**

**(ii) Some persons are not politician.**

**(iii) Some members of the present Indian cricket are not committed.**

**(iv) No child is an adult.**

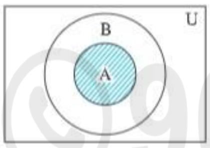
**Solution :**

**(i)** Let U : set of all human being

A : set of all men

B : set of all mortals.

Then the Venn diagram represents the truth of the given statement is as below :



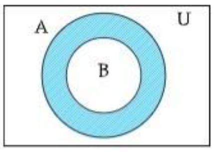

$A \subset B$

**(ii)** Let U : set of all human being

A : set of all persons

B : set of all politicians.

Then the Venn diagram represents the truth of the given statement is as follows :



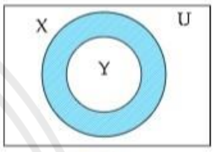

$A - B \neq \phi$

**(iii)** Let U : set of all human being

X : set of all members of present Indian cricket

Y : set of all committed members of the present Indian cricket.

Then the Venn diagram represents the truth of the given statement is as below :



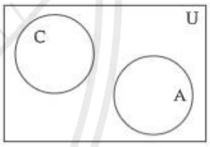

$X - Y \neq \phi$

**(iv)** Let U : set of all human beings

C : set of all children

A : set of all adults.

Then the Venn diagram represents the truth of the given statement is as below :




$C \cap A = \phi$

**21. If $A = \{2, 3, 4, 5, 6, 7, 8\}$, determine the truth value of each of the following statements :**

**(i) $\exists x \in A$, such that $3x + 2 > 9$.**

**(ii) $\forall x \in A$, $x^2 < 18$.**

**(iii) $\exists x \in A$, such that $x + 3 < 11$.**

**(iv) $\forall x \in A$, $x^2 + 2 \geqslant 5$.**

**Solution :**

**(i)** Clearly $x = 3, 4, 5, 6, 7, 8 \in A$ satisfy $3x + 2 > 9$. So, the given statement is **true**, hence its truth value is T.

**(ii)** $x = 5, 6, 7, 8 \in A$ do not satisfy $x^2 < 18$. So, the given statement is **false**, hence its truth value is F.

**(iii)** Clearly $x = 2, 3, 4, 5, 6, 7 \in A$ which satisfy $x + 3 < 11$. So, the given statement is **True**, hence its truth value is T.

**(iv)** $x^2 + 2 \geqslant 5$ for all $x \in A$.

So, the given statement is **true**, hence its truth value is T.

**22. Write the negations of the following statements :**

**(i) 7 is prime number and Taj Mahal is in Agra.**

**(ii) $10 > 5$ and $3 < 8$.**

**(iii) I will have tea or coffee.**

**(iv) $\forall n \in N$, $n + 3 > 9$.**

**(v) $\exists x \in A$, such that $x + 5 < 11$.**

**Solution :**

**(i)** Let $p$ : 7 is prime number.

$q$ : Taj Mahal is in Agra.

Then the symbolic form of the given statement is $p \wedge q$.

Since, $\sim (p \wedge q) \equiv \sim p \vee \sim q$, the negation of the given statement is :

'7 is not prime number or Taj Mahal is not in Agra.'

**(ii)** Let $p : 10 > 5$.

$q : 3 < 8$.

Then the symbolic form of the given statement is $p \wedge q$.

Since, $\sim (p \wedge q) \equiv \sim p \vee \sim q$, the negation of the given statement is :

'$10 \leqslant 5$ or $3 \geqslant 8$' **OR** '$10 \ngtr 5$ or $3 \nless 8$'

**(iii)** The negation of given statement is :

'I will not have tea and coffee.'

**(iv)** The negation of given statement is :

'$\exists n \in N$, such that $n + 3 \ngtr 9$.'

**OR** '$\exists n \in N$, such that $n + 3 \leqslant 9$.'

**(v)** The negation of given statement is :

'$\forall x \in A$, $x + 5 \nless 11$.'

**OR** '$\forall x \in A$, $x + 5 \geqslant 11$.'

**ACTIVITIES** | **Textbook page 34**

**1. Complete the truth table for $\sim [p \vee (\sim q)] \equiv \sim p \wedge q$; justify it.**

**Solution :**

| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| $p$ | $q$ | $\sim p$ | $\sim q$ | $p \vee \sim q$ | $\sim [p \vee (\sim q)]$ | $\sim p \wedge q$ |
| T | T | F | F | T | F | F |
| T | F | F | T | T | F | F |
| F | T | T | F | F | T | T |
| F | F | T | T | T | F | F |

Justification : The entries in columns 6 and 7 are identical.

$\therefore \sim [p \vee (\sim q)] \equiv \sim p \wedge q$.

**2. If $p \leftrightarrow q$ and $p \rightarrow q$ both are true, then find truth values of following with the help of activity :**

**(i) $p \vee q$ (ii) $p \wedge q$.**

**Solution :** $p \leftrightarrow q$ and $p \rightarrow q$ are true if $p$ and $q$ has truth values T, T or F, F.

**(i)** $p \vee q$

(a) If both $p$ and $q$ are true, then

$p \vee q = \text{T} \vee \text{T} = \text{T}$

(b) If both $p$ and $q$ are false, then

$p \vee q = \text{F} \vee \text{F} = \text{F}$.

**(ii)** $p \wedge q$

(a) If both $p$ and $q$ are true, then

$p \wedge q = \text{T} \wedge \text{T} = \text{T}$

(b) If both $p$ and $q$ are false, then

$p \wedge q = \text{F} \wedge \text{F} \equiv \text{F}$.

**3. Represent following statements by Venn diagrams :**

**(i) Many students are not hard working.**

**(ii) Some students are hard working.**

**(iii) Sunday implies holiday.**

**Solution :**

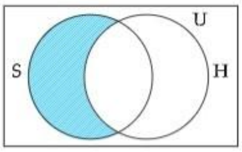
Let U : set of all human being

S : set of all students

H : set of all hard working persons.

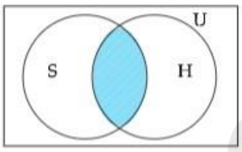
Then the Venn diagrams represent the truth of the statements (i) and (ii) are as follows :

(i)




$S - H \neq \phi$

(ii)



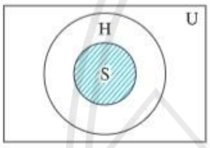

$S \cap H \neq \phi$

(iii) Let U : set of days

S : set of Sundays

H : set of holidays

Then the Venn diagram represents the truth of the given statement is :




$S \subset H$

**4. You have given following statements :**

**$p : 9 \times 5 = 45.$**

**$q$ : Pune is in Maharashtra.**

**$r$ : 3 is smallest prime number.**

**Then write truth values by activity :**

(i) $(p \wedge q) \wedge r = ($ □ $\wedge$ □ $) \wedge$ □

$=$ □ $\wedge$ □

$=$ □

(ii) $\sim [p \wedge r] = \sim ($ □ $\wedge$ □ $)$

$= \sim$ □

$=$ □

(iii) $p \to q$ $=$ □ $\to$ □

$=$ □

(iv) $p \to r$ $=$ □ $\leftrightarrow$ □

$=$ □

**Solution :** The truth values of $p$, $q$, $r$ are T, T, F respectively.

(i) $(p \wedge q) \wedge r = (\boxed{T} \wedge \boxed{T}) \wedge \boxed{F}$

$= \boxed{T} \wedge \boxed{F}$

$= \boxed{F}$

Hence, the truth value of $(p \wedge q) \wedge r$ is false, i.e. **F.**

(ii) $\sim [p \wedge r] = \sim (\boxed{T} \wedge \boxed{F})$

$= \sim \boxed{F} = \boxed{T}$

Hence, the truth value of $\sim [p \wedge q]$ is true, i.e. **T.**

(iii) $p \to q$ $= \boxed{T} \to \boxed{T}$

$= \boxed{T}$

Hence, the truth value of $p \to q$ is true, i.e. **T.**

(iv) $p \to r$ $= \boxed{T} \leftrightarrow \boxed{F}$

$= \boxed{F}$

Hence, the truth value of $p \to r$ is false, i.e. **F.**

## ACTIVITIES FOR PRACTICE

**1. Complete the following truth table and give your conclusion :**

| $p$ | $q$ | $r$ | $\sim q$ | $\sim q \vee r$ | $p \to (\sim q \vee r)$ |
|---|---|---|---|---|---|
| T | T | T | F | T | T |
| T | □ | F | □ | □ | □ |
| □ | F | T | T | T | □ |
| T | □ | F | □ | □ | □ |
| F | T | □ | F | □ | T |
| □ | T | F | F | F | □ |
| □ | □ | □ | □ | □ | □ |
| F | □ | □ | □ | □ | T |

Conclusion : .........

**2.** Let $p$ : The train reaches on time.

$q$ : I can catch the connecting flight.

Therefore the symbolic form of the statement :

'If the train reaches on time, then I can catch the connecting flight' is $p \to q$.

Then, Converse is $q \to p$, i.e. □

Inverse is □, i.e. □

Contrapositive is □, i.e. □

**3. Complete the following activity :**

$[(p \vee q) \wedge \sim p] \to q$

$\equiv [(p \wedge \sim p)$ □ $(q \wedge \sim p)] \to q$ ... [Distributive Law]

$\equiv [$ □ □ $(q \wedge \sim p)] \to q$ ... [....................]

$\equiv (q \wedge \sim p) \to q$ ... [Identity Law]

$\equiv (\sim p \wedge q) \to q$ ... [....................]

$\equiv \sim(\square)\ \square\ q$ ... [Conditional Law]

$\equiv (p \vee \square)\ \square\ q$ ... [De Morgan's Law]

$\equiv p \vee (\square\ \square\ q)$ ... [..............]

$\equiv p \vee \square \equiv t$

**4. Express the truth of each of the following statements by Venn diagram :**

(i) Equilateral triangles are isosceles.

(ii) Many servants are not graduates.

(iii) Some rational numbers are not integers.

(iv) Some quadratic equations have equal roots.

**5. Consider the following statements :**

$p$ : 2 is even prime number.

$q$ : Mumbai is the capital of Maharashtra.

$r$ : 1 is the smallest prime number.

Then write the truth values of :

**(i)** $\sim p \leftrightarrow (\sim q \wedge r) \equiv \sim \square \leftrightarrow (\sim \square \wedge \square)$

$\equiv \square \leftrightarrow (\square \wedge \square)$

$\equiv \square \leftrightarrow \square \equiv \square$

**(ii)** $\sim q \wedge (p \rightarrow q) \equiv \sim \square \wedge (\square \rightarrow \square)$

$\equiv \square \wedge \square$

$\equiv \square$

**(iii)** $p \vee (\sim q \leftrightarrow r) \equiv \square \vee (\sim \square \leftrightarrow \square)$

$\equiv \square \vee (\square \leftrightarrow \square)$

$\equiv \square \vee \square$

$\equiv \square$

**6. Complete truth table for $(p \wedge q) \rightarrow r \equiv p \rightarrow (q \rightarrow r)$.**

**Justify it :**

| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| $p$ | $q$ | $r$ | $p \wedge q$ | $(p \wedge q) \rightarrow r$ | $q \rightarrow r$ | $p \rightarrow (q \rightarrow r)$ |
| T | T | T | T | T | T | T |
| T | $\square$ | F | $\square$ | $\square$ | $\square$ | $\square$ |
| $\square$ | F | T | $\square$ | T | T | $\square$ |
| T | F | $\square$ | $\square$ | $\square$ | $\square$ | $\square$ |
| F | T | T | F | T | T | T |
| $\square$ | T | F | $\square$ | $\square$ | F | $\square$ |
| F | $\square$ | $\square$ | $\square$ | $\square$ | $\square$ | $\square$ |
| $\square$ | $\square$ | $\square$ | $\square$ | T | $\square$ | T |

Justification : ........................

**7. Complete the following truth table and give your conclusion :**

| $p$ | $q$ | $\sim p$ | $\sim p \wedge q$ | $q \rightarrow p$ | $(\sim p \wedge q) \wedge (q \rightarrow p)$ |
|---|---|---|---|---|---|
| T | T | F | $\square$ | T | $\square$ |
| T | F | F | $\square$ | $\square$ | $\square$ |
| F | T | T | T | F | F |
| $\square$ | $\square$ | $\square$ | $\square$ | $\square$ | $\square$ |

Conclusion : ........................

**8. Complete the following activity :**

$(p \wedge q) \vee (\sim p \wedge q) \vee (\sim q \wedge r)$

$\equiv [(\square \vee \square) \wedge q] \vee (\sim q \wedge r)$ ... (Distributive Law)

$\equiv (\square \wedge q) \vee (\sim q \wedge r)$ ... (..................)

$\equiv \square \vee (\sim q \wedge r)$ ... (Identity Law)

$\equiv (\square \vee \sim q) \wedge (\square \vee r)$ ... (..................)

$\equiv \square \wedge (\square \vee r)$ ... (Complement Law)

$\equiv q \vee r.$ ... (Identity Law)

## OBJECTIVE SECTION

### MULTIPLE CHOICE QUESTIONS

**Select and write the correct answer from the given alternatives in each of the following question :**

**1.** Which of the following is a statement in logic?

(a) Where are you?

(b) May God bless you.

(c) Mumbai is the capital of India.

(d) I am lying.

**2.** Which of the following is not a statement?

(a) Delhi is the capital of India.

(b) Mumbai is in Maharashtra.

(c) $x + 3 = 5$.

(d) The sum of two odd integers is an odd integer.

**3.** Which of the following statement is true?

(a) $3 + 7 = 4$ or $3 - 7 = 4$.

(b) If Pune is in Maharashtra, then Hyderabad is in Kerala.

(c) It is a false that 12 is not divisible by 3.

(d) The square of any odd integer is even.

4. A biconditional statement is the conjunction of two ......... statements.
(a) negative (b) compound
(c) connective (d) conditional

5. The words which combine two simple statements are called
(a) disjunction (b) conjunction
(c) logical connectives (d) components

6. If $p \to q$ is true and $p \wedge q$ is false, then the truth value of $\sim p \vee q$ is
(a) T, F (b) T, T (c) F, T (d) F, F

7. If $p \vee q$ is true and $p \wedge q$ is false, then which of the following is not true?
(a) $p \vee q$ (b) $p \leftrightarrow q$ (c) $\sim p \vee \sim q$ (d) $q \vee \sim q$

8. If $\sim p \vee q$ is F, then which of the following is correct?
(a) $p \leftrightarrow q$ is T (b) $p \to q$ is T
(c) $q \to p$ is T (d) $q \to p$ is F

9. If $p$ : Ram is tall and $q$ : Ram is handsome, then the symbolic form of 'Ram is handsome but not tall' is
(a) $p \wedge \sim q$ (b) $q \vee \sim p$ (c) $q \wedge \sim p$ (d) $p \vee \sim q$

10. If $p$ : Rohit is tall and $q$ : Rohit is handsome, then the symbolic form of 'It is not true that Rohit is short or not handsome' is
(a) $\sim(\sim p \wedge \sim q)$ (b) $\sim(\sim p \to \sim q)$
(c) $\sim p \vee \sim q$ (d) $p \wedge q$

11. The converse of contrapositive of $\sim p \to q$ is
(a) $q \to p$ (b) $\sim q \to p$
(c) $p \to \sim q$ (d) $\sim q \to \sim p$

12. Inverse of statement pattern $(p \vee q) \to (p \wedge q)$ is
(a) $(p \wedge q) \to (p \vee q)$
(b) $\sim (p \vee q) \to (p \wedge q)$
(c) $(\sim p \vee \sim q) \to (\sim p \wedge \sim q)$
(d) $(\sim p \wedge \sim q) \to (\sim p \vee \sim q)$

13. If $p \wedge q = F$, $p \to q = F$, then truth values of $p$ and $q$ are
(a) T, T (b) T, F (c) F, T (d) F, F

14. Consider the following statements :
(i) If $x > 10$, then $x > 5$
(ii) If $x \ngtr 10$, then $x \ngtr 5$
(iii) If $x > 5$, then $x > 10$
(iv) If $x \ngtr 5$, tehn $x \ngtr 10$
The pairs of statement having the same meaning is
(a) (i) and (ii) (b) (i) and (iii)
(c) (i) and (iv) (d) (ii) and (iv)

15. Negation of '$A \cup B = B$ is the sufficient condition for $A \subseteq B$' is
(a) $A \cup B = B$ and $A \nsubseteq B$
(b) $A \cup B \neq B$ but $A \subseteq B$
(c) If $A \nsubseteq B$, then $A \cup B \neq B$
(d) $A \cap B = B$ and $A \subseteq B$

16. Negation of $p \to (p \vee \sim q)$ is
(a) $\sim p \to (\sim p \vee q)$ (b) $p \wedge (\sim p \wedge q)$
(c) $\sim p \vee (\sim p \vee \sim q)$ (d) $\sim p \leftarrow (\sim p \to q)$

17. $p \to (q \to r)$ is logically equivalent to
(a) $(p \vee q) \to \sim r$ (b) $(p \wedge q) \to \sim r$
(c) $(p \vee q) \to r$ (d) $(p \wedge q) \to r$

18. If $p$ : 4 is an even prime number
$q$ : 6 is divisor of 12
and $r$ : HCF of 4 and 6 is 2,
then which of the following is true?
(a) $p \wedge q$ (b) $(p \vee q) \wedge (\sim r)$
(c) $\sim (q \wedge r) \vee p$ (d) $\sim p \vee (q \wedge r)$

19. The statement $(p \wedge \sim p) \to q$ is
(a) always true (b) always false
(c) false if $q$ is true (d) none of these

20. If '$r \to s$' is an implication, then the implication '$\sim s \to \sim r$' is called its
(a) converse (b) contrapositive
(c) inverse (d) alternative

21. The statement $\sim (p \leftrightarrow \sim q)$ is
(a) equivalent to $p \leftrightarrow q$ (b) equivalent to $\sim p \leftrightarrow q$
(c) a tautology (d) a contradiction

22. Which of the following is true?
(a) $p \to q \equiv \sim p \to \sim q$
(b) $\sim (p \to \sim q \equiv \sim p \wedge q$
(c) $\sim (\sim p \to \sim q) \equiv \sim p \wedge q$
(d) $\sim (p \leftrightarrow q) \equiv [\sim (p \to q) \wedge \sim (q \to p)]$

23. Let $p$ and $q$ be two statements, then $\sim(\sim p \wedge q) \wedge (p \vee q)$ is logically equivalent to
(a) $q$ (b) $p \wedge q$ (c) $p$ (d) $p \vee \sim q$

24. The negation of disjunction of two statements is logically equivalent to the conjunction of their
(a) disjunctions (b) contradictions
(c) alternatives (d) negations

25. If $p$ : It is an Amir Khan movie.

$q$ : It is hit.

Then the symbolic form of the negation of the statement 'It is an Amir Khan movie but not a hit' is

(a) $p \wedge \sim q$ (b) $\sim p \vee \sim q$

(c) $\sim p \wedge \sim q$ (d) $\sim p \vee q$

26. Negation of '$3+5=8$ and $2<7$' is

(a) $3+5 \neq 8$ and $2 \nless 7$

(b) $3+5 \neq 8$ or $2 \nless 7$

(c) $3+5=8$ and $2 \nless 7$

(d) $3+5 \neq 8$ or $2<7$

27. Negation of '$\forall x \in N$, $x^2+x$ is even number' is

(a) $\forall x \in N$, $x^2+x$ is not an even number.

(b) $\exists x \in N$ such that $x^2+x$ is not an even number.

(c) $\exists x \in N$, such that $x^2+x$ is an even number.

(d) $\forall x \in N$, $x^2+x$ is not an odd number

28. The negation of '$\exists x \in$ N, such that $x^2=x$' is

(a) $\exists x \in N$, such that $x^2 \neq x$

(b) $\forall x \in N$, $x^2=x$

(c) $\forall x \in N$, $x^2 \neq x$

(d) $\exists x \in N$, such that $x^2>x$

29. The negation of $\sim p \vee (\sim q \wedge p)$ is equivalent to

(a) $p \wedge \sim q$ (b) $p \wedge (q \wedge \sim p)$

(c) $p \vee (q \vee \sim p)$ (d) $p \wedge q$

30. The negation of contrapositive of $p \wedge (q \rightarrow r)$ is

(a) $(\sim p) \vee (q \wedge \sim r)$ (b) $(\sim p) \vee (q \vee \sim r)$

(c) $(\sim p) \vee (q \rightarrow r)$ (d) $(\sim p) \rightarrow (q \wedge r)$

### Answers

1. (c) Mumbai is the capital of India.
2. (c) $x+3=5$
3. (c) It is a false that 12 is not divisible by 3.
4. (d) conditional
5. (c) logical connectives
6. (b) T, T  7. (b) $p \leftrightarrow q$
8. (c) $q \rightarrow p$ is T  9. (c) $q \wedge \sim p$
10. (d) $p \wedge q$  11. (c) $p \rightarrow \sim q$
12. (d) $(\sim p \wedge \sim q) \rightarrow (\sim p \vee \sim q)$
13. (b) T, F
14. (c) (i) and (iv)
15. (a) $A \cup B = B$ and $A \subseteq B$
16. (b) $p \wedge (\sim p \wedge q)$  17. (d) $(p \wedge q) \rightarrow r$
18. (d) $\sim p \vee (q \wedge r)$  19. (a) always true
20. (b) contrapositive  21. (a) equivalent to $p \leftrightarrow q$
22. (c) $\sim(\sim p \rightarrow \sim q) \equiv \sim p \wedge q$  23. (c) $p$
24. (d) negations  25. (d) $\sim p \vee q$
26. (b) $3+5 \neq 8$ or $2 \nless 7$
27. (b) $\exists x \in N$ such that $x^2+x$ is not an even number.
28. (c) $\forall x \in N$, $x^2 \neq x$
29. (d) $p \wedge q$  30. (a) $(\sim p) \vee (q \wedge \sim r)$.

### TRUE OR FALSE

**State whether the following statements are *True* or *False* :**

1. Mathematical identities are statements.
2. $x+4=8$ is a true statement.
3. $p \rightarrow q$ is false if $p$ is false and $q$ is true.
4. The truth value of : 'If Rome is not in Italy, then Paris is in France' is T.
5. The bicondition statement $p \leftrightarrow q$ is the disjunction of $p \rightarrow q$ and $q \rightarrow p$.
6. If $p$ and $q$ are true and $r$ and $s$ are false, then the truth value of $(p \leftrightarrow \sim q) \wedge (r \leftrightarrow \sim s)$ is true.
7. If a statement pattern involves 4 component statement $p, q, r, s$ and each of $p, q, r, s$ has 2 possible truth values namely T and F, then the truth table of the statement pattern consists of 16 rows.
8. The symbol $\sim$ is not changed while finding the dual.
9. The dual of the statement $\sim(p \wedge q) \vee (\sim q \wedge \sim p)$ is $(p \vee q) \wedge (q \vee p)$.
10. $p \rightarrow \sim q \equiv \sim p \vee q$.
11. $(p \vee q) \wedge (\sim p)$ is a contradiction.
12. $p \vee [\sim(p \wedge q)]$ is a tautology.
13. The negation of $p \wedge (q \rightarrow r)$ is $p \vee (q \wedge \sim r)$.
14. The dual of $(p \wedge q) \vee \sim q$ is $(p \vee q) \wedge \sim q$.
15. The converse of inverse of $\sim p \rightarrow q$ is $q \rightarrow \sim p$.
16. The symbolic form of the statement 'Sandeep neither likes tea nor coffee but enjoys a soft drink' is $(\sim p \vee \sim q) \wedge r$.
17. 'It is false that 27 is not divisible by 9' is a false statement.
18. The statement 'The centre of a circle bisect each chord of the circle' is a true statement.
19. In the truth table for the statement $(p \vee q) \vee \sim p$, the last column has the truth value T, T, T, T.
20. If $\sim(p \vee q) \rightarrow r$ has truth value F, then truth values of $p, q, r$ are FTF.

### Answers

1. True 2. False 3. False 4. True 5. False
6. False 7. True 8. True 9. False 10. False
11. False 12. True 13. False 14. True 15. False
16. False 17. False 18. False 19. True 20. False.

## FILL IN THE BLANKS

**Fill in the following blanks with an appropriate words/numbers :**

1. The truth value of the statement 'India is not a democratic country or China is a communist country' is .........
2. The truth value of the statement 'A quadratic equation has two distinct roots or 6 has three prime factors' is .........
3. The symbolic form of the statement 'It is false that $3^2+4^2=5^2$ or $\sqrt{2}$ is not a rational number but $3^2+4^2=5^2$ and $8>3$' is .........
4. If $p$ and $q$ are true and $r$ and $s$ are false, then the truth value of $(p \to r) \vee (q \to s)$ is .........
5. If a statement pattern contains '$m$' connectives and '$n$' component statements, then the truth table of the statement pattern consists of ......... columns.
6. The dual of $[(p \wedge q) \vee r] \wedge (q \wedge r) \vee s]$ is .........
7. $\sim p \to q \equiv p$ ...... $q$.
8. $\sim (p \leftrightarrow q) \equiv (p \wedge \sim q)$ ...... $(\sim p \wedge q)$.
9. $\sim (\sim p) \equiv p$ is called ......... law.
10. $\sim (p \vee q) \vee (\sim p \wedge q) \equiv$ .........
11. The truth value of the statement 'Neither 21 is a prime number nor it is divisible by 3' is .........
12. The statement pattern $(p \wedge q) \wedge (\sim q)$ is a .........
13. The dual of $(p \wedge t) \vee (c \wedge \sim q)$ is .........
14. The negation of 'Every student has paid the fees' is .........
15. The negation of $(p \wedge q) \to (\sim p \vee r)$ is .........
16. The contrapositive of $p \to \sim q$ is .........
17. If $p \vee q$ is T and $(p \vee q) \to q$ is F, then truth values of $p$ and $q$ are ......... and ......... respectively.
18. The negation of $(p \to q) \wedge r$ is .........
19. The negation of 'If $10>5$ and $5<8$, then $8<7$' is .........
20. The dual of $(p \to \sim q) \vee q$ is .........

### Answers

1. True (T) 2. False (F) 3. $(\sim p \vee \sim q) \wedge (p \wedge r)$
4. false 5. $m+n$ 6. $[(p \vee q) \wedge r] \vee [q \vee r) \wedge s]$
7. $\vee$ 8. $\vee$ 9. involution 10. $\sim p$ 11. F
12. contradiction 13. $(p \vee c) \wedge (t \vee \sim q)$
14. Some students have not paid the fees.
15. $(p \wedge q) \wedge (p \wedge \sim r)$ 16. $q \to \sim p$ 17. T, F
18. $(p \wedge \sim q) \vee (\sim r)$ 19. $10>5$ and $5<8$ but $8 \nless 7$
20. $(\sim p \wedge \sim q) \wedge q$.

❄ ❄ ❄

# 2 MATRICES

CHAPTER OUTLINE



## IMPORTANT FORMULAE

1. An arrangement of $mn$ numbers in $m$ rows and $n$ columns, written within bracket is called a **matrix** of order $m \times n$ which is generally denoted as $A = [a_{ij}]_{m \times n}$, where $a_{ij}$ = element of A in $i^{th}$ row and $j^{th}$ column.
2. Two matrices A and B are said to be **equal** if they are of same order and their corresponding elements are same.
3. A matrix $A = [a_{ij}]_{m \times n}$ is called a square matrix if $m = n$, i.e. number of rows = number of columns.
4. If A is a square matrix, then determinant of A is denoted by $|A|$.
5. A square matrix A is called *singular matrix* if $|A| = 0$ and it is called *non-singular matrix* if $|A| \neq 0$.
6. A matrix obtained from the matrix A by interchanging its rows and columns is called *transpose of A*, which is denoted by $A'$ or $A^T$.
7. A square matrix A is called *symmetric matrix* if $A = A'$ and it is called *skew-symmetric matrix* if $A = -A'$.
8. The diagonal elements of a skew-symmetric matrix are zero.
9. Two matrices A and B can be added or subtracted, if their orders are same.
10. If A and B are two matrices, then product AB exist, if the number of columns of A is equal to the number of rows of B.
11. If A is a square matrix, then
    (i) $A + A'$ is a symmetric matrix
    (ii) $A - A'$ is a skew-symmetric matrix.
12. Every square matrix A can be expressed as the sum of a symmetric and skew-symmetric matrix as
    $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$.
13. **Elementary Transformations :**
    (1) Interchanging of any two rows (or columns). If $i^{th}$ and $j^{th}$ rows of a matrix are interchanged, then it is denoted by $R_{ij}$ or $R_i \leftrightarrow R_j$.
    If $i^{th}$ and $j^{th}$ columns of a matrix are interchanged, then it is denoted by $C_{ij}$ or $C_i \leftrightarrow C_j$.

(2) Multiplying the elements of any row (or column) by a non-zero real number.

If the elements of $i^{th}$ row are multiplied by a non-zero real number $k$, then we denote it as $kR_i$.

If the elements of $i^{th}$ column are multiplied by a non-zero real number $k$, then we denote it as $kC_i$.

(3) Multiplying all the elements of any row (or column) by a non-zero real number $k$ and adding with the corresponding elements of any other row (or column).

If the elements of $j^{th}$ row are multiplied by $k$ and added with the corresponding elements of $i^{th}$ row, then we denote it as $R_i + kR_j$.

If the elements of $j^{th}$ column are multiplied by $k$ and added with the corresponding elements of $i^{th}$ column, then we denote it as $C_i + kC_j$.

**14. Rules for applying Elementary Transformations :**

(1) If A, B, P are three matrices such that AB = P, then any row transformation which is performed on pre-factor A, must also be performed on P to preserve the equality, i.e. if AB = P and A is changed to $A_R$ by any row transformation and P is changed to $P_R$ by the same row transformation, then $A_R B = P_R$.

(2) If A, B, P are three matrices such that AB = P, then any column transformation which is performed on post-factor B must also be performed on P to presurve the equality, i.e. if AB = P and B is changed to $B_C$ by any column transformation and P is changed to $P_C$ by the same column transformation, then $AB_C = P_C$.

**15. Inverse of a Matrix :** If A and B are two square matrices of same order such that AB = BA = I, then A and B are said to be inverses to each other. The inverse of A, i.e. B is denoted by $A^{-1}$. Thus $AA^{-1} = A^{-1}A = I$.

**16.** The inverse of a square matrix A exists, if it is non-singular, i.e. $|A| \neq 0$.

**17.** For finding the inverse of A, if row transformations are to be used, then we consider $AA^{-1} = I$ and if column transformations are to be used, then we consider $A^{-1}A = I$.

**18. Minor and cofactor of an element :** Minor of an elements $a_{ij}$ of a square matrix A is the determinant obtained by deleting $i^{th}$ row and $j^{th}$ column of the matrix A and is denoted by $M_{ij}$.

Cofactor of $a_{ij}$ is given by $A_{ij} = (-1)^{i+j} M_{ij}$.

**19. Adjoint of a Matrix :**

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then adjoint of A is denoted by adj A, where

$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$, where $A_{ij}$ is the cofactor of $a_{ij}$.

**20.** If A is a non-singular matrix, then $A^{-1} = \frac{1}{|A|}(\text{adj } A)$.

**21.** The matrix form of the equations

$a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$, $a_3x + b_3y + c_3z = d_3$ is

$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$, i.e. AX = B.

**22.** The solution of the equation AX = B is given by $X = A^{-1}B$, where $|A| \neq 0$.

## INTRODUCTION

In determinants, you have studied about a square arrangement of the numbers which has a value.

Let us now study a new type of arrangement which does not necessarily contain equal numbers of rows and columns. Also, it does not have the value. Such an arrangement is called a 'Matrix'.

## 2.1 : DEFINITION OF A MATRIX

An arrangement of $mn$ numbers in $m$ rows and $n$ columns written within brackets, is called a **matrix**.

If a matrix contains $m$ rows and $n$ columns, we say that it is an $m \times n$ (read $m$ by $n$) matrix and $m \times n$ is called the order of the matrix.

The numbers which form a matrix are called the elements of the matrix.

Matrices are usually denoted by capital letters A, B, C, ..., and the elements of the matrix are denoted by small letters.

**For example :** $\begin{bmatrix} 2 & 3 & -1 \\ 4 & 5 & 6 \end{bmatrix}$, $\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$ and $[a \ b \ c]$ are matrices of orders $2 \times 3$, $2 \times 2$ and $1 \times 3$ respectively.

**Notation :** The element of the matrix A which belongs to the first row and first column is denoted by $a_{11}$. In $a_{11}$, the first suffix is for the row and second suffix is for the column, in which the element lies. In general, the element which belongs to the $i^{th}$ row and the $j^{th}$ column of matrix A will be denoted by $a_{ij}$.

Using this notation, a matrix of order $m \times n$ can be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

This is generally written as,

$A = [a_{ij}]_{m \times n}$, where $i = 1, 2, 3, ..., m$ and $j = 1, 2, 3, ..., n$.

e.g. (i) $A = [a_{ij}]_{2 \times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

(ii) $B = [b_{ij}]_{3 \times 3} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$

## 2.2 : TYPES OF MATRICES

1. **Row Matrix :** A matrix which contains only one row is called a row matrix.

   **For example :** [2], [3 2], [2 −1 3] are row matices of orders $1 \times 1$, $1 \times 2$, $1 \times 3$ respectively.

   **Note :** The order of a row matrix is of the type $1 \times n$.

2. **Column Matrix :** A matrix which contains only one column is called a column matrix.

   **For example :** [3], $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ are column matrices of orders $1 \times 1$, $2 \times 1$, $3 \times 1$ respectively.

   **Note :** The order of a column matrix is of the type $m \times 1$.

3. **Rectangular Matrix :** A matrix $A = [a_{ij}]_{m \times n}$ is called a rectangular matrix if $m \neq n$.

   i.e. number of rows ≠ number of columns.

   **For example :** $\begin{bmatrix} 2 & 3 & 1 \\ 4 & 5 & 6 \end{bmatrix}$, $\begin{bmatrix} 2 & 1 \\ -3 & -1 \\ 1 & 4 \end{bmatrix}$, [2 3 1] are rectangular matrices of orders $2 \times 3$, $3 \times 2$, $1 \times 3$ respectively.

4. **Square Matrix :** A matrix $A = [a_{ij}]_{m \times n}$ is called a square matrix if $m = n$.

   i.e. number of rows = number of columns.

   **For example :** [2], $\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, $\begin{bmatrix} 3 & 1 & 2 \\ 4 & 5 & 1 \\ 6 & -2 & 4 \end{bmatrix}$ are square matrices of orders $1 \times 1$, $2 \times 2$, $3 \times 3$ respectively.

   A matrix of order $n \times n$ can be referred as a square matrix of order $n$. Hence, by a square matrix of order 2, we mean a matrix of order $2 \times 2$.

   In a square matrix $A = [a_{ij}]$, the elements $a_{ij}$ where $i = j$ are called **diagonal elements** and the elements $a_{ij}$ where $i \neq j$ are called **non-diagonal elements.**

   **For example :** In a square matrix

   $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

   $a_{11}$, $a_{22}$, $a_{33}$ are diagonal elements and all the remaining elements are non-diagonal elements.

   **Notes :**

   (i) The elements $a_{ij}$, where $i < j$ are called the elements above the diagonal.

   (ii) The elements $a_{ij}$, where $i > j$ are called the elements below the diagonal.

5. **Zero Matrix or Null Matrix :** If every element of a matrix is zero, it is called a zero matrix. It is denoted by 0.

**For example :** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are zero matrices of orders $2 \times 2$, $2 \times 3$ respectively.

**Note :** A zero matrix may or may not be a square matrix.

6. **Diagonal Matrix :** A square matrix $A = [a_{ij}]$ is called a diagonal matrix, if all its non-diagonal elements are zero.

i.e. $a_{ij} = 0$, when $i \neq j$.

**For example :** $\begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}$, $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ are diagonal matrices.

**Note :** If $a_{11}, a_{22}, a_{33}$ are diagonal elements of a diagonal matrix A of order 3, then we write $A = \text{Diag}\ [a_{11}\ a_{22}\ a_{33}]$.

7. **Scalar Matrix :** A square matrix $A = [a_{ij}]$ is called scalar matrix, if all its non-diagonal elements are zero and diagonal elements are same.

i.e. $a_{ij} = 0$, when $i \neq j$

$= k$, when $i = j$, where $k$ is a real number.

**For example :** $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ are scalar matrices.

**Note :** A scalar matrix is always a diagonal matrix but a diagonal matrix is not necessarily a scalar matrix.

8. **Identity Matrix or Unit Matrix :** A square matrix $A = [a_{ij}]$ is called an identity matrix, if all its non-diagonal elements are zero and diagonal elements are one.

i.e. $a_{ij} = 0$, when $i \neq j$

$= 1$, when $i = j$

It is denoted by $I$.

**For example :** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are identity matrices of orders 2 and 3 respectively.

**Note :** A unit matrix is always a diagonal matrix as well as a scalar matrix.

9. **Triangular Matrices :**

- **Upper Triangular Matrix :** A square matrix $A = [a_{ij}]$ is called an upper triangular matrix, if every element below the diagonal is zero.

  i.e. $a_{ij} = 0$, when $i > j$.

  **For example :** $\begin{bmatrix} 2 & 3 \\ 0 & -4 \end{bmatrix}$, $\begin{bmatrix} 3 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ are upper triangular matrices.

- **Lower Triangular Matrix :** A square matrix $A = [a_{ij}]$ is called a lower triangular matrix, if every element above the diagonal is zero.

  i.e. $a_{ij} = 0$, when $i < j$.

  **For example :** $\begin{bmatrix} 3 & 0 \\ 1 & 5 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 0 \\ 4 & 5 & 7 \end{bmatrix}$ are lower triangular matrices.

**Notes :**

(i) A matrix which is either an upper triangular matrix or a lower triangular matrix is called a triangular matrix.

(ii) The diagonal matrix, unit matrix, scalar matrix and square null matrix are also triangular matrices.

10. **Determinant of a Square Matrix :** If A is a square matrix, then determinant of A is denoted by $|A|$ or det A.

**For example :** If $A = \begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}$, then $|A| = \begin{vmatrix} 2 & -3 \\ 4 & 5 \end{vmatrix}$.

11. **Non-singular Matrix :** A square matrix A is called a non-singular matrix, if $|A| \neq 0$.

**For example :** If $A = \begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}$,

then $|A| = \begin{vmatrix} 2 & -3 \\ 4 & 5 \end{vmatrix} = 10 + 12 = 22 \neq 0$

$\therefore$ A is a non-singular matrix.

12. **Singular Matrix :** A square matrix A is called a singular matrix, if $|A| = 0$.

**For example :** If $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$, then

$|A| = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 4 - 4 = 0$

$\therefore$ A is a singular matrix.

**EXERCISE 2.1** | **Textbook pages 39 and 40**

**1. Construct a matrix $A = [a_{ij}]_{3 \times 2}$ whose elements $a_{ij}$ is given by**

**(i)** $a_{ij} = \frac{(i-j)^2}{5-i}$ **(ii)** $a_{ij} = i - 3j$

**(iii)** $a_{ij} = \frac{(i+j)^3}{5}$.

**Solution :**

(i) $A = [a_{ij}]_{3\times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$

Now, $a_{ij} = \dfrac{(i-j)^2}{5-i}$

$\therefore\ a_{11} = \dfrac{(1-1)^2}{5-1} = \dfrac{0}{4} = 0$

$a_{12} = \dfrac{(1-2)^2}{5-1} = \dfrac{1}{4}$

$a_{21} = \dfrac{(2-1)^2}{5-2} = \dfrac{1}{3}$

$a_{22} = \dfrac{(2-2)^2}{5-2} = \dfrac{0}{3} = 0$

$a_{31} = \dfrac{(3-1)^2}{5-3} = \dfrac{4}{2} = 2$

$a_{32} = \dfrac{(3-2)^2}{5-3} = \dfrac{1}{2}$

$\therefore\ A = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{3} & 0 \\ 2 & \frac{1}{2} \end{bmatrix}$

**(ii) and (iii) :** *Refer to the solution of Q. 1 (i).*

**Ans.** **(ii)** $\begin{bmatrix} -2 & -5 \\ -1 & -4 \\ 0 & -3 \end{bmatrix}$ **(iii)** $\begin{bmatrix} \frac{8}{5} & \frac{27}{5} \\ \frac{27}{5} & \frac{64}{5} \\ \frac{64}{5} & \frac{125}{5} \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 8 & 27 \\ 27 & 64 \\ 64 & 125 \end{bmatrix}$

**2. Classify each of the following matrices as a row, a column, a square, a diagonal, a scalar, a unit, an upper triangular, a lower triangular matrix :**

(i) $\begin{bmatrix} 3 & -2 & 4 \\ 0 & 0 & -5 \\ 0 & 0 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 5 \\ 4 \\ -3 \end{bmatrix}$ (iii) $[9 \ \ \sqrt{2} \ \ -3]$

(iv) $\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$ (v) $\begin{bmatrix} 2 & 0 & 0 \\ 3 & -1 & 0 \\ -7 & 3 & 1 \end{bmatrix}$ (vi) $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$

(vii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

**Solution :**

(i) Since, all the elements below the diagonal are zero, it is an **upper tringular matrix.**

(ii) This matrix has only one column, it is a **column matrix.**

(iii) This matrix has only one row, it is a **row matrix.**

(iv) Since, diagonal elements are equal and non-diagonal elements are zero, it is a **scalar matrix.**

(v) Since, all the elements above the diagonal are zero, it is a **lower triangular matrix.**

(vi) Since, all the non-diagonal elements are zero, it is a **diagonal matrix.**

(vii) Since, diagonal elements are 1 and non-diagonal elements are 0, it is an **identity (or unit) matrix.**

**3. Which of the following matrices are singular or non-singular :**

(i) $\begin{bmatrix} a & b & c \\ p & q & r \\ 2a-p & 2b-q & 2c-r \end{bmatrix}$ (ii) $\begin{bmatrix} 5 & 0 & 5 \\ 1 & 99 & 100 \\ 6 & 99 & 105 \end{bmatrix}$

(iii) $\begin{bmatrix} 3 & 5 & 7 \\ -2 & 1 & 4 \\ 3 & 2 & 5 \end{bmatrix}$ (iv) $\begin{bmatrix} 7 & 5 \\ -4 & 7 \end{bmatrix}$.

**Solution :**

(i) Let $A = \begin{bmatrix} a & b & c \\ p & q & r \\ 2a-p & 2b-q & 2c-r \end{bmatrix}$

$\therefore\ |A| = \begin{vmatrix} a & b & c \\ p & q & r \\ 2a-p & 2b-q & 2c-r \end{vmatrix}$

By $R_3 + R_2$, we get,

$|A| = \begin{vmatrix} a & b & c \\ p & q & r \\ 2a & 2b & 2c \end{vmatrix}$

$= 2\begin{vmatrix} a & b & c \\ p & q & r \\ a & b & c \end{vmatrix}$

$= 2 \times 0$ ... $[\because R_1 \equiv R_3]$

$= 0$

$\therefore$ A is a **singular** matrix.

(ii) Let $B = \begin{bmatrix} 5 & 0 & 5 \\ 1 & 99 & 100 \\ 6 & 99 & 105 \end{bmatrix}$

$\therefore\ |B| = \begin{vmatrix} 5 & 0 & 5 \\ 1 & 99 & 100 \\ 6 & 99 & 105 \end{vmatrix}$

By $R_3 - R_2$, we get

$$|B| = \begin{vmatrix} 5 & 0 & 5 \\ 1 & 99 & 100 \\ 5 & 0 & 5 \end{vmatrix}$$

$= 0 \quad \ldots [\because R_1 \equiv R_3]$

$\therefore$ B is a **singular** matrix.

**(iii)** Let $C = \begin{bmatrix} 3 & 5 & 7 \\ -2 & 1 & 4 \\ 3 & 2 & 5 \end{bmatrix}$

$\therefore |C| = \begin{vmatrix} 3 & 5 & 7 \\ -2 & 1 & 4 \\ 3 & 2 & 5 \end{vmatrix}$

$= 3(5-8) - 5(-10-12) + 7(-4-3)$

$= -9 + 110 - 49 = 52 \neq 0$

$\therefore$ C is a **non-singular** matrix.

**(iv)** Let $D = \begin{bmatrix} 7 & 5 \\ -4 & 7 \end{bmatrix}$

$\therefore |D| = \begin{vmatrix} 7 & 5 \\ -4 & 7 \end{vmatrix}$

$= 49 - (-20) = 69 \neq 0$

$\therefore$ D is a **non-singular** matrix.

**4. Find $k$, if the following matrices are singular :**

**(i)** $\begin{bmatrix} 7 & 3 \\ -2 & k \end{bmatrix}$ **(ii)** $\begin{bmatrix} 4 & 3 & 1 \\ 7 & k & 1 \\ 10 & 9 & 1 \end{bmatrix}$

**(iii)** $\begin{bmatrix} k-1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & -2 & 4 \end{bmatrix}$.

**Solution :**

(i) Let $A = \begin{bmatrix} 7 & 3 \\ -2 & k \end{bmatrix}$

Since, A is a singular matrix, $|A| = 0$

$\therefore \begin{vmatrix} 7 & 3 \\ -2 & k \end{vmatrix} = 0$

$\therefore 7k - (-6) = 0$

$\therefore 7k = -6 \quad \therefore k = -\frac{6}{7}$

**(ii)** Let $B = \begin{bmatrix} 4 & 3 & 1 \\ 7 & k & 1 \\ 10 & 9 & 1 \end{bmatrix}$

Since, B is a singular matrix, $|B| = 0$

$\therefore \begin{vmatrix} 4 & 3 & 1 \\ 7 & k & 1 \\ 10 & 9 & 1 \end{vmatrix} = 0$

$\therefore 4(k-9) - 3(7-10) + 1(63-10k) = 0$

$\therefore 4k - 36 + 9 + 63 - 10k = 0$

$\therefore -6k + 36 = 0$

$\therefore 6k = 36 \quad \therefore k = 6.$

**(iii)** Let $C = \begin{bmatrix} k-1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & -2 & 4 \end{bmatrix}$

Since, C is a singular matrix, $|C| = 0$

$\therefore \begin{vmatrix} k-1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & -2 & 4 \end{vmatrix} = 0$

$\therefore (k-1)(4+4) - 2(12-2) + 3(-6-1) = 0$

$\therefore 8k - 8 - 20 - 21 = 0$

$\therefore 8k = 49$

$\therefore k = \frac{49}{8}$.

## EXAMPLES FOR PRACTICE 2.1

**1.** Find the elements $a_{12}$, $a_{22}$, $a_{31}$, $a_{32}$ in the matrix

$A = \begin{bmatrix} 2 & 3 & -5 \\ 1 & -2 & 4 \\ 6 & 7 & -8 \end{bmatrix}$

**2.** State the order of the following matrices :

(i) $\begin{bmatrix} 2 & 3 & -1 \\ 4 & 5 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & -1 \\ 4 & 5 \\ 6 & 2 \end{bmatrix}$

(iv) $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ (v) $[3 \quad -1]$ (vi) $\begin{bmatrix} 2 & -1 & 4 \\ 6 & 2 & 8 \\ 7 & 9 & 4 \end{bmatrix}$

**3.** State the types of the following matrices :

(i) $\begin{bmatrix} 1 & 2 & 3 \\ -2 & 5 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 \\ -3 \\ 4 \\ 9 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

(v) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (vi) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 3 \end{bmatrix}$

4. Construct a matrix $A = [a_{ij}]$ whose elements $a_{ij}$ is given by

(i) $a_{ij} = \dfrac{i+4j}{2}$ (ii) $a_{ij} = \dfrac{1}{2}(i-2j)$.

**5. Determine whether the following matrices are singular or non-singular :**

(i) $\begin{bmatrix} 3 & 4 \\ -3 & -4 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & -1 & 2 \\ 7 & 0 & 7 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & -2 & 2 \\ 3 & 0 & 4 \\ 5 & -1 & 8 \end{bmatrix}$

(v) $\begin{bmatrix} x+y & y+z & z+x \\ 1 & 1 & 1 \\ z & x & y \end{bmatrix}$.

6. Find $k$, if the following matrices are singular :

(i) $\begin{bmatrix} 6 & -5 & 1 \\ 4 & 2 & -1 \\ 14 & -1 & k \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 2 & 1 \\ -1 & 0 & 2 \\ k & 1 & 2 \end{bmatrix}$.

### Answers

1. $a_{12} = 3$, $a_{22} = -2$, $a_{31} = 6$, $a_{32} = 7$

2. (i) $2 \times 3$ (ii) $3 \times 1$ (iii) $3 \times 2$ (iv) $2 \times 2$

(v) $1 \times 2$ (vi) $3 \times 3$.

3. (i) Rectangular matrix

(ii) Column matrix

(iii) Identity matrix

(iv) Square matrix

(v) Zero matrix

(vi) Upper triangular matrix.

4. (i) $\begin{bmatrix} \frac{5}{2} & \frac{9}{2} \\ 3 & 5 \\ \frac{7}{2} & \frac{11}{2} \end{bmatrix}$ (ii) $\begin{bmatrix} -\frac{1}{2} & -\frac{3}{2} \\ 0 & -1 \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

5. (i) singular (ii) singular

(iii) singular (iv) non-singular

(v) singular.

6. (i) $k = -1$ (ii) $k = \dfrac{3}{4}$.

## 2.3 : ALGEBRA OF MATRICES

1. **Equality of Matrices :** Two matrices A and B are said to be equal $(A = B)$ if

(i) they are of same order

(ii) their corresponding elements are equal.

Thus if two matrices $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are equal, then $a_{ij} = b_{ij}$ for all $i$ and $j$.

**For example :**

(i) If $A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 4 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 5 & 4 \end{bmatrix}$, then $A = B$ but $A \neq C$ and $B \neq C$.

(ii) If $\begin{bmatrix} a & -1 \\ 2 & b \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$, then $a = 3$ and $b = 5$.

2. **Negative of a Matrix :** Let A be a given matrix. Then the matrix, whose elements are the negatives of the corresponding elements of the matrix A, is called the **negative** of the matrix A and is denoted by $-A$.

Thus if $A = [a_{ij}]_{m \times n}$ then $-A = [-a_{ij}]_{m \times n}$

**For example :**

If $A = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$, then $-A = \begin{bmatrix} -2 & 3 \\ 0 & -1 \end{bmatrix}$

3. **Transpose of a Matrix :** A matrix obtained from the matrix A by interchanging its rows and columns is called the transpose of A, which is denoted by $A'$ or $A^T$.

**For example :**

If $A = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 5 & 6 \end{bmatrix}$, then $A' = \begin{bmatrix} 2 & 4 \\ -3 & 5 \\ 1 & 6 \end{bmatrix}$

**Notes :** (i) If the order of A is $m \times n$, then the order of $A'$ is $n \times m$.

(ii) If $A'$ is the transpose of the square matrix A, then $\det A = \det A'$ i.e. $|A| = |A'|$.

(iii) $(A')' = A$, for every matrix A.

Using the above concept of the transpose of a matrix, we have the following two types of matrices :





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**(i) Symmetric Matrix :** A square matrix $A = [a_{ij}]$ is called symmetric matrix if $A = A'$, i.e. $a_{ij} = a_{ji}$ for all $i$ and $j$.

**For example :**

If $A = \begin{bmatrix} 2 & 3 \\ 3 & -1 \end{bmatrix}$, then $A' = \begin{bmatrix} 2 & 3 \\ 3 & -1 \end{bmatrix}$

$\therefore$ $A = A'$, so A is symmetric matrix.

**(ii) Skew-symmetric Matrix :** A square matrix $A = [a_{ij}]$ is called skew-symmetric matrix if $A = -A'$.

i.e. $a_{ij} = -a_{ji}$ for all $i$ and $j$.

**For example :** $\begin{bmatrix} 0 & -4 \\ 4 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix}$ are skew-symmetric matrices.

**Notes :** (i) The diagonal elements of a skew-symmetric matrix are always zero.

(ii) The non-diagonal elements of a skew-symmetric matrix are symmetric (in magnitude) about the diagonal but are opposite in signs.

(iii) The non-diagonal elements of a symmetric matrix are symmetric about the diagonal.

**4. Addition of Matrices :** If two matrices are of same order, then only they can be added. If they are of different orders, we cannot add them.

**Definition :** If $A = [a_{ij}]$ and $B = [b_{ij}]$ are the two matrices of same order $m \times n$, then their sum $A + B$ is a matrix of order $m \times n$, where $A + B = [a_{ij} + b_{ij}]$ for all $i$ and $j$.

**For example :**

If $A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 4 \\ 6 & -2 & 3 \end{bmatrix}$

then $A + B = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 4 \\ 6 & -2 & 3 \end{bmatrix}$

$= \begin{bmatrix} 2+1 & -1+(-1) & 3+4 \\ 4+6 & 5+(-2) & 6+3 \end{bmatrix}$

$= \begin{bmatrix} 3 & -2 & 7 \\ 10 & 3 & 9 \end{bmatrix}$

**Note :** If A, B, C are three matrices of same order, then

(i) $A + B = B + A$, commutative law.

(ii) $A + (B + C) = (A + B) + C$, associative law.

(iii) $A + O = O + A = A$, where $O$ is a null matrix of the same order as A.

**5. Subtraction of Matrices :** As in the case of addition, we can subtract two matrices if and only if they are of same order.

**Definition :** If $A = [a_{ij}]$ and $B = [b_{ij}]$ are two matrices of same order $m \times n$, then their difference $A - B$ is a matrix of order $m \times n$, where

$A - B = [a_{ij} - b_{ij}]$ for all $i$ and $j$.

**For example :**

If $A = \begin{bmatrix} 3 & 1 & 4 \\ -2 & 8 & 6 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 1 & -3 \\ -1 & 4 & 5 \end{bmatrix}$

then $A - B = \begin{bmatrix} 3 & 1 & 4 \\ -2 & 8 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 1 & -3 \\ -1 & 4 & 5 \end{bmatrix}$

$= \begin{bmatrix} 3-2 & 1-1 & 4-(-3) \\ -2-(-1) & 8-4 & 6-5 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 7 \\ -1 & 4 & 1 \end{bmatrix}$

**6. Scalar Multiplication :** Let A be a matrix and $k$ be a number (scalar), then the scalar multiple of A by $k$ is denoted by $kA$ and is obtained by multiplying every element of the matrix A by $k$.

Thus, if $A = [a_{ij}]_{m \times n}$, then

$kA = [ka_{ij}]_{m \times n}$ for all $i$ and $j$.

**For example :** If $A = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 4 & 2 \end{bmatrix}$, then

$3A = 3\begin{bmatrix} 1 & -1 & 4 \\ 3 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -3 & 12 \\ 9 & 12 & 6 \end{bmatrix}$

$-A = \begin{bmatrix} -1 & 1 & -4 \\ -3 & -4 & -2 \end{bmatrix}$.

**Note :** If A and B are two matrices of same order, $O$ is a zero matrix of the same order as that of A and B and $p$, $q$ are scalars, then

(i) $p(A \pm B) = pA \pm pB$.

(ii) $(p \pm q)A = pA \pm qA$.

(iii) $0 \cdot A = O$

(iv) $p \cdot O = O$

(v) $p(qA) = q(pA) = (pq)A$.

**EXERCISE 2.2** | Textbook pages 46 and 47

**1. If** $A = \begin{bmatrix} 2 & -3 \\ 5 & -4 \\ -6 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 2 & 2 \\ 0 & 3 \end{bmatrix}$ **and** $C = \begin{bmatrix} 4 & 3 \\ -1 & 4 \\ -2 & 1 \end{bmatrix}$

**show that**

**(i)** $A + B = B + A$ **(ii)** $(A + B) + C = A + (B + C)$.

**Solution :**

(i) $A + B = \begin{bmatrix} 2 & -3 \\ 5 & -4 \\ -6 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 2 & 2 \\ 0 & 3 \end{bmatrix}$

$= \begin{bmatrix} 2-1 & -3+2 \\ 5+2 & -4+2 \\ -6+0 & 1+3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 7 & -2 \\ -6 & 4 \end{bmatrix}$ ... (1)

$B + A = \begin{bmatrix} -1 & 2 \\ 2 & 2 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 2 & -3 \\ 5 & -4 \\ -6 & 1 \end{bmatrix}$

$= \begin{bmatrix} -1+2 & 2-3 \\ 2+5 & 2-4 \\ 0-6 & 3+1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 7 & -2 \\ -6 & 4 \end{bmatrix}$ ... (2)

From (1) and (2), we get

$A + B = B + A$.

(ii) $A + B = \begin{bmatrix} 2 & -3 \\ 5 & -4 \\ -6 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 2 & 2 \\ 0 & 3 \end{bmatrix}$

$= \begin{bmatrix} 2-1 & -3+2 \\ 5+2 & -4+2 \\ -6+0 & 1+3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 7 & -2 \\ -6 & 4 \end{bmatrix}$

$\therefore (A + B) + C = \begin{bmatrix} 1 & -1 \\ 7 & -2 \\ -6 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ -1 & 4 \\ -2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1+4 & -1+3 \\ 7-1 & -2+4 \\ -6-2 & 4+1 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 6 & 2 \\ -8 & 5 \end{bmatrix}$ ... (1)

Also, $B + C = \begin{bmatrix} -1 & 2 \\ 2 & 2 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ -1 & 4 \\ -2 & 1 \end{bmatrix}$

$= \begin{bmatrix} -1+4 & 2+3 \\ 2-1 & 2+4 \\ 0-2 & 3+1 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & 6 \\ -2 & 4 \end{bmatrix}$

$\therefore A + (B + C) = \begin{bmatrix} 2 & -3 \\ 5 & -4 \\ -6 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 5 \\ 1 & 6 \\ -2 & 4 \end{bmatrix}$

$= \begin{bmatrix} 2+3 & -3+5 \\ 5+1 & -4+6 \\ -6-2 & 1+4 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 6 & 2 \\ -8 & 5 \end{bmatrix}$ ... (2)

From (1) and (2), we get

$(A + B) + C = A + (B + C)$.

**2. If** $A = \begin{bmatrix} 1 & -2 \\ 5 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -3 \\ 4 & -7 \end{bmatrix}$, **then find the matrix** $A - 2B + 6I$, **where I is the unit matrix of order 2.**

**Solution :**

$A - 2B + 6I = \begin{bmatrix} 1 & -2 \\ 5 & 3 \end{bmatrix} - 2\begin{bmatrix} 1 & -3 \\ 4 & -7 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & -2 \\ 5 & 3 \end{bmatrix} - \begin{bmatrix} 2 & -6 \\ 8 & -14 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$

$= \begin{bmatrix} 1-2+6 & -2-(-6)+0 \\ 5-8+0 & 3-(-14)+6 \end{bmatrix}$

$= \begin{bmatrix} 5 & 4 \\ -3 & 23 \end{bmatrix}$.

**3. If** $A = \begin{bmatrix} 1 & 2 & -3 \\ -3 & 7 & -8 \\ 0 & -6 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 9 & -1 & 2 \\ -4 & 2 & 5 \\ 4 & 0 & -3 \end{bmatrix}$, **then find the matrix C such that A + B + C is a zero matrix.**

**Solution :** $A + B + C = 0$

$\therefore C = -A - B$

$= -\begin{bmatrix} 1 & 2 & -3 \\ -3 & 7 & -8 \\ 0 & -6 & 1 \end{bmatrix} - \begin{bmatrix} 9 & -1 & 2 \\ -4 & 2 & 5 \\ 4 & 0 & -3 \end{bmatrix}$

$= \begin{bmatrix} -1 & -2 & 3 \\ 3 & -7 & 8 \\ 0 & 6 & -1 \end{bmatrix} - \begin{bmatrix} 9 & -1 & 2 \\ -4 & 2 & 5 \\ 4 & 0 & -3 \end{bmatrix}$

$= \begin{bmatrix} -1-9 & -2-(-1) & 3-2 \\ 3-(-4) & -7-2 & 8-5 \\ 0-4 & 6-0 & -1-(-3) \end{bmatrix}$

$\therefore C = \begin{bmatrix} -10 & -1 & 1 \\ 7 & -9 & 3 \\ -4 & 6 & 2 \end{bmatrix}$.

**4. If** $A = \begin{bmatrix} 1 & -2 \\ 3 & -5 \\ -6 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -2 \\ 4 & 2 \\ 1 & 5 \end{bmatrix}$ **and**

$C = \begin{bmatrix} 2 & 4 \\ -1 & -4 \\ -3 & 6 \end{bmatrix}$,

**find the matrix X such that** $3A - 4B + 5X = C$.

**Solution :** $3A - 4B + 5X = C$

$\therefore 5X = C - 3A + 4B$

$$= \begin{bmatrix} 2 & 4 \\ -1 & -4 \\ -3 & 6 \end{bmatrix} - 3\begin{bmatrix} 1 & -2 \\ 3 & -5 \\ -6 & 0 \end{bmatrix} + 4\begin{bmatrix} -1 & -2 \\ 4 & 2 \\ 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 \\ -1 & -4 \\ -3 & 6 \end{bmatrix} - \begin{bmatrix} 3 & -6 \\ 9 & -15 \\ -18 & 0 \end{bmatrix} + \begin{bmatrix} -4 & -8 \\ 16 & 8 \\ 4 & 20 \end{bmatrix}$$

$$= \begin{bmatrix} 2-3+(-4) & 4-(-6)-8 \\ -1-9+16 & -4-(-15)+8 \\ -3-(-18)+4 & 6-0+20 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 2 \\ 6 & 19 \\ 19 & 26 \end{bmatrix}$$

$$\therefore\ X = \frac{1}{5}\begin{bmatrix} -5 & 2 \\ 6 & 19 \\ 19 & 26 \end{bmatrix} = \begin{bmatrix} -1 & \frac{2}{5} \\ \frac{6}{5} & \frac{19}{5} \\ \frac{19}{5} & \frac{26}{5} \end{bmatrix}$$

**5. If $A = \begin{bmatrix} 5 & 1 & -4 \\ 3 & 2 & 0 \end{bmatrix}$, find $(A^T)^T$.**

**Solution :** $A = \begin{bmatrix} 5 & 1 & -4 \\ 3 & 2 & 0 \end{bmatrix}$

$$\therefore\ A^T = \begin{bmatrix} 5 & 3 \\ 1 & 2 \\ -4 & 0 \end{bmatrix}$$

$$\therefore\ (A^T)^T = \begin{bmatrix} 5 & 1 & -4 \\ 3 & 2 & 0 \end{bmatrix} = A.$$

**6. If $A = \begin{bmatrix} 7 & 3 & 1 \\ -2 & -4 & 1 \\ 5 & 9 & 1 \end{bmatrix}$, find $(A^T)^T$.**

**Solution :** $A = \begin{bmatrix} 7 & 3 & 1 \\ -2 & -4 & 1 \\ 5 & 9 & 1 \end{bmatrix}$

$$\therefore\ A^T = \begin{bmatrix} 7 & -2 & 5 \\ 3 & -4 & 9 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\therefore\ (A^T)^T = \begin{bmatrix} 7 & 3 & 1 \\ -2 & -4 & 1 \\ 5 & 9 & 1 \end{bmatrix} = A.$$

**7. Find $a, b, c$ if $\begin{bmatrix} 1 & \frac{3}{5} & a \\ b & -5 & -7 \\ -4 & c & 0 \end{bmatrix}$ is a symetric matrix.**

**Solution :** Let $A = \begin{bmatrix} 1 & \frac{3}{5} & a \\ b & -5 & -7 \\ -4 & c & 0 \end{bmatrix}$

Since, A is a symmetric matrix, $a_{ij} = a_{ji}$ for all $i'$ and $j$

$\therefore\ a_{13} = a_{31},\ a_{12} = a_{21}$ and $a_{23} = a_{32}$

$\therefore\ a = -4,\ \frac{3}{5} = b$ and $-7 = c$

$\therefore\ a = -4,\ b = \frac{3}{5}$ and $c = -7.$

*Alternative Method :*

Let $A = \begin{bmatrix} 1 & \frac{3}{5} & a \\ b & -5 & -7 \\ -4 & c & 0 \end{bmatrix}$

Then $A^T = \begin{bmatrix} 1 & b & -4 \\ \frac{3}{5} & -5 & -7 \\ a & -7 & 0 \end{bmatrix}$

Since, A is symmetric matrix, $A = A^T$

$$\therefore\ \begin{bmatrix} 1 & \frac{3}{5} & a \\ b & -5 & -7 \\ -4 & c & 0 \end{bmatrix} = \begin{bmatrix} 1 & b & -4 \\ \frac{3}{5} & -5 & -7 \\ a & -7 & 0 \end{bmatrix}$$

By equality of matrices

$a = -4,\ b = \frac{3}{5}$ and $c = -7.$

**8. Find $x, y, z$ if $\begin{bmatrix} 0 & -5i & x \\ y & 0 & z \\ \frac{3}{2} & -\sqrt{2} & 0 \end{bmatrix}$ is a skew symmetric matrix.**

**Solution :** Let $A = \begin{bmatrix} 0 & -5i & x \\ y & 0 & z \\ \frac{3}{2} & -\sqrt{2} & 0 \end{bmatrix}$

Since, A is skew-symmetric matrix,

$a_{ij} = -a_{ji}$ for all $i$ and $j$.

$\therefore\ a_{13} = -a_{31},\ a_{12} = -a_{21}$ and $a_{23} = -a_{32}$

$\therefore\ x = -\frac{3}{2},\ -5i = -y$ and $z = -(-\sqrt{2})$

$\therefore\ x = -\frac{3}{2},\ y = 5i$ and $z = \sqrt{2}.$

*Alternative Method :*

Let $A = \begin{bmatrix} 0 & -5i & x \\ y & 0 & z \\ \frac{3}{2} & -\sqrt{2} & 0 \end{bmatrix}$



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Then $A^T = \begin{bmatrix} 0 & y & \frac{3}{2} \\ -5i & 0 & -\sqrt{2} \\ x & z & 0 \end{bmatrix}$

Since, A is skew-symmetric matrix, $A = -A^T$

$$\therefore \begin{bmatrix} 0 & -5i & x \\ y & 0 & z \\ \frac{3}{2} & -\sqrt{2} & 0 \end{bmatrix} = -\begin{bmatrix} 0 & y & \frac{3}{2} \\ -5i & 0 & -\sqrt{2} \\ x & z & 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 0 & -5i & x \\ y & 0 & z \\ \frac{3}{2} & -\sqrt{2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -y & -\frac{3}{2} \\ 5i & 0 & \sqrt{2} \\ -x & -z & 0 \end{bmatrix}$$

By equality of matrices

$x = -\frac{3}{2}$, $y = 5i$ and $z = \sqrt{2}$.

**9. For each of the following matrices, find its transpose and state whether it is symmetric, skew-symmetric or neither :**

**(i)** $\begin{bmatrix} 1 & 2 & -5 \\ 2 & -3 & 4 \\ -5 & 4 & 9 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 2 & 5 & 1 \\ -5 & 4 & 6 \\ -1 & -6 & 3 \end{bmatrix}$

**(iii)** $\begin{bmatrix} 0 & 1+2i & i-2 \\ -1-2i & 0 & -7 \\ 2-i & 7 & 0 \end{bmatrix}$.

**Solution :**

**(i)** Let $A = \begin{bmatrix} 1 & 2 & -5 \\ 2 & -3 & 4 \\ -5 & 4 & 9 \end{bmatrix}$

Then $A^T = \begin{bmatrix} 1 & 2 & -5 \\ 2 & -3 & 4 \\ -5 & 4 & 9 \end{bmatrix}$

Since, $A = A^T$, A is a symmetric matrix.

**(ii)** Let $B = \begin{bmatrix} 2 & 5 & 1 \\ -5 & 4 & 6 \\ -1 & -6 & 3 \end{bmatrix}$

Then $B^T = \begin{bmatrix} 2 & -5 & -1 \\ 5 & 4 & -6 \\ 1 & 6 & 3 \end{bmatrix}$

$\therefore B \neq B^T$

Also,

$$-B^T = -\begin{bmatrix} 2 & -5 & -1 \\ 5 & 4 & -6 \\ 1 & 6 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 5 & 1 \\ -5 & -4 & 6 \\ -1 & -6 & -3 \end{bmatrix}$$

$\therefore B \neq -B^T$.

Hence, B is neither symmetric nor skew-symmetric matrix.

**(iii)** Let $C = \begin{bmatrix} 0 & 1+2i & i-2 \\ -1-2i & 0 & -7 \\ 2-i & 7 & 0 \end{bmatrix}$

Then $C^T = \begin{bmatrix} 0 & -1-2i & 2-i \\ 1+2i & 0 & 7 \\ i-2 & -7 & 0 \end{bmatrix}$

$\therefore -C^T = \begin{bmatrix} 0 & 1+2i & i-2 \\ -1-2i & 0 & -7 \\ 2-i & 7 & 0 \end{bmatrix}$

$\therefore C = -C^T$

Hence, C is skew-symmetric matrix.

**10. Construct the matrix $A = [a_{ij}]_{3 \times 3}$, where $a_{ij} = i - j$. State whether A is symmetric or skew-symmetric.**

**Solution :** $A = [a_{ij}]_{3 \times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Now, $a_{ij} = i - j$ for all $i$ and $j$

$\therefore a_{11} = 1 - 1 = 0$, $a_{12} = 1 - 2 = -1$

$a_{13} = 1 - 3 = -2$, $a_{21} = 2 - 1 = 1$

$a_{22} = 2 - 2 = 0$, $a_{23} = 2 - 3 = -1$

$a_{31} = 3 - 1 = 2$, $a_{32} = 3 - 2 = 1$, $a_{33} = 3 - 3 = 0$

$\therefore A = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$

Since, $a_{ij} = i - j = -(j - i) = -a_{ji}$ for all $i$ and $j$,

A is skew-symmetric matrix.

**11. Solve the following equations for X and Y, if**

$3X - Y = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ **and** $X - 3Y = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix}$.

**Solution :**

$3X - Y = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ ... (1)

$X - 3Y = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix}$ ... (2)

Multiplying (1) by 3, we get

$9X - 3y = 3\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$ ... (3)

Subtracting (2) from (3), we get

$$8X = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3-0 & -3-(-1) \\ -3-0 & 3-(-1) \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -3 & 4 \end{bmatrix}$$

$$\therefore \ X = \frac{1}{8}\begin{bmatrix} 3 & -2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} \frac{3}{8} & -\frac{1}{4} \\ -\frac{3}{8} & \frac{1}{2} \end{bmatrix}$$

Substituting the value of X in (1), we get

$$3\begin{bmatrix} \frac{3}{8} & -\frac{1}{4} \\ -\frac{3}{8} & \frac{1}{2} \end{bmatrix} - Y = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\therefore \ Y = \begin{bmatrix} \frac{9}{8} & -\frac{3}{4} \\ -\frac{9}{8} & \frac{3}{2} \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{9}{8}-1 & -\frac{3}{4}-(-1) \\ -\frac{9}{8}-(-1) & \frac{3}{2}-1 \end{bmatrix} = \begin{bmatrix} \frac{1}{8} & \frac{1}{4} \\ -\frac{1}{8} & \frac{1}{2} \end{bmatrix}$$

Hence, $X = \begin{bmatrix} \frac{3}{8} & -\frac{1}{4} \\ -\frac{3}{8} & \frac{1}{2} \end{bmatrix}$ and $Y = \begin{bmatrix} \frac{1}{8} & \frac{1}{4} \\ -\frac{1}{8} & \frac{1}{2} \end{bmatrix}$.

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**12. Find matrices A and B, if**

$2A - B = \begin{bmatrix} 6 & -6 & 0 \\ -4 & 2 & 1 \end{bmatrix}$ and

$A - 2B = \begin{bmatrix} 3 & 2 & 8 \\ -2 & 1 & -7 \end{bmatrix}$.

**Solution :** *Refer to the solution of Q. 11.*

**Ans.** $A = \begin{bmatrix} 3 & -\frac{14}{3} & -\frac{8}{3} \\ -2 & 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -\frac{10}{3} & -\frac{16}{3} \\ 0 & 0 & 5 \end{bmatrix}$.

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**13. Find $x$ and $y$, if**

$$\begin{bmatrix} 2x+y & -1 & 1 \\ 3 & 4y & 4 \end{bmatrix} + \begin{bmatrix} -1 & 6 & 4 \\ 3 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 5 \\ 6 & 18 & 7 \end{bmatrix}$$

**Solution :**

$$\begin{bmatrix} 2x+y & -1 & 1 \\ 3 & 4y & 4 \end{bmatrix} + \begin{bmatrix} -1 & 6 & 4 \\ 3 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 5 \\ 6 & 18 & 7 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 2x+y-1 & -1+6 & 1+4 \\ 3+3 & 4y+0 & 4+3 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 5 \\ 6 & 18 & 7 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 2x+y-1 & 5 & 5 \\ 6 & 4y & 7 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 5 \\ 6 & 18 & 7 \end{bmatrix}$$

By equality of matrices, we get

$2x + y - 1 = 3$ ... (1)

and $4y = 18$ ... (2)

From (2), $y = \frac{9}{2}$

Substituting $y = \frac{9}{2}$ in (1), we get

$2x + \frac{9}{2} - 1 = 3$

$\therefore \ 2x = 3 - \frac{7}{2} = -\frac{1}{2}$

$\therefore \ x = -\frac{1}{4}$

Hence, $x = -\frac{1}{4}$ and $y = \frac{9}{2}$.

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**14. If $\begin{bmatrix} 2a+b & 3a-b \\ c+2d & 2c-d \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$, find $a$, $b$, $c$ and $d$.**

**Solution :** $\begin{bmatrix} 2a+b & 3a-b \\ c+2d & 2c-d \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$

By equality of matrices,

$2a + b = 2$ ... (1)

$3a - b = 3$ ... (2)

$c + 2d = 4$ ... (3)

$2c - d = -1$ ... (4)

Adding (1) and (2), we get

$5a = 5 \quad \therefore \ a = 1$

Substituting $a = 1$ in (1), we get

$2(1) + b = 2 \quad \therefore \ b = 0$

Multiplying equation (4) by 2, we get

$4c - 2d = -2$ ... (5)

Adding (3) and (5), we get

$5c = 2 \quad \therefore \ c = \frac{2}{5}$

Substituting $c = \frac{2}{5}$ in (4), we get

$$2\left(\frac{2}{5}\right) - d = -1$$

$$\therefore\ d = \frac{4}{5} + 1 = \frac{9}{5}$$

Hence, $a = 1$, $b = 0$, $c = \frac{2}{5}$ and $d = \frac{9}{5}$.

**15. There are two book shops owned by Suresh and Ganesh. Their sales (in ₹) for books in three subjects – Physics, Chemistry and Mathematics for two months, July and August 2017 are given by two matrices A and B :**

**July sales (in ₹), Physics, Chemistry, Mathematics**

$$A = \begin{bmatrix} 5600 & 6750 & 8500 \\ 6650 & 7055 & 8905 \end{bmatrix} \begin{matrix} \text{First Row Suresh,} \\ \text{Second Row Ganesh} \end{matrix}$$

**August Sales (in ₹), Physics, Chemistry, Mathematics**

$$B = \begin{bmatrix} 6650 & 7055 & 8905 \\ 7000 & 7500 & 10200 \end{bmatrix} \begin{matrix} \text{First Row Suresh,} \\ \text{Second Row Ganesh} \end{matrix}$$

**(i) Find the increase in sales in ₹ from July to August 2017.**

**(ii) If both book shops get 10% profit in the month of August 2017, find the profit for each book seller in each subject in that month.**

**Solution :** The sales for the July and August 2017 for Suresh and Ganesh are given by the matrices A and B as :

**July Sales (in ₹)**

$$A = \begin{matrix} \text{Physics} & \text{Chemistry} & \text{Mathematics} & \\ \end{matrix}$$
$$A = \begin{bmatrix} 5600 & 6750 & 8500 \\ 6650 & 7055 & 8905 \end{bmatrix} \begin{matrix} \text{Suresh} \\ \text{Ganesh} \end{matrix}$$

**August Sales (in ₹)**

Physics Chemistry Mathematics

$$B = \begin{bmatrix} 6650 & 7055 & 8905 \\ 7000 & 7500 & 10200 \end{bmatrix} \begin{matrix} \text{Suresh} \\ \text{Ganesh} \end{matrix}$$

(i) The increase in sales (in ₹) from July to August 2017 is obtained by subtracting the matrix A from B.

$$\text{Now, } B - A = \begin{bmatrix} 6650 & 7055 & 8905 \\ 7000 & 7500 & 10200 \end{bmatrix} - \begin{bmatrix} 5600 & 6750 & 8500 \\ 6650 & 7055 & 8905 \end{bmatrix}$$

$$= \begin{bmatrix} 6650-5600 & 7055-6750 & 8905-8500 \\ 7000-6650 & 7500-7055 & 10200-8905 \end{bmatrix}$$

Physics Chemistry Mathematics

$$= \begin{bmatrix} 1050 & 305 & 405 \\ 350 & 445 & 1295 \end{bmatrix} \begin{matrix} \text{Suresh} \\ \text{Ganesh} \end{matrix}$$

Hence, the increase in sales (in ₹) from July to August 2017 for :

Suresh book shop : ₹ 1050 in Physics, ₹ 305 in Chemistry and ₹ 405 in Mathematics.

Ganesh book shop : ₹ 350 in Physics, ₹ 445 in Chemistry and ₹ 1295 in Mathematics.

(ii) Both the book shops get 10% profit in August 2017, the profit for each book seller in each subject in August 2017 is obtained by the scalar multiplication of matrix B by 10%, i.e. $\frac{10}{100} = \frac{1}{10}$.

$$\text{Now, } \frac{1}{10}B = \frac{1}{10}\begin{bmatrix} 6650 & 7055 & 8905 \\ 7000 & 7500 & 10200 \end{bmatrix}$$

Physics Chemistry Mathematics

$$= \begin{bmatrix} 665 & 705.5 & 890.5 \\ 700 & 750 & 1020 \end{bmatrix} \begin{matrix} \text{Suresh} \\ \text{Ganesh} \end{matrix}$$

Hence, the profit for Suresh book shop are ₹ 665 in Physics, ₹ 705.50 in Chemistry and ₹ 890.50 in Mathematics and for Ganesh book shop are ₹ 700 in Physics, ₹ 750 in Chemistry and ₹ 1020 in Mathematics.

## EXAMPLES FOR PRACTICE 2.2

1. If $A = \begin{bmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -3 & 0 \\ 3 & 3 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & -1 & 4 \\ 2 & 1 & 0 \end{bmatrix}$, verify the following :

   (i) $(A+B)+C = A+(B+C)$
   (ii) $4(A+B) = 4A+4B$
   (iii) $5(B-C) = 5B-5C$
   (iv) $(A+B)' = A'+B'$.

2. (i) If $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 \\ -2 & 0 \end{bmatrix}$, find the matrix $3A - 2B$.

   (ii) If $A = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 4 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 & 3 \\ -2 & 5 & 0 \end{bmatrix}$, then find $2A + 3B$.

3. If $A = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}$, find (i) $A+B$ (ii) $B-A$ (iii) $A+B+C$ (iv) $(C-B)-A$.

4. If $A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & -1 & -3 \\ 1 & 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 0 & 5 \\ 6 & 9 & -1 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & 4 & 4 \\ 5 & -1 & 0 \\ 7 & 8 & 1 \end{bmatrix}$, then find

(i) $2A - B + C$ (ii) $A + C - 3B$ (iii) $2C - B + 3A$.

5. If $A = \text{diag}(2, \ -5, \ 9)$, $B = \text{diag}(-3, \ 7, \ -14)$ and $C = \text{diag}(1, 0, 3)$, find $B - A - C$.

6. (i) If $A = \begin{bmatrix} 3 & -5 \\ 4 & 2 \end{bmatrix}$, find the matrix X such that $A - 2X = 0$.

(ii) If $A = \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}$, find the matrix X such that $A - 3X = \begin{bmatrix} 3 & 5 \\ -8 & 2 \end{bmatrix}$.

**7. Find the matrix X, such that**

(i) $2A - B + X = 0$, where $A = \begin{bmatrix} 2 & 4 & 3 \\ -3 & -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 & 4 \\ 5 & -6 & 0 \end{bmatrix}$

(ii) $2A - 3B + X$ is a zero matrix, where $A = \begin{bmatrix} 1 & 2 & 2 \\ -3 & -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 & 4 \\ 5 & -6 & 0 \end{bmatrix}$

(iii) $2X + 3A - 4B = 0$, where $A = \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix}$

(iv) $2X + 3A - 2B = 0$, where $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$

(v) $2A - 3B + I = X$, where I is the unit matrix of order 2 and $A = \begin{bmatrix} 3 & 7 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 5 \\ 1 & -3 \end{bmatrix}$

(vi) $3A - 2B + 4X = 5C$, where $A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 7 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 6 & -1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -1 & 6 \\ 0 & 2 & -5 \end{bmatrix}$.

**8. Solve for X and Y, if**

(i) $X + Y = \begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix}$, $X - Y = \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$

(ii) $X - 2Y = \begin{bmatrix} -1 & 4 \\ -3 & 1 \end{bmatrix}$, $2X - Y = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}$

(iii) $2X + 3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}$, $3X + 2Y = \begin{bmatrix} 2 & -2 \\ -1 & 5 \end{bmatrix}$

(iv) $X + Y = \begin{bmatrix} 2 & -1 \\ 1 & 3 \\ -3 & -2 \end{bmatrix}$, $X - 2Y = \begin{bmatrix} -2 & 1 \\ 3 & -1 \\ 4 & -2 \end{bmatrix}$.

9. (i) Find $a, b, c$ if $\begin{bmatrix} 2 & 3 & a \\ b & -1 & 4 \\ -5 & c & 7 \end{bmatrix}$ is a symmetric matrix.

(ii) Find $a, b, c$ if $\begin{bmatrix} 2 & a & 3 \\ -7 & 4 & 5 \\ c & b & 6 \end{bmatrix}$ is a symmetric matrix.

(iii) Find $a, b, c, x, y, z$ if $\begin{bmatrix} a & x & y \\ -3 & b & z \\ 2 & -4 & c \end{bmatrix}$ is a skew-symmetric matrix.

**10. For the following matrices, find its transpose and state whether it is symmetric, skew-symmetric or neither :**

(i) $\begin{bmatrix} 2 & -3 & 1 \\ -3 & 4 & 5 \\ 1 & 5 & 6 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -1 & 4 \\ 1 & 6 & 2 \\ 4 & 5 & 3 \end{bmatrix}$

(iii) $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ (iv) $\begin{bmatrix} 0 & 3 & -4 \\ -3 & 0 & 6 \\ 4 & -6 & 0 \end{bmatrix}$

11. If $A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \\ 5 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 7 \\ 4 & 5 & 6 \end{bmatrix}$, find the matrix $(3A - B')$.

**12. Find the values of $x$ and $y$, if**

(i) $\begin{bmatrix} 1 & x & 0 \\ y & 2 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 2 \\ 4 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 6 & 5 & 2 \end{bmatrix}$

(ii) $2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$

(iii) $x\begin{bmatrix} 2 \\ 3 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \end{bmatrix}$

(iv) $2\begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$

(v) $\begin{bmatrix} 2x+1 & -1 \\ 3 & 4y \end{bmatrix} + \begin{bmatrix} -1 & 6 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 6 & 12 \end{bmatrix}$

**13.** If $\begin{bmatrix} x+y & y-z \\ z-2x & y-x \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$, find $x, y, z$.

**14.** Find the value of $x$, $y$, $z$ and $w$, if

$$2\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}.$$

### Answers

**2.** (i) $\begin{bmatrix} 0 & -7 \\ 13 & 12 \end{bmatrix}$ (ii) $\begin{bmatrix} 5 & 16 & 3 \\ 4 & 23 & 0 \end{bmatrix}$

**3.** (i) $\begin{bmatrix} 3 & 7 \\ 1 & 7 \end{bmatrix}$ (ii) $\begin{bmatrix} -1 & -1 \\ -5 & 3 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 12 \\ 4 & 11 \end{bmatrix}$

(iv) $\begin{bmatrix} -5 & -2 \\ 2 & -3 \end{bmatrix}$

**4.** (i) $\begin{bmatrix} 5 & 5 & 5 \\ 6 & -3 & -11 \\ 3 & -1 & 8 \end{bmatrix}$ (ii) $\begin{bmatrix} -3 & -3 & -3 \\ -2 & -2 & -18 \\ -10 & -19 & 6 \end{bmatrix}$

(iii) $\begin{bmatrix} 11 & 11 & 11 \\ 13 & -5 & -14 \\ 11 & 7 & 11 \end{bmatrix}$

**5.** $\begin{bmatrix} -6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & -26 \end{bmatrix}$

**6.** (i) $\begin{bmatrix} 3/2 & -5/2 \\ 2 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & -1 \\ 3 & 1 \end{bmatrix}$

**7.** (i) $\begin{bmatrix} -2 & -11 & -2 \\ 11 & -4 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 4 & -13 & 8 \\ 21 & -16 & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} -5/2 & 11/2 \\ -9 & -6 \end{bmatrix}$ (iv) $\begin{bmatrix} -4 & -5/2 \\ 3/2 & -5 \end{bmatrix}$

(v) $\begin{bmatrix} -5 & -1 \\ 1 & 18 \end{bmatrix}$ (vi) $\begin{bmatrix} \frac{1}{4} & -2 & \frac{37}{4} \\ -1 & \frac{1}{4} & -\frac{21}{2} \end{bmatrix}$

**8.** (i) $X = \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix}$, $Y = \begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix}$

(ii) $X = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$, $Y = \begin{bmatrix} 1 & -3 \\ 3 & 0 \end{bmatrix}$

(iii) $X = \begin{bmatrix} \frac{2}{5} & -\frac{12}{5} \\ -\frac{11}{5} & 3 \end{bmatrix}$, $Y = \begin{bmatrix} \frac{2}{5} & \frac{13}{5} \\ \frac{14}{5} & -2 \end{bmatrix}$

(iv) $X = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{5}{3} & \frac{5}{3} \\ -\frac{2}{3} & -2 \end{bmatrix}$, $Y = \begin{bmatrix} \frac{4}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{4}{3} \\ -\frac{7}{3} & 0 \end{bmatrix}$

**9.** (i) $a = -5, b = 3, c = 4$

(ii) $a = -7, b = 5, c = 3$

(iii) $a = 0, b = 0, c = 0, x = 3, y = -2, z = 4$.

**10.** (i) symmatric

(ii) neither symmetric nor skew-symmetric

(iii) skew-symmetric (iv) skew-symmetric.

**11.** $\begin{bmatrix} 1 & -4 \\ -9 & -2 \\ 8 & 6 \end{bmatrix}$

**12.** (i) $x = 1, y = 2$ (ii) $x = 2, y = 9$

(iii) $x = 3, y = -4$ (iv) $x = 3, y = 3$

(v) $x = 2, y = 3$.

**13.** $x = 1, y = 2, z = 3$

**14.** $x = 4, y = 10, z = 2, w = 3$.

## 7. Multiplication of Matrices :

### (a) Multiplication of a Row Matrix by a Column Matrix :

If A is a row matrix and B is a column matrix, then product A × B can be found if number of columns of A is equal to the number of rows of B.

Hence, if A is row matrix of order $1 \times n$ and B is a column matrix of order $n \times 1$, then we can find AB and order of AB is $1 \times 1$.

If $A = [a_{11} \ a_{12} \ a_{13}]$ and $B = \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix}$,

then $AB = [a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}]$

**For example :**

(i) $[2 \ 1 \ 5] \times \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = [2(3) + 1(-1) + 5(2)]$

$= [15]$

(ii) $[2 \ 1] \times \begin{bmatrix} 2 \\ -3 \end{bmatrix} = [4 - 3]$

$= [1]$

(iii) If $[x \ 2 \ \ 3] \times \begin{bmatrix} x \\ 5 \\ -3 \end{bmatrix} = [5]$, then

$[x^2 + 10 - 9] = [5]$

$\therefore \ [x^2 + 1] = [5]$

$\therefore \ x^2 + 1 = 5 \quad x^2 = 4 \quad \therefore \ x = \pm 2.$

### (b) Multiplication in general :

If the number of columns of a matrix A is equal to the number of rows of a matrix B, then only we can multiply the matrices A and B and obtain their product AB.

Thus if A is an $m \times n$ matrix and B is an $n \times p$ matrix, then we can multiply them and their product AB is an $m \times p$ matrix. The product AB is obtained by multiplying the rows of A by the corresponding columns of B.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$

Here, A is a $2 \times 3$ matrix and B is a $3 \times 2$ matrix. Hence, their product AB is $2 \times 2$ matrix.

Let us take two rows of A as $R_1$, $R_2$ and two columns of B as $C_1$, $C_2$. Hence, we can write

$A = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$, and $B = [C_1 \ \ C_2]$

$\therefore \ AB = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \times [C_1 \ \ C_2] = \begin{bmatrix} R_1C_1 & R_1C_2 \\ R_2C_1 & R_2C_2 \end{bmatrix}$

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix}$$

Also, B is a $3 \times 2$ matrix and A is a $2 \times 3$ matrix. Hence their product BA is a $3 \times 3$ matrix. In this case, we can write

$B = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}$ and $A = [C_1 \ \ C_2 \ \ C_3]$

$$\therefore \ BA = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} \times [C_1 \ \ C_2 \ \ C_3]$$

$$= \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \\ R_3C_1 & R_3C_2 & R_3C_3 \end{bmatrix}$$

$$BA = \begin{bmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{22} & b_{11}a_{13} + b_{12}a_{23} \\ b_{21}a_{11} + b_{22}a_{21} & b_{21}a_{12} + b_{22}a_{22} & b_{21}a_{13} + b_{22}a_{23} \\ b_{31}a_{11} + b_{32}a_{21} & b_{31}a_{12} + b_{32}a_{22} & b_{31}a_{13} + b_{32}a_{23} \end{bmatrix}$$

**For example :**

Let $A = \begin{bmatrix} 1 & -1 & 4 \\ 2 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 4 & 2 \\ 1 & -5 \end{bmatrix}$.

Then $R_1 = [1 \ \ -1 \ \ 4]$, $R_2 = [2 \ \ 3 \ \ 1]$

$C_1 = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$, $C_2 = \begin{bmatrix} 0 \\ 2 \\ -5 \end{bmatrix}$

$\therefore \ R_1C_1 = 1(2) + (-1)(4) + 4(1) = 2 - 4 + 4 = 2$

$R_1C_2 = 1(0) + (-1)(2) + 4(-5)$

$= 0 - 2 - 20 = -22$

$R_2C_1 = 2(2) + 3(4) + 1(1) = 4 + 12 + 1 = 17$

$R_2C_2 = 2(0) + 3(2) + 1(-5) = 0 + 6 - 5 = 1$

$\therefore \ AB = \begin{bmatrix} R_1C_1 & R_1C_2 \\ R_2C_1 & R_2C_2 \end{bmatrix} = \begin{bmatrix} 2 & -22 \\ 17 & 1 \end{bmatrix}$

$$BA = \begin{bmatrix} 2 & 0 \\ 4 & 2 \\ 1 & -5 \end{bmatrix} \times \begin{bmatrix} 1 & -1 & 4 \\ 2 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2(1)+0(2) & 2(-1)+0(3) & 2(4)+0(1) \\ 4(1)+2(2) & 4(-1)+2(3) & 4(4)+2(1) \\ 1(1)+(-5)(2) & 1(-1)+(-5)(3) & 1(4)+(-5)(1) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & 8 \\ 8 & 2 & 18 \\ -9 & -16 & -1 \end{bmatrix}$$

**Note :** In the above example, we observe that AB is a $2 \times 2$ matrix and BA is a $3 \times 3$ matrix, i.e. their orders are not equal. Hence, $AB \neq BA$. This is true in general i.e. matrix multiplication is not commutative. Even when AB and BA have the same order, they may not be equal in general.

The definition for product AB is as follows :

**Definition :** If $A = [a_{ij}]$ is a matrix of order $m \times n$ and $B = [b_{jk}]$ is a matrix of order $n \times p$, then we define $C = AB$ as a matrix of order $m \times p$ by $C = [c_{ik}]$ where $c_{ik}$ is obtained by multiplying the $i^{th}$ row of A and $k^{th}$ column of B.

$\therefore \ c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + a_{i3}b_{3k} + \ldots + a_{in}b_{nk}$

$= \sum_{j=1}^{n} a_{ij} \cdot b_{jk}.$

In the product AB, A is called **pre-factor** and B is called **post-factor.**

## 2.4 : PROPERTIES OF MATRIX MULTIPLICATION

**1.** For matrices A and B, matrix multiplication is not commutative.

i.e. $AB \neq BA$ (in general)

**2.** For three matrices A, B, C, matrix multiplication is associative.

i.e. $A(BC) = (AB)C$ if all the products exist.

**For example :**

Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}$.

Then

$$AB = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2-1 & 4+0 \\ 1-2 & 2+0 \\ 1+1 & 2-0 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -1 & 2 \\ 2 & 2 \end{bmatrix}$$

$$\therefore \ (AB)C = \begin{bmatrix} 1 & 4 \\ -1 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+8 & -3+4 \\ -1+4 & 3+2 \\ 2+4 & -6+2 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 1 \\ 3 & 5 \\ 6 & -4 \end{bmatrix} \quad \ldots (1)$$

$$BC = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4 & -3+2 \\ -1+0 & 3+0 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ -1 & 3 \end{bmatrix}$$

$$\therefore \ A(BC) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 10-1 & -2+3 \\ 5-2 & -1+6 \\ 5+1 & -1-3 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 1 \\ 3 & 5 \\ 6 & -4 \end{bmatrix} \quad \ldots (2)$$

From (1) and (2), we get

$(AB)C = A(BC)$.

**3.** For three matrices A, B, C, matrix multiplication is distributive over addition.

i.e. $A(B+C) = AB + AC$, $(A+B)C = AC + BC$.

**4.** For given square matrix A,

$A \cdot I = I \cdot A = A$, where I is an identity matrix of the same order as that of A.

**For example :**

Let $A = \begin{bmatrix} -2 & 3 \\ 4 & 1 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\text{Then } A \cdot I = \begin{bmatrix} -2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2+0 & 0+3 \\ 4+0 & 0+1 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 4 & 1 \end{bmatrix} = A$$

$$\text{and } I \cdot A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2+0 & 3+0 \\ 0+4 & 0+1 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 4 & 1 \end{bmatrix} = A$$

$\therefore \ A \cdot I = I \cdot A = A$.

**5.** For any matrix A, there exists a null matrix O such that $A \cdot O = O \cdot A = O$.

**6.** If A and B are two matrices such that the product $AB = 0$, then it need not imply that $A = 0$ or $B = 0$. Such matrices A and B are called **Non-zero divisors of zero.**

**For example :**

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}$,

$$\text{then } AB = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-2 & -2+2 \\ 6-6 & -6+6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus the product AB is zero matrix but neither A nor B is zero matrix.

**7.** In number theory, if $a$, $b$, $c$ are three numbers such that $ab = ac$, then either $a = 0$ or $b = c$. Matrix do not always satisfy this condition, i.e. if A, B, C are three matrices such that $AB = AC$, then it need not imply that $A = O$ or $B = C$.

For example : If $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \end{bmatrix}$,

$B = \begin{bmatrix} -2 & 3 \\ 1 & -5 \\ 4 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 4 & 1 \\ -4 & -3 \\ 3 & 1 \end{bmatrix}$, then

$$AB = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 1 & -5 \\ 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2+1+4 & 3-5+1 \\ -6+3+12 & 9-15+3 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix}$$

and $AC = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ -4 & -3 \\ 3 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 4-4+3 & 1-3+1 \\ 12-12+9 & 3-9+3 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix}$$

Thus, $AB = AC$ but neither A is a zero matrix nor $B = C$.

8. Positive integer powers of square matrix A are obtained by repeated multiplication of A by itself.
Thus, $A^2 = A \cdot A$, $A^3 = A \cdot A \cdot A$, ...,
$A^n = A \cdot A \cdot A$ ... $n$ times.

**Some useful results :**

If A and B are square matrices of same order, then

(1) $(A+B)^2 = (A+B)(A+B)$
$= A(A+B) + B(A+B)$
$= A^2 + AB + BA + B^2$

Note that : $(A+B)^2 \neq A^2 + 2AB + B^2$ unless $AB = BA$

(2) $(A-B)^2 = (A-B)(A-B)$
$= A(A-B) - B(A-B)$
$= A^2 - AB - BA + B^2$

Note that : $(A-B)^2 \neq A^2 - 2AB + B^2$ unless $AB = BA$.

(3) $(A+B)(A-B) = A(A-B) + B(A-B)$
$= A^2 - AB + BA - B^2$

Note that : $(A+B)(A-B) \neq A^2 - B^2$ unless $AB = BA$.

## EXERCISE 2.3 — Textbook pages 55 and 56

**1. Evaluate : (i)** $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} [2 \ \ -4 \ \ 3]$ **(ii)** $[2 \ \ -1 \ \ 3] \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$.

**Solution :**

(i) $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} [2 \ \ -4 \ \ 3] = \begin{bmatrix} 6 & -12 & 9 \\ 4 & -8 & 6 \\ 2 & -4 & 3 \end{bmatrix}$.

(ii) $[2 \ \ -1 \ \ 3] \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} = [8-3+3] = [8]$.

**2. If** $A = \begin{bmatrix} -1 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & -3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 0 & 2 \\ 1 & 2 & 1 \end{bmatrix}$.

**State whether AB = BA? Justify your answer.**

**Solution :**

$$AB = \begin{bmatrix} -1 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 4 \\ 3 & 0 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2+3+1 & -1+0+2 & -4+2+1 \\ 4+9+0 & 2+0+0 & 8+6+0 \\ 2-9+1 & 1-0+2 & 4-6+1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & -1 \\ 13 & 2 & 14 \\ -6 & 3 & -1 \end{bmatrix} \quad \ldots (1)$$

$$BA = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 0 & 2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2+2+4 & 2+3-12 & 2+0+4 \\ -3+0+2 & 3+0-6 & 3+0+2 \\ -1+4+1 & 1+6-3 & 1+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -7 & 6 \\ -1 & -3 & 5 \\ 4 & 4 & 2 \end{bmatrix} \quad \ldots (2)$$

From (1) and (2), $AB \neq BA$.

**3. Show that AB = BA, where** $A = \begin{bmatrix} -2 & 3 & -1 \\ -1 & 2 & -1 \\ -6 & 9 & -4 \end{bmatrix}$,

$B = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & -1 \\ 3 & 0 & -1 \end{bmatrix}$.

**Solution :**

$$AB = \begin{bmatrix} -2 & 3 & -1 \\ -1 & 2 & -1 \\ -6 & 9 & -4 \end{bmatrix} \begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & -1 \\ 3 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -2+6-3 & -6+6-0 & 2-3+1 \\ -1+4-3 & -3+4-0 & 1-2+1 \\ -6+18-12 & -18+18-0 & 6-9+4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \ldots (1)$$

$$BA = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & -1 \\ 3 & 0 & -1 \end{bmatrix} \begin{bmatrix} -2 & 3 & -1 \\ -1 & 2 & -1 \\ -6 & 9 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} -2-3+6 & 3+6-9 & -1-3+4 \\ -4-2+6 & 6+4-9 & -2-2+4 \\ -6-0+6 & 9+0-9 & -3-0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \ldots (2)$$

From (1) and (2), $AB = BA$.

**4. Verify $A(BC) = (AB)C$, if $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 0 \\ 0 & 4 & 5 \end{bmatrix}$,**

**$B = \begin{bmatrix} 2 & -2 \\ -1 & 1 \\ 0 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 0 & -2 \end{bmatrix}$.**

**Solution :**

$$BC = \begin{bmatrix} 2 & -2 \\ -1 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 & -1 \\ 2 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 6-4 & 4-0 & -2+4 \\ -3+2 & -2+0 & 1-2 \\ 0+6 & 0+0 & 0-6 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 2 \\ -1 & -2 & -1 \\ 6 & 0 & -6 \end{bmatrix}$$

$$\therefore \ A(BC) = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 0 \\ 0 & 4 & 5 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ -1 & -2 & -1 \\ 6 & 0 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 2-0+6 & 4-0+0 & 2-0-6 \\ 4-3+0 & 8-6+0 & 4-3-0 \\ 0-4+30 & 0-8+0 & 0-4-30 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 4 & -4 \\ 1 & 2 & 1 \\ 26 & -8 & -34 \end{bmatrix} \quad \ldots (1)$$

$$\text{Also, } AB = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 0 \\ 0 & 4 & 5 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -1 & 1 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2-0+0 & -2+0+3 \\ 4-3+0 & -4+3+0 \\ 0-4+0 & 0+4+15 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ -4 & 19 \end{bmatrix}$$

$$\therefore \ (AB)C = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ -4 & 19 \end{bmatrix} \begin{bmatrix} 3 & 2 & -1 \\ 2 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 6+2 & 4+0 & -2-2 \\ 3-2 & 2-0 & -1+2 \\ -12+38 & -8+0 & 4-38 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 4 & -4 \\ 1 & 2 & 1 \\ 26 & -8 & -34 \end{bmatrix} \quad \ldots (2)$$

From (1) and (2), $A(BC) = (AB)C$.

**5. Verify that $A(B+C) = AB + AC$, if**

**$A = \begin{bmatrix} 4 & -2 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & 1 \\ 2 & -1 \end{bmatrix}$.**

**Solution :**

$$B + C = \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix} + \begin{bmatrix} 4 & 1 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1+4 & 1+1 \\ 3+2 & -2-1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 5 & -3 \end{bmatrix}$$

$$\therefore \ A(B+C) = \begin{bmatrix} 4 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 5 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 12-10 & 8+6 \\ 6+15 & 4-9 \end{bmatrix} = \begin{bmatrix} 2 & 14 \\ 21 & -5 \end{bmatrix} \quad \ldots (1)$$

$$\text{Also, } AB = \begin{bmatrix} 4 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -4-6 & 4+4 \\ -2+9 & 2-6 \end{bmatrix} = \begin{bmatrix} -10 & 8 \\ 7 & -4 \end{bmatrix}$$

$$AC = \begin{bmatrix} 4 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 16-4 & 4+2 \\ 8+6 & 2-3 \end{bmatrix} = \begin{bmatrix} 12 & 6 \\ 14 & -1 \end{bmatrix}$$

$$\therefore \ AB + AC = \begin{bmatrix} -10 & 8 \\ 7 & -4 \end{bmatrix} + \begin{bmatrix} 12 & 6 \\ 14 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -10+12 & 8+6 \\ 7+14 & -4-1 \end{bmatrix} = \begin{bmatrix} 2 & 14 \\ 21 & -5 \end{bmatrix} \ldots (2)$$

From (1) and (2), $A(B+C) = AB + AC$.

**6. If $A = \begin{bmatrix} 4 & 3 & 2 \\ -1 & 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 1 & -2 \end{bmatrix}$, show that matrix AB is non-singular.**

**Solution :**

$$AB = \begin{bmatrix} 4 & 3 & 2 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 4-3+2 & 8+0-4 \\ -1-2+0 & -2+0-0 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -3 & -2 \end{bmatrix}$$

$$\therefore \ |AB| = \begin{vmatrix} 3 & 4 \\ -3 & -2 \end{vmatrix}$$

$$= -6 - (-12) = 6 \neq 0$$

Hence, AB is a non-singular matrix.

**7. If $A + I = \begin{bmatrix} 1 & 2 & 0 \\ 5 & 4 & 2 \\ 0 & 7 & -3 \end{bmatrix}$, find the product $(A+I)(A-I)$.**

**Solution :**

$A - I = (A + I) - 2I$

$$= \begin{bmatrix} 1 & 2 & 0 \\ 5 & 4 & 2 \\ 0 & 7 & -3 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 0 \\ 5 & 4 & 2 \\ 0 & 7 & -3 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 2 & 0 \\ 5 & 2 & 2 \\ 0 & 7 & -5 \end{bmatrix}$$

$$\therefore\ (A + I)(A - I) = \begin{bmatrix} 1 & 2 & 0 \\ 5 & 4 & 2 \\ 0 & 7 & -3 \end{bmatrix}\begin{bmatrix} -1 & 2 & 0 \\ 5 & 2 & 2 \\ 0 & 7 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} -1+10+0 & 2+4+0 & 0+4-0 \\ -5+20+0 & 10+8+14 & 0+8-10 \\ 0+35-0 & 0+14-21 & 0+14+15 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 6 & 4 \\ 15 & 32 & -2 \\ 35 & -7 & 29 \end{bmatrix}.$$

**8. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, show that $A^2 - 4A$ is a scalar matrix.**

**Solution :**

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$\therefore\ A^2 - 4A = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix},$$

which is a scalar matrix.

**9. If $A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$, find $k$ so that $A^2 - 8A - kI = 0$, where I is a $2 \times 2$ unit matrix and 0 is null matrix of order 2.**

**Solution :**

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1-0 & 0+0 \\ -1-7 & 0+49 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix}$$

$$\therefore\ A^2 - 8A - kI = \begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix} - 8\begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix} - k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix} - \begin{bmatrix} 8 & 0 \\ -8 & 56 \end{bmatrix} - \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

$$= \begin{bmatrix} 1-8-k & 0-0-0 \\ -8+8-0 & 49-56-k \end{bmatrix}$$

$$= \begin{bmatrix} -k-7 & 0 \\ 0 & -k-7 \end{bmatrix}$$

But, $A^2 - 8A - kI = 0$

$$\therefore\ \begin{bmatrix} -k-7 & 0 \\ 0 & -k-7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

By equality of matrices,

$-k - 7 = 0 \quad \therefore\ k = -7.$

**10. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, prove that $A^2 - 5A + 7I = 0$, where I is a $2 \times 2$ unit matrix.**

**Solution :**

$$A^2 = A \cdot A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

$$\therefore\ A^2 - 5A + 7I = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore\ A^2 - 5A + 7I = 0.$





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**11. If $A = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & a \\ -1 & b \end{bmatrix}$ and if $(A+B)^2 = A^2 + B^2$, find values of $a$ and $b$.**

**Solution :** $(A+B)^2 = A^2 + B^2$

$\therefore\ (A+B)(A+B) = A^2 + B^2$

$\therefore\ A^2 + AB + BA + B^2 = A^2 + B^2$

$\therefore\ AB + BA = 0$

$\therefore\ AB = -BA$

$$\therefore \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 2 & a \\ -1 & b \end{bmatrix} = -\begin{bmatrix} 2 & a \\ -1 & b \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 2-2 & a+2b \\ -2+2 & -a-2b \end{bmatrix} = -\begin{bmatrix} 2-a & 4-2a \\ -1-b & -2-2b \end{bmatrix}$$

$$\therefore \begin{bmatrix} 0 & a+2b \\ 0 & -a-2b \end{bmatrix} = \begin{bmatrix} a-2 & 2a-4 \\ 1+b & 2+2b \end{bmatrix}$$

By the equality of matrices, we get

$0 = a - 2$ ... (1)

$0 = 1 + b$ ... (2)

$a + 2b = 2a - 4$ ... (3)

$-a - 2b = 2 + 2b$ ... (4)

From equations (1) and (2), we get

$a = 2$ and $b = -1$

The values of $a$ and $b$ satisfy equations (3) and (4) also.

Hence, $a = 2$ and $b = -1$.

**12. Find $k$, if $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$ and $A^2 = kA - 2I$.**

**Solution :**

$$A^2 = A \cdot A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 9-8 & -6+4 \\ 12-8 & -8+4 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix}$$

$$kA - 2I = k\begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3k & -2k \\ 4k & -2k \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3k-2 & -2k \\ 4k & -2k-2 \end{bmatrix}$$

But, $A^2 = kA - 2I$

$$\therefore \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix} = \begin{bmatrix} 3k-2 & -2k \\ 4k & -2k-2 \end{bmatrix}$$

By equality of matrices,

$1 = 3k - 2$ ... (1)

$-2 = -2k$ ... (2)

$4 = 4k$ ... (3)

$-4 = -2k - 2$ ... (4)

From (2), $k = 1$.

$k = 1$ also satisfies equation (1), (3) and (4).

Hence, $k = 1$.

**13. Find $x$ and $y$, if**

$$\left\{4\begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 3 & -3 & 4 \\ 2 & 1 & 1 \end{bmatrix}\right\}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

**Solution :**

$$\left\{4\begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 3 & -3 & 4 \\ 2 & 1 & 1 \end{bmatrix}\right\}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\therefore \left\{\begin{bmatrix} 8 & -4 & 12 \\ 4 & 0 & 8 \end{bmatrix} - \begin{bmatrix} 3 & -3 & 4 \\ 2 & 1 & 1 \end{bmatrix}\right\}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\therefore \begin{bmatrix} 5 & -1 & 8 \\ 2 & -1 & 7 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\therefore \begin{bmatrix} 10+1+8 \\ 4+1+7 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\therefore \begin{bmatrix} 19 \\ 12 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

By equality of matrices,

$x = 19$ and $y = 12$.

**14. Find $x, y, z$, if**

$$\left\{3\begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 2 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 3 & 1 \end{bmatrix}\right\}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} x-3 \\ y-1 \\ 2z \end{bmatrix}$$

**Solution :**

$$\left\{3\begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 2 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 3 & 1 \end{bmatrix}\right\}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} x-3 \\ y-1 \\ 2z \end{bmatrix}$$

$$\therefore \left\{\begin{bmatrix} 6 & 0 \\ 0 & 6 \\ 6 & 6 \end{bmatrix} - \begin{bmatrix} 4 & 4 \\ -4 & 8 \\ 12 & 4 \end{bmatrix}\right\}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} x-3 \\ y-1 \\ 2z \end{bmatrix}$$

$$\therefore \begin{bmatrix} 2 & -4 \\ 4 & -2 \\ -6 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} x-3 \\ y-1 \\ 2z \end{bmatrix}$$

$$\therefore \begin{bmatrix} 2-8 \\ 4-4 \\ -6+4 \end{bmatrix} = \begin{bmatrix} x-3 \\ y-1 \\ 2z \end{bmatrix}$$

$$\therefore \begin{bmatrix} -6 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} x-3 \\ y-1 \\ 2z \end{bmatrix}$$

By equality of matrices,

$-6 = x-3$, $0 = y-1$ and $-2 = 2z$

$\therefore$ $x = -3$, $y = 1$ and $z = -1$.

**15. Jay and Ram are two friends. Jay wants to buy 4 pens and 8 notebooks. Ram wants to buy 5 pens and 12 notebooks. The price of one pen and one notebook was ₹ 6 and ₹ 10 respectively. Using matrix multiplication, find the amount each one of them requires for buying the pens and notebooks.**

**Solution :** The given data can be written in matrix form as :

**Number of Pens and Notebooks**

$$A = \begin{array}{c} \text{Pens} \quad \text{Notebooks} \\ \begin{bmatrix} 4 & 8 \\ 5 & 12 \end{bmatrix} \begin{array}{l} \text{Jay} \\ \text{Ram} \end{array} \end{array}$$

**Price in ₹**

$$B = \begin{bmatrix} 6 \\ 10 \end{bmatrix} \begin{array}{l} \text{Pen} \\ \text{Notebook} \end{array}$$

For finding the amount each one of them requires to buy the pens and notebook, we require the multiplication of the two matrices A and B.

$$\text{Consider } AB = \begin{bmatrix} 4 & 8 \\ 5 & 12 \end{bmatrix} \begin{bmatrix} 6 \\ 10 \end{bmatrix}$$

$$= \begin{bmatrix} 24+80 \\ 30+120 \end{bmatrix} = \begin{bmatrix} 104 \\ 150 \end{bmatrix}$$

Hence, Jay requires ₹ 104 and Ram requires ₹ 150 to buy the pens and notebooks.

## EXAMPLES FOR PRACTICE 2.3

**1. Find AB and BA whenever they exist in each of following cases :**

(i) $A = \begin{bmatrix} 1 & 3 \\ -1 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 6 & -6 \\ -2 & 2 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 0 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 3 & -1 & 2 \\ 4 & -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -2 \\ 1 & 4 \\ 3 & 6 \end{bmatrix}$

(iv) $A = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 & 3 \\ -2 & -3 & 1 \end{bmatrix}$.

**2.** If $A = \begin{bmatrix} 3 & 4 & 3 \\ 1 & 0 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 5 & -1 & 2 \\ 0 & 7 & 3 \end{bmatrix}$, $C = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$,

then find the matrix X such that $X = (3A - B)C$.

**3.** If $A = \begin{bmatrix} -2 & 4 \\ 3 & 2 \\ 1 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -3 & 6 \\ 1 & 0 & 5 \end{bmatrix}$, find AB and without finding BA, show that $AB \neq BA$.

**4. Verify that :**

(i) $A(BC) = (AB)C$, where

$A = \begin{bmatrix} 1 & 3 & 2 \\ -2 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 \\ -2 & -1 \\ 0 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$.

(ii) $A(B+C) = AB + AC$, where

$A = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 12 & -3 & 2 \\ 4 & 6 & 8 \end{bmatrix}$,

$C = \begin{bmatrix} 0 & -2 & 4 \\ 8 & 7 & 5 \end{bmatrix}$.

(iii) $A(B-C) = AB - AC$, where

$A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 & 1 \\ 6 & 2 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 & -4 \\ 0 & 6 & 5 \end{bmatrix}$.

**5.** If $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$, prove that $|AB| = |A||B|$.

**6.** If $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then find $|AB|$.

**7.** Show that AB is a singular matrix, where

$A = \begin{bmatrix} 3 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ -4 & 2 \\ 1 & 0 \end{bmatrix}$.

**8.** Show that AB is a singular matrix but BA is non-singular, where

$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 & 2 \\ 4 & -1 & -3 \end{bmatrix}$.

**9. Show that**

(i) $A^2 - 2A$ is a scalar matrix, if $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

(ii) $A^2 - 7A$ is a scalar matrix, if $A = \begin{bmatrix} 1 & 5 & 5 \\ 5 & 1 & 5 \\ 5 & 5 & 1 \end{bmatrix}$.

(iii) $A^2 - 5A$ is a scalar matrix, if $A = \begin{bmatrix} 1 & 3 & 3 \\ 3 & 1 & 3 \\ 3 & 3 & 1 \end{bmatrix}$.

**10.** If $A = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}$, show that $A^2$ is a null matrix.

**11.** If $A = \begin{bmatrix} 5 & 2 \\ -1 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, then find the matrix X and Y such that $XA = B$, $AY = B$.

**12.** (i) If $A = \begin{bmatrix} 1 & 3 \\ 0 & 3 \end{bmatrix}$, show that $A^2 - 4A + 3I = 0$.

(ii) If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, show that $A^2 - 5A - 2I = 0$.

(iii) If $A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = 0$.

(iv) If $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, show that $A^2 - 3A + I = 0$.

**13.** If $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$ and $A^2 - 7A + kI = 0$, find $k$.

**14.** If $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & -3 \\ 1 & -1 & 0 \end{bmatrix}$, find $A^2 - 5A + 6I$.

**15.** If $A = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$, find a matrix X such that $AX = I_2$.

**16.** If $A + I = \begin{bmatrix} 1 & 3 & 4 \\ -1 & 1 & 3 \\ -2 & -3 & 1 \end{bmatrix}$, find the matrix $(A + I)(A - I)$.

[Hint : $A - I = (A + I) - 2I$.]

**17.** If $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, compute the matrix $A^2 + 2A$.

**18.** If $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, find $A^2 - 7A$.

**19.** If $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$, find $A^2 - 3A$.

**20.** If $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, prove that $A^2 = 2A$.

**21.** (i) If $A = \begin{bmatrix} 2 & a \\ 2 & b \end{bmatrix}$, $B = \begin{bmatrix} 5 & -5 \\ -2 & 2 \end{bmatrix}$ such that $(A - B)^2 = A^2 + B^2$, find $a$ and $b$.

(ii) If $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & x \\ 4 & y \end{bmatrix}$ such that $(A + B)^2 = A^2 + B^2$, find $x$ and $y$.

(iii) If $A = \begin{bmatrix} 2 & 6 \\ 3 & -4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & x \\ y & 2 \end{bmatrix}$ such that $(A + B)(A - B) = A^2 - B^2$, find $x$ and $y$.

(iv) If $A = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 3 & -1 \end{bmatrix}$ such that $(A + B)^2 = A^2 + B^2$, find $a$ and $b$.

(v) If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & x \\ y & 0 \end{bmatrix}$ and $(A + B)(A - B) = A^2 - B^2$, find $x$ and $y$.

(vi) If $A = \begin{bmatrix} -3 & 2 \\ 2 & -4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & x \\ y & 0 \end{bmatrix}$ and $(A + B)(A - B) = A^2 - B^2$, then find $x$ and $y$.

**22. Find $x$, $y$, if**

(i) $\left\{4\begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 & -1 \\ 2 & -3 & 4 \end{bmatrix}\right\}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

(ii) $\left\{4\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 3 \end{bmatrix} - 2\begin{bmatrix} 1 & 3 & 1 \\ 2 & 3 & 8 \end{bmatrix}\right\}\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

(iii) $\left\{3\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 5 & -2 \\ -3 & -4 & 4 \end{bmatrix}\right\}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

**23. Find $x$, $y$, $z$, if**

(i) $\left\{4\begin{bmatrix} 1 & 2 \\ 0 & 3 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ 1 & 4 \\ -2 & 4 \end{bmatrix}\right\}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

(ii) $\left\{\begin{bmatrix} 3 & 2 & 5 \\ 2 & 1 & 3 \\ 1 & 2 & 3 \end{bmatrix} - 2\begin{bmatrix} 1 & 2 & 5 \\ 1 & 6 & 1 \\ 2 & 3 & 1 \end{bmatrix}\right\}\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

(iii) $(5A - 3B)C = X$, where

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ -2 & 3 \\ 3 & 1 \end{bmatrix}, C = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

**24.** Find $x$, $y$, $z$, if $\begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} [1\ 2] \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

**25.** A fruit-stall has 10 dozen mangoes, 8 dozen apples and 10 dozen bananas. Their selling prices are ₹ 200, ₹ 160 and ₹ 30 per dozen respectively. Find the total amount which will be received by selling all the fruits. (Use matrix algebra.)

### Answers

**1.** (i) $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $BA = \begin{bmatrix} 12 & 36 \\ -4 & -12 \end{bmatrix}$

(ii) $AB = \begin{bmatrix} 14 & -2 & 5 \\ -1 & 3 & -2 \end{bmatrix}$, BA does not exist

(iii) $AB = \begin{bmatrix} 11 & 2 \\ 15 & 2 \end{bmatrix}$, $BA = \begin{bmatrix} -2 & 2 & -2 \\ 19 & -9 & 14 \\ 33 & -15 & 24 \end{bmatrix}$

(iv) $AB = \begin{bmatrix} -7 & -8 & 6 \\ -1 & -4 & -2 \end{bmatrix}$, BA does not exist.

**2.** $\begin{bmatrix} 29 \\ -29 \end{bmatrix}$

**3.** $AB = \begin{bmatrix} -4 & 6 & 8 \\ 14 & -9 & 28 \\ 10 & -3 & 36 \end{bmatrix}$, order of BA $\neq$ order of AB.

**6.** 0

**11.** $X = \begin{bmatrix} 1/32 & 5/32 \\ 1/32 & 5/32 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & 1/8 \\ 0 & 3/16 \end{bmatrix}$

**13.** $k = 10$ **14.** $\begin{bmatrix} 1 & -1 & -3 \\ -7 & 5 & 14 \\ -5 & 4 & 10 \end{bmatrix}$ **15.** $\begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$

**16.** $\begin{bmatrix} -12 & -12 & 9 \\ -6 & -13 & -4 \\ 3 & -6 & -18 \end{bmatrix}$ **17.** $\begin{bmatrix} 17 & 12 \\ 6 & 5 \end{bmatrix}$

**18.** $\begin{bmatrix} -10 & -3 \\ 0 & -10 \end{bmatrix}$ **19.** $\begin{bmatrix} 0 & 2 \\ 1 & 2 \end{bmatrix}$

**21.** (i) $a = 5, b = 5$ (ii) $x = 1, y = -1$

(iii) $x = -3, y = -3/2$ (iv) $a = 1, b = 5$

(v) $x = -\frac{2}{3}, y = -1$ (vi) $x = 2, y = 2$.

**22.** (i) $x = 33, y = 5$ (ii) $x = 6, y = 12$ (iii) $x = 6, y = 10$.

**23.** (i) $x = 12, y = 16, z = 8$

(ii) $x = -9, y = -32, z = -17$

(iii) $x = -2, y = 8, z = -6$.

**24.** $x = 3, y = 2, z = 2$.

**25.** ₹ 3580.

### • Properties of Transpose of a Matrix :

**1.** If A and B are two matrices of same order, then $(A+B)^T = A^T + B^T$.

**2.** If A is a matrix and $k$ is a constant, then $(kA)^T = kA^T$.

**3.** If A and B are conformable for the product AB, then $(AB)^T = B^T A^T$.

**For example :**

Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & 5 \end{bmatrix}$, $B \begin{bmatrix} 3 & 1 \\ 2 & 0 \\ 5 & -2 \end{bmatrix}$.

Then AB is defined and

$$AB = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 0 \\ 5 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3-2+10 & 1-0-4 \\ 9+8+25 & 3+0-10 \end{bmatrix} = \begin{bmatrix} 11 & -3 \\ 42 & -7 \end{bmatrix}$$

$$\therefore (AB)^T = \begin{bmatrix} 11 & 42 \\ -3 & -7 \end{bmatrix} \quad \ldots (1)$$

Now, $A^T = \begin{bmatrix} 1 & 3 \\ -1 & 4 \\ 2 & 5 \end{bmatrix}$ and $B^T = \begin{bmatrix} 3 & 2 & 5 \\ 1 & 0 & -2 \end{bmatrix}$

$$\therefore B^T A^T = \begin{bmatrix} 3 & 2 & 5 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & 4 \\ 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3-2+10 & 9+8+25 \\ 1-0-4 & 3+0-10 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 42 \\ -3 & -7 \end{bmatrix} \quad \ldots (2)$$

From (1) and (2), $(AB)^T = B^T A^T$.

**Note :** In general,
$(A_1 \cdot A_2 \cdot A_3 \cdot \ldots \cdot A_n)^T = (A_n)^T \cdot (A_{n-1})^T \ldots (A_2)^T \cdot (A_1)^T$.

**4.** If A is symmetric, then $A^T = A$.

**5.** If A is skew-symmetric, then $A^T = -A$.

**6.** If A is a square matrix, then

(i) $A + A^T$ is symmetric

(ii) $A - A^T$ is skew-symmetric.

**For example :**

Let $A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 0 & 2 \\ 5 & 1 & 6 \end{bmatrix}$.

Then $A^T = \begin{bmatrix} 2 & 4 & 5 \\ 3 & 0 & 1 \\ -1 & 2 & 6 \end{bmatrix}$

$\therefore\ A + A^T = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 0 & 2 \\ 5 & 1 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 4 & 5 \\ 3 & 0 & 1 \\ -1 & 2 & 6 \end{bmatrix}$

$= \begin{bmatrix} 4 & 7 & 4 \\ 7 & 0 & 3 \\ 4 & 3 & 12 \end{bmatrix}$

This is a symmetric matrix (by definition).

Also, $A - A^T = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 0 & 2 \\ 5 & 1 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 4 & 5 \\ 3 & 0 & 1 \\ -1 & 2 & 6 \end{bmatrix}$

$= \begin{bmatrix} 0 & -1 & -6 \\ 1 & 0 & 1 \\ 6 & -1 & 0 \end{bmatrix}$

This is a skew-symmetric matrix (by definition).

**Remark :** A square matrix A can be expressed as the sum of a symmetric and a skew-symmetric matrix as follows :

$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$.

**EXERCISE 2.4** | **Textbook pages 59 and 60**

**1. Find $A^T$, if (i) $A = \begin{bmatrix} 1 & 3 \\ -4 & 5 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 2 & -6 & 1 \\ -4 & 0 & 5 \end{bmatrix}$.**

**Solution :**

(i) $A = \begin{bmatrix} 1 & 3 \\ -4 & 5 \end{bmatrix}$

$\therefore\ A^T = \begin{bmatrix} 1 & -4 \\ 3 & 5 \end{bmatrix}$.

(ii) $A = \begin{bmatrix} 2 & -6 & 1 \\ -4 & 0 & 5 \end{bmatrix}$

$\therefore\ A^T = \begin{bmatrix} 2 & -4 \\ -6 & 0 \\ 1 & 5 \end{bmatrix}$.

**2. If $A = [a_{ij}]_{3 \times 3}$ where $a_{ij} = 2(i - j)$. Find A and $A^T$. State whether A and $A^T$ both are symmetric or skew-symmetric matrices.**

**Solution :** $A = [a_{ij}]_{3 \times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Given : $a_{ij} = 2(i - j)$

$\therefore\ a_{11} = 2(1-1) = 0,\ a_{12} = 2(1-2) = -2,$

$a_{13} = 2(1-3) = -4,\ a_{21} = 2(2-1) = 2,$

$a_{22} = 2(2-2) = 0,\ a_{23} = 2(2-3) = -2,$

$a_{31} = 2(3-1) = 4,\ a_{32} = 2(3-2) = 2,$

$a_{33} = 2(3-3) = 0$

$\therefore\ A = \begin{bmatrix} 0 & -2 & -4 \\ 2 & 0 & -2 \\ 4 & 2 & 0 \end{bmatrix}$

$\therefore\ A^T = \begin{bmatrix} 0 & 2 & 4 \\ -2 & 0 & 2 \\ -4 & -2 & 0 \end{bmatrix}$

$\therefore\ -A^T = -\begin{bmatrix} 0 & 2 & 4 \\ -2 & 0 & 2 \\ -4 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 & -4 \\ 2 & 0 & -2 \\ 4 & 2 & 0 \end{bmatrix}$

$\therefore\ A = -A^T$ and $A^T = -A$

Hence, A and $A^T$ are both skew-symmetric matrices.

**3. If $A = \begin{bmatrix} 5 & -3 \\ 4 & -3 \\ -2 & 1 \end{bmatrix}$, prove that $(A^T)^T = A$.**

**Solution :** $A = \begin{bmatrix} 5 & -3 \\ 4 & -3 \\ -2 & 1 \end{bmatrix}$

$\therefore\ A^T = \begin{bmatrix} 5 & 4 & -2 \\ -3 & -3 & 1 \end{bmatrix}$

$\therefore\ (A^T)^T = \begin{bmatrix} 5 & -3 \\ 4 & -3 \\ -2 & 1 \end{bmatrix} = A.$

**4. If $A = \begin{bmatrix} 1 & 2 & -5 \\ 2 & -3 & 4 \\ -5 & 4 & 9 \end{bmatrix}$, prove that $A^T = A$.**

**Solution :** $A = \begin{bmatrix} 1 & 2 & -5 \\ 2 & -3 & 4 \\ -5 & 4 & 9 \end{bmatrix}$ ... (1)

$\therefore\ A^T = \begin{bmatrix} 1 & 2 & -5 \\ 2 & -3 & 4 \\ -5 & 4 & 9 \end{bmatrix}$ ... (2)

From (1) and (2), $A^T = A$.

**5. If $A = \begin{bmatrix} 2 & -3 \\ 5 & -4 \\ -6 & 1 \end{bmatrix}$, $B \begin{bmatrix} 2 & 1 \\ 4 & -1 \\ -3 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ -2 & 3 \end{bmatrix}$, then show that**

**(i) $(A + B)^T = A^T + B^T$ (ii) $(A - C)^T = A^T - C^T$.**

**Solution :**

(i) $A+B=\begin{bmatrix}2&-3\\5&-4\\-6&1\end{bmatrix}+\begin{bmatrix}2&1\\4&-1\\-3&3\end{bmatrix}$

$\begin{bmatrix}2+2&-3+1\\5+4&-4-1\\-6-3&1+3\end{bmatrix}=\begin{bmatrix}4&-2\\9&-5\\-9&4\end{bmatrix}$

$\therefore\ (A+B)^T=\begin{bmatrix}4&9&-9\\-2&-5&4\end{bmatrix}$ ... (1)

$A^T=\begin{bmatrix}2&5&-6\\-3&-4&1\end{bmatrix}$, $B^T=\begin{bmatrix}2&4&-3\\1&-1&3\end{bmatrix}$

$\therefore\ A^T+B^T=\begin{bmatrix}2&5&-6\\-3&-4&1\end{bmatrix}+\begin{bmatrix}2&4&-3\\1&-1&3\end{bmatrix}$

$=\begin{bmatrix}2+2&5+4&-6-3\\-3+1&-4-1&1+3\end{bmatrix}$

$=\begin{bmatrix}4&9&-9\\-2&-5&4\end{bmatrix}$ ... (2)

From (1) and (2),

$(A+B)^T=A^T+B^T$.

(ii) $A-C=\begin{bmatrix}2&-3\\5&-4\\-6&1\end{bmatrix}-\begin{bmatrix}1&2\\-1&4\\-2&3\end{bmatrix}$

$=\begin{bmatrix}2-1&-3-2\\5-(-1)&-4-4\\-6-(-2)&1-3\end{bmatrix}=\begin{bmatrix}1&-5\\6&-8\\-4&-2\end{bmatrix}$

$\therefore\ (A-C)^T=\begin{bmatrix}1&6&-4\\-5&-8&-2\end{bmatrix}$ ... (1)

$A^T=\begin{bmatrix}2&5&-6\\-3&-4&1\end{bmatrix}$, $C^T=\begin{bmatrix}1&-1&-2\\2&4&3\end{bmatrix}$

$\therefore\ A^T-C^T=\begin{bmatrix}2&5&-6\\-3&-4&1\end{bmatrix}-\begin{bmatrix}1&-1&-2\\2&4&3\end{bmatrix}$

$=\begin{bmatrix}2-1&5-(-1)&-6-(-2)\\-3-2&-4-4&1-3\end{bmatrix}$

$=\begin{bmatrix}1&6&-4\\-5&-8&-2\end{bmatrix}$ ... (2)

From (1) and (2),

$(A-C)^T=A^T-C^T$.

**6. If $A=\begin{bmatrix}5&4\\-2&3\end{bmatrix}$ and $B=\begin{bmatrix}-1&3\\4&-1\end{bmatrix}$, then find $C^T$, such that $3A-2B+C=I$, where I is the unit matrix of order 2.**

**Solution :** $3A-2B+C=I$

$\therefore\ C=I-3A+2B$

$=\begin{bmatrix}1&0\\0&1\end{bmatrix}-3\begin{bmatrix}5&4\\-2&3\end{bmatrix}+2\begin{bmatrix}-1&3\\4&-1\end{bmatrix}$

$=\begin{bmatrix}1&0\\0&1\end{bmatrix}-\begin{bmatrix}15&12\\-6&9\end{bmatrix}+\begin{bmatrix}-2&6\\8&-2\end{bmatrix}$

$=\begin{bmatrix}1-15+(-2)&0-12+6\\0-(-6)+8&1-9-2\end{bmatrix}$

$\therefore\ C=\begin{bmatrix}-16&-6\\14&-10\end{bmatrix}$

$\therefore\ C^T=\begin{bmatrix}-16&14\\-6&-10\end{bmatrix}$.

**7. If $A=\begin{bmatrix}7&3&0\\0&4&-2\end{bmatrix}$, $B=\begin{bmatrix}0&-2&3\\2&1&-4\end{bmatrix}$, then find (i) $A^T+4B^T$ (ii) $5A^T-5B^T$.**

**Solution :**

$A=\begin{bmatrix}7&3&0\\0&4&-2\end{bmatrix}$, $B=\begin{bmatrix}0&-2&3\\2&1&-4\end{bmatrix}$

$\therefore\ A^T=\begin{bmatrix}7&0\\3&4\\0&-2\end{bmatrix}$, $B=\begin{bmatrix}0&2\\-2&1\\3&-4\end{bmatrix}$

(i) $A^T+4B^T=\begin{bmatrix}7&0\\3&4\\0&-2\end{bmatrix}+4\begin{bmatrix}0&2\\-2&1\\3&-4\end{bmatrix}$

$=\begin{bmatrix}7&0\\3&4\\0&-2\end{bmatrix}+\begin{bmatrix}0&8\\-8&4\\12&-16\end{bmatrix}$

$=\begin{bmatrix}7+0&0+8\\3-8&4+4\\0+12&-2-16\end{bmatrix}=\begin{bmatrix}7&8\\-5&8\\12&-18\end{bmatrix}$.

(ii) $5A^T-5B^T=5\begin{bmatrix}7&0\\3&4\\0&-2\end{bmatrix}-5\begin{bmatrix}0&2\\-2&1\\3&-4\end{bmatrix}$

$=\begin{bmatrix}35&0\\15&20\\0&-10\end{bmatrix}-\begin{bmatrix}0&10\\-10&5\\15&-20\end{bmatrix}$

$=\begin{bmatrix}35-0&0-10\\15-(-10)&20-5\\0-15&-10-(-20)\end{bmatrix}$

$=\begin{bmatrix}35&-10\\25&15\\-15&10\end{bmatrix}$.

**8. If $A = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & -4 \\ 3 & 5 & -2 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 2 & 3 \\ -1 & -1 & 0 \end{bmatrix}$, verify that $(A + 2B + 3C)^T = A^T + 2B^T + 3C^T$.**

**Solution :**

$A + 2B + 3C$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix} + 2\begin{bmatrix} 2 & 1 & -4 \\ 3 & 5 & -2 \end{bmatrix} + 3\begin{bmatrix} 0 & 2 & 3 \\ -1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 2 & -8 \\ 6 & 10 & -4 \end{bmatrix} + \begin{bmatrix} 0 & 6 & 9 \\ -3 & -3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+0 & 0+2+6 & 1-8+9 \\ 3+6-3 & 1+10-3 & 2-4+0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 8 & 2 \\ 6 & 8 & -2 \end{bmatrix}$$

$$\therefore (A + 2B + 3C)^T = \begin{bmatrix} 5 & 6 \\ 8 & 8 \\ 2 & -2 \end{bmatrix} \quad \dots (1)$$

$$A^T = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}, B^T = \begin{bmatrix} 2 & 3 \\ 1 & 5 \\ -4 & -2 \end{bmatrix}, C^T = \begin{bmatrix} 0 & -1 \\ 2 & -1 \\ 3 & 0 \end{bmatrix}$$

$\therefore A^T + 2B^T + 3C^T$

$$= \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} + 2\begin{bmatrix} 2 & 3 \\ 1 & 5 \\ -4 & -2 \end{bmatrix} + 3\begin{bmatrix} 0 & -1 \\ 2 & -1 \\ 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 6 \\ 2 & 10 \\ -8 & -4 \end{bmatrix} + \begin{bmatrix} 0 & -3 \\ 6 & -3 \\ 9 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+0 & 3+6-3 \\ 0+2+6 & 1+10-3 \\ 1-8+9 & 2-4+0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 6 \\ 8 & 8 \\ 2 & -2 \end{bmatrix} \quad \dots (2)$$

From (1) and (2),

$(A + 2B + 3C)^T = A^T + 2B^T + 3C^T$.

**9. If $A = \begin{bmatrix} -1 & 2 & 1 \\ -3 & 2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ -3 & 2 \\ -1 & 3 \end{bmatrix}$, prove that $(A + B^T)^T = A^T + B$.**

**Solution :**

$$A = \begin{bmatrix} -1 & 2 & 1 \\ -3 & 2 & -3 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ -3 & 2 \\ -1 & 3 \end{bmatrix}$$

$$\therefore A^T = \begin{bmatrix} -1 & -3 \\ 2 & 2 \\ 1 & -3 \end{bmatrix}, B^T = \begin{bmatrix} 2 & -3 & -1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\therefore A + B^T = \begin{bmatrix} -1 & 2 & 1 \\ -3 & 2 & -3 \end{bmatrix} + \begin{bmatrix} 2 & -3 & -1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1+2 & 2-3 & 1-1 \\ -3+1 & 2+2 & -3+3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ -2 & 4 & 0 \end{bmatrix}$$

$$\therefore (A + B^T)^T = \begin{bmatrix} 1 & -2 \\ -1 & 4 \\ 0 & 0 \end{bmatrix} \quad \dots (1)$$

$$A^T + B = \begin{bmatrix} -1 & -3 \\ 2 & 2 \\ 1 & -3 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ -3 & 2 \\ -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1+2 & -3+1 \\ 2-3 & 2+2 \\ 1-1 & -3+3 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 4 \\ 0 & 0 \end{bmatrix} \quad \dots (2)$$

From (1) and (2),

$(A + B^T)^T = A^T + B$.

**10. Prove that $A + A^T$ is a symmetric and $A - A^T$ is a skew-symmetric matrix, where**

**(i) $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 2 & 1 \\ -2 & -3 & 2 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 5 & 2 & -4 \\ 3 & -7 & 2 \\ 4 & -5 & -3 \end{bmatrix}$.**

**Solution :**

**(i)** $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 2 & 1 \\ -2 & -3 & 2 \end{bmatrix}$ $\therefore A^T = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 2 & -3 \\ 4 & 1 & 2 \end{bmatrix}$

$$\therefore A + A^T = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 2 & 1 \\ -2 & -3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 3 & -2 \\ 2 & 2 & -3 \\ 4 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1 & 2+3 & 4-2 \\ 3+2 & 2+2 & 1-3 \\ -2+4 & -3+1 & 2+2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 5 & 2 \\ 5 & 4 & -2 \\ 2 & -2 & 4 \end{bmatrix}$$

This is a symmetric matrix (by definition).

Also, $A - A^T = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 2 & 1 \\ -2 & -3 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 3 & -2 \\ 2 & 2 & -3 \\ 4 & 1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 1-1 & 2-3 & 4-(-2) \\ 3-2 & 2-2 & 1-(-3) \\ -2-4 & -3-1 & 2-2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 6 \\ 1 & 0 & 4 \\ -6 & -4 & 0 \end{bmatrix}$$

This is a skew-symmetric matrix (by definition).

**(ii)** *Refer to the solution of Q. 10 (i).*

**11. Express each of the following matrix as the sum of a symmetric and a skew-symmetric matrix :**

**(i)** $\begin{bmatrix} 4 & -2 \\ 3 & -5 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$.

**Solution :**

**(i)** Let $A = \begin{bmatrix} 4 & -2 \\ 3 & -5 \end{bmatrix}$

Then $A^T = \begin{bmatrix} 4 & 3 \\ -2 & -5 \end{bmatrix}$

$\therefore\ A + A^T = \begin{bmatrix} 4 & -2 \\ 3 & -5 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ -2 & -5 \end{bmatrix}$

$$= \begin{bmatrix} 4+4 & -2+3 \\ 3-2 & -5-5 \end{bmatrix} = \begin{bmatrix} 8 & 1 \\ 1 & -10 \end{bmatrix}$$

This is a symmetric matrix.

Also, $A - A^T = \begin{bmatrix} 4 & -2 \\ 3 & -5 \end{bmatrix} - \begin{bmatrix} 4 & 3 \\ -2 & -5 \end{bmatrix}$

$$= \begin{bmatrix} 4-4 & -2-3 \\ 3-(-2) & -5-(-5) \end{bmatrix} = \begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}$$

This is a skew-symmetric matrix.

Now, $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

$$= \frac{1}{2}\begin{bmatrix} 8 & 1 \\ 1 & -10 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}$$

$$\therefore\ A = \begin{bmatrix} 4 & \frac{1}{2} \\ \frac{1}{2} & -5 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{5}{2} \\ \frac{5}{2} & 0 \end{bmatrix}.$$

**(ii)** Let $A = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$

Then $A^T = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$

$\therefore\ A + A^T = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} - \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 3+3 & 3-2 & -1-4 \\ -2+3 & -2-2 & 1-5 \\ -4-1 & -5+1 & 2+2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix}$$

This is a symmetric matrix.

Also, $A - A^T$

$$= \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} - \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3-3 & 3-(-2) & -1-(-4) \\ -2-3 & -2-(-2) & 1-(-5) \\ -4-(-1) & -5-1 & 2-2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 5 & 3 \\ -5 & 0 & 6 \\ -3 & -6 & 0 \end{bmatrix}$$

This is a skew-symmtric matrix.

Now, $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

$$\therefore\ A = \frac{1}{2}\begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 5 & 3 \\ -5 & 0 & 6 \\ -3 & -6 & 0 \end{bmatrix}.$$

**12. If** $A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \\ 4 & 1 \end{bmatrix}$ **and** $B = \begin{bmatrix} 0 & 3 & -4 \\ 2 & -1 & 1 \end{bmatrix}$,

**verify that (i)** $(AB)^T = B^T A^T$ **(ii)** $(BA)^T = A^T B^T$.

**Solution :**

$A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \\ 4 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 3 & -4 \\ 2 & -1 & 1 \end{bmatrix}$

$\therefore\ A^T = \begin{bmatrix} 2 & 3 & 4 \\ -1 & -2 & 1 \end{bmatrix}$, $B^T = \begin{bmatrix} 0 & 2 \\ 3 & -1 \\ -4 & 1 \end{bmatrix}$

**(i)** $AB = \begin{bmatrix} 2 & -1 \\ 3 & -2 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 0 & 3 & -4 \\ 2 & -1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 0-2 & 6+1 & -8-1 \\ 0-4 & 9+2 & -12-2 \\ 0+2 & 12-1 & -16+1 \end{bmatrix} = \begin{bmatrix} -2 & 7 & -9 \\ -4 & 11 & -14 \\ 2 & 11 & -15 \end{bmatrix}$$

$$\therefore\ (AB)^T = \begin{bmatrix} -2 & -4 & 2 \\ 7 & 11 & 11 \\ -9 & -14 & -15 \end{bmatrix} \quad \dots (1)$$

$$B^TA^T = \begin{bmatrix} 0 & 2 \\ 3 & -1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \\ -1 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0-2 & 0-4 & 0+2 \\ 6+1 & 9+2 & 12-1 \\ -8-1 & -12-2 & -16+1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -4 & 2 \\ 7 & 11 & 11 \\ -9 & -14 & -15 \end{bmatrix} \quad \dots (2)$$

From (1) and (2),

$(AB)^T = B^TA^T$.

**(ii)** $BA = \begin{bmatrix} 0 & 3 & -4 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & -2 \\ 4 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 0+9-16 & 0-6-4 \\ 4-3+4 & -2+2+1 \end{bmatrix} = \begin{bmatrix} -7 & -10 \\ 5 & 1 \end{bmatrix}$$

$$\therefore\ (BA)^T = \begin{bmatrix} -7 & 5 \\ -10 & 1 \end{bmatrix} \quad \dots (1)$$

$$A^TB^T = \begin{bmatrix} 2 & 3 & 4 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 3 & -1 \\ -4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0+9-16 & 4-3+4 \\ 0-6-4 & -2+2+1 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 5 \\ -10 & 1 \end{bmatrix} \quad \dots (2)$$

From (1) and (2), $(BA)^T = A^TB^T$.

## EXAMPLES FOR PRACTICE 2.4

1. Find the transpose of the following matrices :

(i) $\begin{bmatrix} 3 & -1 \\ 4 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -1 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 6 \\ -1 & 1 \\ 2 & 7 \end{bmatrix}$ (iv) $\begin{bmatrix} 0 & 2 & 1 \\ 3 & 4 & 5 \\ 6 & -1 & 8 \end{bmatrix}$.

2. If $A = \begin{bmatrix} 5 & 5 & 5 \\ 6 & -3 & -11 \\ 3 & -1 & 8 \end{bmatrix}$, prove that $(A^T)^T = A$.

3. If $A = \begin{bmatrix} 2 & 3 & -5 \\ 1 & -2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 5 & 2 \end{bmatrix}$,

$C = \begin{bmatrix} 5 & -1 & 0 \\ 7 & 8 & -1 \end{bmatrix}$, verify that

(i) $(A+B)^T = A^T + B^T$

(ii) $(B-C)^T = B^T - C^T$

(iii) $(A+2B-3C)^T = A^T + 2B^T - 3C^T$.

4. If $A = \begin{bmatrix} 7 & 0 & 3 \\ 2 & 5 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 & 2 \\ 4 & 0 & -1 \end{bmatrix}$, then find

(i) $A^T + 3B^T$ (ii) $2A^T - 5B^T$.

5. If $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}$, verify that

(i) $(AB)^T = B^TA^T$ (ii) $(BA)^T = A^TB^T$.

6. Express each of the following matrix as the sum of a symmetric and a skew-symmetric matrix :

(i) $\begin{bmatrix} 3 & -1 \\ 4 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 4 & -5 & 3 \\ -6 & 2 & 1 \\ 7 & 8 & -9 \end{bmatrix}$.

### Answers

1. (i) $\begin{bmatrix} 3 & 4 \\ -1 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 4 \\ -1 & 5 \\ 3 & 6 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & -1 & 2 \\ 6 & 1 & 7 \end{bmatrix}$ (iv) $\begin{bmatrix} 0 & 3 & 6 \\ 2 & 4 & -1 \\ 1 & 5 & 8 \end{bmatrix}$

4. (i) $\begin{bmatrix} 13 & 14 \\ 9 & 5 \\ 9 & -3 \end{bmatrix}$ (ii) $\begin{bmatrix} 4 & -16 \\ -15 & 10 \\ -4 & 5 \end{bmatrix}$.

6. (i) $\begin{bmatrix} 3 & \frac{3}{2} \\ \frac{3}{2} & 2 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{5}{2} \\ \frac{5}{2} & 0 \end{bmatrix}$

(ii) $\begin{bmatrix} 4 & -\frac{11}{2} & 5 \\ -\frac{11}{2} & 2 & \frac{9}{2} \\ 5 & \frac{9}{2} & -18 \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{2} & -2 \\ -\frac{1}{2} & 0 & -\frac{7}{2} \\ 2 & \frac{7}{2} & 0 \end{bmatrix}$.

## 2.5 : ELEMENTARY TRANSFORMATIONS

The elementary transformations are the operations performed on rows (or columns) of a matrix.

**(a) The following are elementary row transformations :**

**(i) Interchanging any two rows :**

If the $i^{th}$ and the $j^{th}$ rows of a matrix are interchanged, then we denote this by $R_{ij}$ or by $R_i \leftrightarrow R_j$.

Note that R and C symbolically represent the rows and columns of a matrix, respectively.

**Important :** A row cannot be interchanged with a column and a column cannot be interchanged with a row.

**For example :**

If $A = \begin{bmatrix} 3 & -2 & 1 \\ 4 & 0 & -5 \\ 6 & 1 & 8 \end{bmatrix}$, then $R_2 \leftrightarrow R_3$ gives

the matrix $\begin{bmatrix} 3 & -2 & 1 \\ 6 & 1 & 8 \\ 4 & 0 & -5 \end{bmatrix}$

Note that $A \neq \begin{bmatrix} 3 & -2 & 1 \\ 6 & 1 & 8 \\ 4 & 0 & -5 \end{bmatrix}$

We write $A \sim \begin{bmatrix} 3 & -2 & 1 \\ 6 & 1 & 8 \\ 4 & 0 & -5 \end{bmatrix}$

**(ii) Multiplying the elements of any row by a non-zero number :**

If the elements of the $i^{th}$ row are multiplied by a non-zero number $k$, then we denote this by $kR_i$ or by $R_i \to kR_i$.

**For example :**

If $A = \begin{bmatrix} 3 & -2 \\ 6 & 1 \end{bmatrix}$, then $2R_1$ gives $A \sim \begin{bmatrix} 6 & -4 \\ 6 & 1 \end{bmatrix}$

**(iii) Adding to the elements of any row, the same multiples of the corresponding elements of any other row :**

If to the elements of the $i^{th}$ row, are added $k$ times the corresponding elements of the $j^{th}$ row, then we denote this by $R_i + kR_j$ or by $R_i \to R_i + kR_j$.

**For example :**

If $A = \begin{bmatrix} 3 & -2 \\ 6 & 1 \end{bmatrix}$ then $R_2 + 2R_1$ gives

$A \sim \begin{bmatrix} 3 & -2 \\ 6+2(3) & 1+2(-2) \end{bmatrix}$, i.e. $A \sim \begin{bmatrix} 3 & -2 \\ 12 & -3 \end{bmatrix}$

**(b) The following are elementary column transformations :**

**(i) Interchanging any two columns :**

If the $i^{th}$ and the $j^{th}$ columns of a matrix are interchanged, then we denote this by $C_{ij}$ or by $C_i \leftrightarrow C_j$.

**For example :**

If $A = \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}$ then $C_1 \leftrightarrow C_2$ gives $A \sim \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$

**(ii) Multiplying the elements of any column by a non-zero number :**

If the elements of the $i^{th}$ column are multiplied by a non-zero number $k$, then we denote this by $kC_i$ or by $C_i \to kC_i$.

**For example :**

If $A = \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}$ then $3C_2$ gives

$A \sim \begin{bmatrix} 2 & 3 \\ 4 & 15 \end{bmatrix}$

**(iii) Adding to the elements of any column, the same multiples of the corresponding elements of any other column :**

If to the elements of the $i^{th}$ column, are added $k$ times the corresponding elements of the $j^{th}$ column, then we denote this by $C_i + kC_j$ or by $C_i \to C_i + kC_j$.

**For Example :**

If $A = \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}$ then $C_2 - 2C_1$ gives

$A \sim \begin{bmatrix} 2 & 1-2(2) \\ 4 & 5-2(4) \end{bmatrix}$, i.e. $A \sim \begin{bmatrix} 2 & -3 \\ 4 & -3 \end{bmatrix}$

**Notes :**

1. The elements of a row cannot be added to the elements of a column and conversely.
2. After the elementary transformations, the matrix obtained is said to be equivalent to the original matrix.

**(c) Elementary row transformations on the product :**

Suppose A, B and P are matrices such that AB = P. If we perform some elementary row transformation R on the matrix A, we get a new matrix. Let us denote it by $A_R$. Now if we take the product $A_RB$, it is not the same as P. It can be proved that the product matrix

$A_RB$ is the one obtained by performing the same elementary row transformation R on P, i.e. $P_R$ and we get $A_RB = P_R$.

(d) **Elementary column transformations on the product :**

Suppose A, B and P are matrices such that AB = P. If we perform some elementary column transformation C on the matrix B, we get a new matrix. Let us denote it by $B_C$. Now if we take the product $AB_C$, it is not the same as P. It can be proved that the product matrix $AB_C$ is the one obtained by performing the same elementary column transformation C on P, i.e. $P_C$ and we get $AB_C = P_C$.

**Note :** If AB = P, A can be changed using only row transformations and B can be changed using only column transformations.

## 2.6 : INVERSE OF A MATRIX

**Definition :** If for a square matrix A, there exists a square matrix B of same order such that AB = I = BA, then B is called the **inverse** of the matrix A and is denoted by $A^{-1}$.

Thus, $AA^{-1} = I = A^{-1}A$.

Let A be a square matrix. Then its determinant is $|A|$ whose value is either zero or not zero. If $|A| = 0$, then the inverse of the matrix A does not exist. If $|A| \neq 0$, then $A^{-1}$ exists. Hence, we can find the inverse of the matrix A if and only if A is non-singular. i.e. $A^{-1}$ exists if and only if $|A| \neq 0$.

**Remark :** Let A be a square matrix. Let, if possible, B and C both be inverse of A

$\therefore$ AB = BA = I ... (1)

and AC = CA = I ... (2)

Now, B = BI = B(AC) ... [By (2)]

= (BA)C = IC ... [By (1)]

= C

$\therefore$ the inverse of a matrix A, if it exists, is unique.

**Inverse of the Product of Two Matrices :**

1. Suppose A and B are square matrices of the same order. Then AB and BA exist and they are of the same order. Now if $|AB| \neq 0$, then $(AB)^{-1}$ exists. Also if $|BA| \neq 0$, then $(BA)^{-1}$ exists. Both $(AB)^{-1}$ and $(BA)^{-1}$ are of the same order. They may or may not be equal.

Using associativity of multiplication of matrices, we have the following results :

If A and B are square matrices of the same order and $|A| \neq 0$, $|B| \neq 0$, $|AB| \neq 0$, $|BA| \neq 0$, i.e. their inverses exist, then

(1) $(AB)(B^{-1}A^{-1}) = A[B(B^{-1}A^{-1})]$

$= A[(BB^{-1})A^{-1}]$

$= A(IA^{-1}) = AA^{-1} = I$.

This shows that the inverse of AB is $B^{-1}A^{-1}$ i.e. $(AB)^{-1} = B^{-1}A^{-1}$.

(2) $(BA)(A^{-1}B^{-1}) = B[A(A^{-1}B^{-1})]$

$= B[(AA^{-1})B^{-1}]$

$= B(IB^{-1}) = BB^{-1} = I$.

This shows that the inverse of BA is $A^{-1}B^{-1}$ i.e. $(BA)^{-1} = A^{-1}B^{-1}$.

2. Now suppose A is a $2 \times 3$ matrix and B is a $3 \times 2$ matrix. Then AB is a square matrix of order $2 \times 2$ and BA is a square matrix of order $3 \times 3$.

Suppose $|AB| \neq 0$ and $|BA| \neq 0$. Then the inverse of AB which is $(AB)^{-1}$ exists and is of order $2 \times 2$.

Similarly, the inverse of BA which is $(BA)^{-1}$ exists and is of order $3 \times 3$.

Since, the orders of $(AB)^{-1}$ and $(BA)^{-1}$ are not equal, $(AB)^{-1} \neq (BA)^{-1}$.

We note that neither A nor B is a square matrix. Hence their inverses do not exist. But the inverses of their products exist, if $|AB| \neq 0$ and $|BA| \neq 0$.

- **Inverse of a non-singular matrix by elementary transformations :**

**Method to find $A^{-1}$ by Elementary Row Transformations :**

We write $AA^{-1} = I$

Perform suitable row transformations on the matrix A, so that it is converted into the identity matrix I. The same row transformations are to be performed simultaneously on I on the RHS will convert it into the matrix B.

$\therefore$ $AA^{-1} = I$ reduces to $IA^{-1} = B$ i.e. $A^{-1} = B$.

In order to convert the matrix

(i) $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ in to the identity matrix, perform suitable elementary row transformations on A so as to get :

**Step 1 :** 1 in the place of $a_{11}$

**Step 2 :** 0 in the place of $a_{21}$

**Step 3 :** 1 in the place of $a_{22}$

**Step 4 :** 0 in the place of $a_{12}$.

(ii) $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ in to the identity matrix, perform suitable elementary row transformations on A so as to get :

**Step 1 :** 1 in the place of $a_{11}$

**Step 2 :** 0 in the place of $a_{21}$ and $a_{31}$

**Step 3 :** 1 in the place of $a_{22}$

**Step 4 :** 0 in the place of $a_{12}$ and $a_{32}$

**Step 5 :** 1 in the place of $a_{33}$

**Step 6 :** 0 in the place of $a_{13}$ and $a_{23}$.

**Method to find $A^{-1}$ by Elementary Column Transformations :**

We write $A^{-1}A = I$

Perform suitable column transformations on the matrix A, so that it is converted in to the identity matrix I. The same column transformations are to be performed simultaneously on I on the RHS will convert it in to the matrix B.

$\therefore$ $A^{-1}A = I$ reduces to $A^{-1}I = B$ i.e. $A^{-1} = B$.

In order to convert the matrix

(i) $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ in to the identity matrix, perform suitable column transformations on A so as to get :

**Step 1 :** 1 in the place of $a_{11}$

**Step 2 :** 0 in the place of $a_{12}$

**Step 3 :** 1 in the place of $a_{22}$

**Step 4 :** 0 in the place of $a_{21}$.

(ii) $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ in to the identity matrix, perform suitable elementary column transformations on A so as to get :

**Step 1 :** 1 in the place of $a_{11}$

**Step 2 :** 0 in the place of $a_{12}$ and $a_{13}$

**Step 3 :** 1 in the place of $a_{22}$

**Step 4 :** 0 in the place of $a_{21}$ and $a_{23}$

**Step 5 :** 1 in the place of $a_{33}$

**Step 6 :** 0 in the place of $a_{31}$ and $a_{32}$.

**Note :** While evaluating the inverse of a matrix, do not mix row and column transformations in the same example.

- **Inverse of a non-singular matrix by Adjoint Method :**

**Minor of an element :**

The minor of an element $a_{ij}$ of a square matrix $A = [a_{ij}]_{m \times m}$ is the value of the determinant obtained by striking off the $i^{th}$ row and $j^{th}$ column of the matrix A. The minor of $a_{ij}$ is denoted by $M_{ij}$.

Note that, if A is a matrix of order $m \times m$ then the minor of any element of A is the value of the determinant of order $(m-1) \times (m-1)$.

**For example :** Consider the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 4 & 5 \\ 6 & -2 & 1 \end{bmatrix}$$

Here $a_{11} = 2$ (element in 1st row and 1st column). The minor of $a_{11}$ is the value of the determinant obtained from the matrix A by striking off 1st row and 1st column.

Hence, the minor of $a_{11} = M_{11} = \begin{vmatrix} 4 & 5 \\ -2 & 1 \end{vmatrix}$

$= 4 + 10 = 14.$

Similarly, $a_{23} = 5$ and the minor of $a_{23}$ is

$M_{23} = \begin{vmatrix} 2 & -1 \\ 6 & -2 \end{vmatrix} = -4 + 6 = 2.$

**Cofactor of an element :**

The cofactor of an element $a_{ij}$ of a square matrix $A = [a_{ij}]$ is given by $(-1)^{i+j}M_{ij}$, where $M_{ij}$ is the minor of $a_{ij}$. The cofactor of $a_{ij}$ is denoted by $A_{ij}$.

i.e. $A_{ij} = (-1)^{i+j}M_{ij}$

**For example :**

For the matrix $A = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 4 & 5 \\ 6 & -2 & 1 \end{bmatrix}$,

Cofactor of $a_{11} = (-1)^{1+1}M_{11} = (-1)^2 \begin{vmatrix} 4 & 5 \\ -2 & 1 \end{vmatrix}$

$= 4 + 10 = 14$

Cofactor of $a_{23} = (-1)^{2+3}M_{23} = (-1)^5 \begin{vmatrix} 2 & -1 \\ 6 & -2 \end{vmatrix}$

$= -(-4+6) = -2.$

**Cofactor Matrix :**

The cofactor matrix of the square matrix $A = [a_{ij}]_{m \times m}$ is a matrix of order $m \times m$ where each element $a_{ij}$ of the matrix A is replaced by its cofactor $A_{ij}$.

i.e. if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

Cofactor matrix $= \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

**Adjoint of a Matrix :**

The adjoint of a matrix $A = [a_{ij}]$ is the transpose of the cofactor matrix. It is denoted by 'adj A'.

i.e. if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

**For example :** Consider the matrix $A = \begin{bmatrix} 3 & -2 \\ 6 & 8 \end{bmatrix}$

Here, $a_{11} = 3$, $M_{11} = 8$ and $A_{11} = (-1)^{1+1}(8) = 8$

$a_{12} = -2$, $M_{12} = 6$ and $A_{12} = (-1)^{1+2}(6) = -6$

$a_{21} = 6$, $M_{21} = -2$ and $A_{21} = (-1)^{2+1}(-2) = 2$

$a_{22} = 8$, $M_{22} = 3$ and $A_{22} = (-1)^{2+2}(3) = 3$

$\therefore$ cofactor matrix $= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$

$= \begin{bmatrix} 8 & -6 \\ 2 & 3 \end{bmatrix}$

$\therefore \text{adj } A = \begin{bmatrix} 8 & 2 \\ -6 & 3 \end{bmatrix}$

**Note :** The adjoint of a square matrix of order 2 can be obtained by interchanging the diagonal elements and changing the signs of non-diagonal elements.

i.e. if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\text{adj } A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

**Inverse by adjoint method :**

If $A = [a_{ij}]_{m \times m}$ is a non-singular square matrix, i.e. $|A| \neq 0$, then its inverse exists and it is given as

$A^{-1} = \frac{1}{|A|}(\text{adj } A)$.

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|---|---|

**1. Apply the given elementary transformation on each of the following matrices :**

**(i)** $\begin{bmatrix} 3 & -4 \\ 2 & 2 \end{bmatrix}$, $R_1 \leftrightarrow R_2$

**(ii)** $\begin{bmatrix} 2 & 4 \\ 1 & -5 \end{bmatrix}$, $C_1 \leftrightarrow C_2$

**(iii)** $\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$, $3R_2$ and $C_2 \rightarrow C_2 - 4C_1$.

**Solution :**

(i) Let $A = \begin{bmatrix} 3 & -4 \\ 2 & 2 \end{bmatrix}$

By $R_1 \leftrightarrow R_2$, we get

$A \sim \begin{bmatrix} 2 & 2 \\ 3 & -4 \end{bmatrix}$

(ii) Let $B = \begin{bmatrix} 2 & 4 \\ 1 & -5 \end{bmatrix}$

By $C_1 \leftrightarrow C_2$, we get

$B \sim \begin{bmatrix} 4 & 2 \\ -5 & 1 \end{bmatrix}$.

(iii) Let $C = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$

By $3R_2$, we get

$C \sim \begin{bmatrix} 3 & 1 & -1 \\ 3 & 9 & 3 \\ -1 & 1 & 3 \end{bmatrix}$

By $C_2 - 4C_1$ on C, we get

$C \sim \begin{bmatrix} 3 & -11 & -1 \\ 1 & -1 & 1 \\ -1 & 5 & 3 \end{bmatrix}$.

**2. Transform $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 4 \end{bmatrix}$ into an upper triangular matrix by suitable row transformations.**

**Solution :** Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 4 \end{bmatrix}$

By $R_2 - 2R_1$ and $R_3 - 3R_1$, we get

$A \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -1 \\ 0 & 5 & -2 \end{bmatrix}$

By $\left(\frac{1}{3}\right) R_2$, we get

$A \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -\frac{1}{3} \\ 0 & 5 & -2 \end{bmatrix}$

By $R_3 - 5R_2$, we get

$$A \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & -\frac{1}{3} \end{bmatrix}$$

This is an upper triangular matrix.

**3. Find the cofactor matrix of the following matrices :**

**(i)** $\begin{bmatrix} 1 & 2 \\ 5 & -8 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 5 & 8 & 7 \\ -1 & -2 & 1 \\ -2 & 1 & 1 \end{bmatrix}$.

**Solution :**

**(i)** Let $A = \begin{bmatrix} 1 & 2 \\ 5 & -8 \end{bmatrix}$

Here, $a_{11} = 1$, $M_{11} = -8$

$\therefore A_{11} = (-1)^{1+1} M_{11} = -8$

$a_{12} = 2$, $M_{12} = 5$

$\therefore A_{12} = (-1)^{1+2} M_{12} = -1(5) = -5$

$a_{21} = 5$, $M_{21} = 2$

$\therefore A_{21} = (-1)^{2+1} M_{21} = -1(2) = -2$

$a_{22} = -8$, $M_{22} = 1$

$\therefore A_{22} = (-1)^{2+2} M_{22} = 1$.

$$\therefore \text{cofactor matrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} -8 & -5 \\ -2 & 1 \end{bmatrix}.$$

**(ii)** Let $A = \begin{bmatrix} 5 & 8 & 7 \\ -1 & -2 & 1 \\ -2 & 1 & 1 \end{bmatrix}$

The cofactor of $a_{ij}$ is given by $A_{ij} = (-1)^{i+j} M_{ij}$

Now, $M_{11} = \begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} = -2 - 1 = -3$

$\therefore A_{11} = (-1)^{1+1} M_{11} = 1(-3) = -3$

$M_{12} = \begin{vmatrix} -1 & 1 \\ -2 & 1 \end{vmatrix} = -1 - (-2) = 1$

$\therefore A_{12} = (-1)^{1+2} M_{12} = -1(1) = -1$

$M_{13} = \begin{vmatrix} -1 & -2 \\ -2 & 1 \end{vmatrix} = -1 - 4 = -5$

$\therefore A_{13} = (-1)^{1+3} M_{13} = 1(-5) = -5$

$M_{21} = \begin{vmatrix} 8 & 7 \\ 1 & 1 \end{vmatrix} = 8 - 7 = 1$

$\therefore A_{21} = (-1)^{2+1} M_{21} = -1(1) = -1$

$M_{22} = \begin{vmatrix} 5 & 7 \\ -2 & 1 \end{vmatrix} = 5 + 14 = 19$

$\therefore A_{22} = (-1)^{2+2} M_{22} = 1(19) = 19$

$M_{23} = \begin{vmatrix} 5 & 8 \\ -2 & 1 \end{vmatrix} = 5 - (-16) = 21$

$\therefore A_{23} = (-1)^{2+3} M_{23} = -1(21) = -21$

$M_{31} = \begin{vmatrix} 8 & 7 \\ -2 & 1 \end{vmatrix} = 8 - (-14) = 22$

$\therefore A_{31} = (-1)^{3+1} M_{31} = 1(22) = 22$

$M_{32} = \begin{vmatrix} 5 & 7 \\ -1 & 1 \end{vmatrix} = 5 - (-7) = 12$

$\therefore A_{32} = (-1)^{3+2} M_{32} = -1(12) = -12$

$M_{33} = \begin{vmatrix} 5 & 8 \\ -1 & -2 \end{vmatrix} = -10 - (-8) = -2$

$\therefore A_{33} = (-1)^{3+3} M_{33} = 1(-2) = -2$

$$\therefore \text{cofactor matrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} -3 & -1 & -5 \\ -1 & 19 & -21 \\ 22 & -12 & -2 \end{bmatrix}.$$

**4. Find the adjoint of the following matrices :**

**(i)** $\begin{bmatrix} 2 & -3 \\ 3 & 5 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 1 & -1 & 2 \\ -2 & 3 & 5 \\ -2 & 0 & -1 \end{bmatrix}$.

**Solution :**

(i) $A = \begin{bmatrix} 2 & -3 \\ 3 & 5 \end{bmatrix}$

Here, $a_{11} = 2$, $M_{11} = 5$

$\therefore A_{11} = (-1)^{1+1}(5) = 5$

$a_{12} = -3$, $M_{12} = 3$

$\therefore A_{12} = (-1)^{1+2}(3) = -3$

$a_{21} = 3$, $M_{21} = -3$

$\therefore A_{21} = (-1)^{2+1}(-3) = 3$

$a_{22} = 5$, $M_{22} = 2$

$\therefore A_{22} = (-1)^{2+2} = 2$

$\therefore$ the cofactor matrix $= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ 3 & 2 \end{bmatrix}$

$\therefore$ adj A $= \begin{bmatrix} 5 & 3 \\ -3 & 2 \end{bmatrix}$.

**(ii)** Let $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 3 & 5 \\ -2 & 0 & -1 \end{bmatrix}$

The cofactor of $a_{ij}$ is given by $A_{ij} = (-1)^{i+j} M_{ij}$

Now, $M_{11} = \begin{vmatrix} 3 & 5 \\ 0 & -1 \end{vmatrix} = -3 - 0 = -3$

$\therefore A_{11} = (-1)^{1+1}(-3) = -3$

$M_{12} = \begin{vmatrix} -2 & 5 \\ -2 & -1 \end{vmatrix} = 2 + 10 = 12$

$\therefore A_{12} = (-1)^{1+2}(12) = -12$

$M_{13} = \begin{vmatrix} -2 & 3 \\ -2 & 0 \end{vmatrix} = 0 + 6 = 6$

$\therefore A_{13} = (-1)^{1+3}(6) = 6$

$M_{21} = \begin{vmatrix} -1 & 2 \\ 0 & -1 \end{vmatrix} = 1 - 0 = 1$

$\therefore A_{21} = (-1)^{2+1}(1) = -1$

$M_{22} = \begin{vmatrix} 1 & 2 \\ -2 & -1 \end{vmatrix} = -1 + 4 = 3$

$\therefore A_{22} = (-1)^{2+2}(3) = 3$

$M_{23} = \begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = 0 - 2 = -2$

$\therefore A_{23} = (-1)^{2+3}(-2) = 2$

$M_{31} = \begin{vmatrix} -1 & 2 \\ 3 & 5 \end{vmatrix} = -5 - 6 = -11$

$\therefore A_{31} = (-1)^{3+1}(-11) = -11$

$M_{32} = \begin{vmatrix} 1 & 2 \\ -2 & 5 \end{vmatrix} = 5 + 4 = 9$

$\therefore A_{32} = (-1)^{3+2}(9) = -9$

$M_{33} = \begin{vmatrix} 1 & -1 \\ -2 & 3 \end{vmatrix} = 3 - 2 = 1$

$\therefore A_{33} = (-1)^{3+3}(1) = 1$.

$\therefore$ the cofactor matrix $= \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

$= \begin{bmatrix} -3 & -12 & 6 \\ -1 & 3 & 2 \\ -11 & -9 & 1 \end{bmatrix}$

$\therefore$ adj A $= \begin{bmatrix} -3 & -1 & -11 \\ -12 & 3 & -9 \\ 6 & 2 & 1 \end{bmatrix}$.

**5. Find the inverses of the following matrices by the adjoint mathod :**

**(i)** $\begin{bmatrix} 3 & -1 \\ 2 & -1 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 2 & -2 \\ 4 & 5 \end{bmatrix}$ **(iii)** $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$.

**Solution :**

**(i)** Let $A = \begin{bmatrix} 3 & -1 \\ 2 & -1 \end{bmatrix}$

Then $|A| = \begin{vmatrix} 3 & -1 \\ 2 & -1 \end{vmatrix} = -3 - (-2) = -1 \neq 0$

$\therefore A^{-1}$ exists.

First we have to find the cofactor matrix

$= [A_{ij}]_{2 \times 2}$, where $A_{ij} = (-1)^{i+j} M_{ij}$

Now, $A_{11} = (-1)^{1+1} M_{11} = -1$

$A_{12} = (-1)^{1+2} M_{12} = -2$

$A_{21} = (-1)^{2+1} M_{21} = -(-1) = 1$

$A_{22} = (-1)^{2+2} M_{22} = 3$

$\therefore$ the cofactor matrix

$= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 1 & 3 \end{bmatrix}$

$\therefore$ adj A $= \begin{bmatrix} -1 & 1 \\ -2 & 3 \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{-1}\begin{bmatrix} -1 & 1 \\ -2 & 3 \end{bmatrix}$

$\therefore A^{-1} = \begin{bmatrix} 1 & -1 \\ 2 & -3 \end{bmatrix}$.

**(ii)** *Refer to the solution of Q. 5 (i).*

**Ans.** $\frac{1}{18}\begin{bmatrix} 5 & 2 \\ -4 & 2 \end{bmatrix}$.

**(iii)** Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

Then $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{vmatrix}$

$= 1(10 - 0) - 2(0 - 0) + 3(0 - 0)$

$= 10 \neq 0$

$\therefore A^{-1}$ exist.

First we have to find the cofactor matrix

$= [A_{ij}]_{3 \times 3}$, where $A_{ij} = (-1)^{i+j} M_{ij}$

Now, $A_{11}=(-1)^{1+1}M_{11}=\begin{vmatrix}2&4\\0&5\end{vmatrix}=10-0=10$

$A_{12}=(-1)^{1+2}M_{12}=-\begin{vmatrix}0&4\\0&5\end{vmatrix}=-(0-0)=0$

$A_{13}=(-1)^{1+3}M_{13}=\begin{vmatrix}0&2\\0&0\end{vmatrix}=0-0=0$

$A_{21}=(-1)^{2+1}M_{21}=-\begin{vmatrix}2&3\\0&5\end{vmatrix}=-(10-0)=-10$

$A_{22}=(-1)^{2+2}M_{22}=\begin{vmatrix}1&3\\0&5\end{vmatrix}=5-0=5$

$A_{23}=(-1)^{2+3}M_{23}=-\begin{vmatrix}1&2\\0&0\end{vmatrix}=-(0-0)=0$

$A_{31}=(-1)^{3+1}M_{31}=\begin{vmatrix}2&3\\2&4\end{vmatrix}=8-6=2$

$A_{32}=(-1)^{3+2}M_{32}=-\begin{vmatrix}1&3\\0&4\end{vmatrix}=-(4-0)=-4$

$A_{33}=(-1)^{3+3}M_{33}=\begin{vmatrix}1&2\\0&2\end{vmatrix}=2-0=2$

$\therefore$ the cofactor matrix

$$=\begin{bmatrix}A_{11}&A_{12}&A_{13}\\A_{21}&A_{22}&A_{23}\\A_{31}&A_{32}&A_{33}\end{bmatrix}=\begin{bmatrix}10&0&0\\-10&5&0\\2&-4&2\end{bmatrix}$$

$$\therefore \text{adj } A=\begin{bmatrix}10&-10&2\\0&5&-4\\0&0&2\end{bmatrix}$$

$$\therefore A^{-1}=\frac{1}{|A|}(\text{adj } A)$$

$$\therefore A^{-1}=\frac{1}{10}\begin{bmatrix}10&-10&2\\0&5&-4\\0&0&2\end{bmatrix}.$$

**6. Find the inverses of the following matrices by the transformation method :**

**(i)** $\begin{bmatrix}1&2\\2&-1\end{bmatrix}$ **(ii)** $\begin{bmatrix}2&0&-1\\5&1&0\\0&1&3\end{bmatrix}$.

**Solution :**

(i) Let $A=\begin{bmatrix}1&2\\2&-1\end{bmatrix}$

Then $|A|=\begin{vmatrix}1&2\\2&-1\end{vmatrix}=-1-4=-5\neq 0$

$\therefore$ $A^{-1}$ exists.

**$A^{-1}$ by Row transformations :**

We write $AA^{-1}=I$

$$\therefore \begin{bmatrix}1&2\\2&-1\end{bmatrix}A^{-1}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

By $R_2-2R_1$, we get

$$\begin{bmatrix}1&2\\0&-5\end{bmatrix}A^{-1}=\begin{bmatrix}1&0\\-2&1\end{bmatrix}$$

By $\left(-\frac{1}{5}\right)R_2$, we get

$$\begin{bmatrix}1&2\\0&1\end{bmatrix}A^{-1}=\begin{bmatrix}1&0\\\frac{2}{5}&-\frac{1}{5}\end{bmatrix}$$

By $R_1-2R_2$, we get

$$\begin{bmatrix}1&0\\0&1\end{bmatrix}A^{-1}=\begin{bmatrix}\frac{1}{5}&\frac{2}{5}\\\frac{2}{5}&-\frac{1}{5}\end{bmatrix}$$

$$\therefore A^{-1}=\frac{1}{5}\begin{bmatrix}1&2\\2&-1\end{bmatrix} \quad \dots (1)$$

**$A^{-1}$ by Column transformations :**

We write $A^{-1}A=I$

$$\therefore A^{-1}\begin{bmatrix}1&2\\2&-1\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

By $C_2-2C_1$, we get

$$A^{-1}\begin{bmatrix}1&0\\2&-5\end{bmatrix}=\begin{bmatrix}1&-2\\0&1\end{bmatrix}$$

By $\left(-\frac{1}{5}\right)C_2$, we get

$$A^{-1}\begin{bmatrix}1&0\\2&1\end{bmatrix}=\begin{bmatrix}1&\frac{2}{5}\\0&-\frac{1}{5}\end{bmatrix}$$

By $C_1-2C_2$, we get

$$A^{-1}\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}\frac{1}{5}&\frac{2}{5}\\\frac{2}{5}&-\frac{1}{5}\end{bmatrix}$$

$$\therefore A^{-1}=\frac{1}{5}\begin{bmatrix}1&2\\2&-1\end{bmatrix} \quad \dots (2)$$

From (1) and (2),

$A^{-1}=\frac{1}{5}\begin{bmatrix}1&2\\2&-1\end{bmatrix}$, which is unique.

Note : If it is asked to find $A^{-1}$ by elementary transformation, we can use either row transformation or column transformation.

**(ii)** Let $A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

Then $|A| = \begin{vmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{vmatrix}$

$= 2(3-0) - 0(15-0) - 1(5-0)$

$= 6 - 0 - 5 = 1 \neq 0$

$\therefore$ $A^{-1}$ exists.

We write $AA^{-1} = I$

$\therefore \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

By $3R_1$, we get

$\begin{bmatrix} 6 & 0 & -3 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} A^{-1} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

By $R_1 - R_2$, we get

$\begin{bmatrix} 1 & -1 & -3 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} A^{-1} = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

By $R_2 - 5R_1$, we get

$\begin{bmatrix} 1 & -1 & -3 \\ 0 & 6 & 15 \\ 0 & 1 & 3 \end{bmatrix} A^{-1} = \begin{bmatrix} 3 & -1 & 0 \\ -15 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

By $R_2 - 5R_3$, we get

$\begin{bmatrix} 1 & -1 & -3 \\ 0 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} A^{-1} = \begin{bmatrix} 3 & -1 & 0 \\ -15 & 6 & -5 \\ 0 & 0 & 1 \end{bmatrix}$

By $R_1 + R_2$ and $R_3 - R_2$, we get

$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} A^{-1} = \begin{bmatrix} -12 & 5 & -5 \\ -15 & 6 & -5 \\ 15 & -6 & 6 \end{bmatrix}$

By $\left(\frac{1}{3}\right)R_3$, we get

$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} -12 & 5 & -5 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$

By $R_1 + 3R_3$, we get

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$

$\therefore A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$

**Note :** $A^{-1}$ can also be obtained by using column transformations taking $A^{-1}A = I$.

**7. Find the inverse of $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ by elementary column transformations.**

**Solution :** $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

$\therefore |A| = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 1 & 2 & 1 \end{vmatrix}$

$= 1(2-6) - 0 + 1(0-2)$

$= -4 - 2 = -6 \neq 0$

$\therefore$ $A^{-1}$ exists.

We write $A^{-1}A = I$

$\therefore A^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

By $C_3 - C_1$, we get

$A^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

By $\left(\frac{1}{2}\right)C_2$, we get

$A^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

By $C_3 - 3C_2$, we get

$A^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & \frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 1 \end{bmatrix}$

By $\left(-\frac{1}{3}\right)C_3$, we get

$A^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{1}{3} \end{bmatrix}$

By $C_1 - C_3$ and $C_2 - C_3$, we get

$$A^{-1}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}\frac{2}{3} & -\frac{1}{3} & \frac{1}{3}\\ -\frac{1}{2} & 0 & \frac{1}{2}\\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3}\end{bmatrix}$$

$$\therefore\ A^{-1} = \frac{1}{6}\begin{bmatrix}4 & -2 & 2\\-3 & 0 & 3\\2 & 2 & -2\end{bmatrix}$$

[**Note :** *Answer in the textbook is incorrect.*]

**8. Find the inverse of $\begin{bmatrix}1 & 2 & 3\\1 & 1 & 5\\2 & 4 & 7\end{bmatrix}$ by the elementary row transformations.**

**Solution :** Let $A = \begin{bmatrix}1 & 2 & 3\\1 & 1 & 5\\2 & 4 & 7\end{bmatrix}$

Then $|A| = \begin{vmatrix}1 & 2 & 3\\1 & 1 & 5\\2 & 4 & 7\end{vmatrix}$

$= 1(7-20) - 2(7-10) + 3(4-2)$

$= -13 + 6 + 6 = -1 \neq 0$

$\therefore\ A^{-1}$ exists.

We write $AA^{-1} = I$

$$\therefore\ \begin{bmatrix}1 & 2 & 3\\1 & 1 & 5\\2 & 4 & 7\end{bmatrix} A^{-1} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

By $R_2 - R_1$ and $R_3 - 2R_1$, we get

$$\begin{bmatrix}1 & 2 & 3\\0 & -1 & 2\\0 & 0 & 1\end{bmatrix} A^{-1} = \begin{bmatrix}1 & 0 & 0\\-1 & 1 & 0\\-2 & 0 & 1\end{bmatrix}$$

By $(-1)R_2$, we get

$$\begin{bmatrix}1 & 2 & 3\\0 & 1 & -2\\0 & 0 & 1\end{bmatrix} A^{-1} = \begin{bmatrix}1 & 0 & 0\\1 & -1 & 0\\-2 & 0 & 1\end{bmatrix}$$

By $R_1 - 2R_2$, we get

$$\begin{bmatrix}1 & 0 & 7\\0 & 1 & -2\\0 & 0 & 1\end{bmatrix} A^{-1} = \begin{bmatrix}-1 & 2 & 0\\1 & -1 & 0\\-2 & 0 & 1\end{bmatrix}$$

By $R_1 - 7R_3$ and $R_3 + 2R_3$, we get,

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} A^{-1} = \begin{bmatrix}13 & 2 & -7\\-3 & -1 & 2\\-2 & 0 & 1\end{bmatrix}$$

$$\therefore\ A^{-1} = \begin{bmatrix}13 & 2 & -7\\-3 & -1 & 2\\-2 & 0 & 1\end{bmatrix}.$$

**9. If $A = \begin{bmatrix}1 & 0 & 1\\0 & 2 & 3\\1 & 2 & 1\end{bmatrix}$ and $B = \begin{bmatrix}1 & 2 & 3\\1 & 1 & 5\\2 & 4 & 7\end{bmatrix}$, then find matrix X such that XA = B.**

**Solution :** XA = B

$$\therefore\ X\begin{bmatrix}1 & 0 & 1\\0 & 2 & 3\\1 & 2 & 1\end{bmatrix} = \begin{bmatrix}1 & 2 & 3\\1 & 1 & 5\\2 & 4 & 7\end{bmatrix}$$

By $C_3 - C_1$, we get

$$X\begin{bmatrix}1 & 0 & 0\\0 & 2 & 3\\1 & 2 & 0\end{bmatrix} = \begin{bmatrix}1 & 2 & 2\\1 & 1 & 4\\2 & 4 & 5\end{bmatrix}$$

By $\left(\frac{1}{2}\right)C_2$, we get

$$X\begin{bmatrix}1 & 0 & 0\\0 & 1 & 3\\1 & 1 & 0\end{bmatrix} = \begin{bmatrix}1 & 1 & 2\\1 & \frac{1}{2} & 4\\2 & 2 & 5\end{bmatrix}$$

By $C_3 - 3C_2$, we get

$$X\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\1 & 1 & -3\end{bmatrix} = \begin{bmatrix}1 & 1 & -1\\1 & \frac{1}{2} & \frac{5}{2}\\2 & 2 & -1\end{bmatrix}$$

By $\left(-\frac{1}{3}\right)C_3$, we get

$$X\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\1 & 1 & 1\end{bmatrix} = \begin{bmatrix}1 & 1 & \frac{1}{3}\\1 & \frac{1}{2} & -\frac{5}{6}\\2 & 2 & \frac{1}{3}\end{bmatrix}$$

By $C_1 - C_3$ and $C_2 - C_3$, we get

$$X\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}\frac{2}{3} & \frac{2}{3} & \frac{1}{3}\\ \frac{11}{6} & \frac{4}{3} & -\frac{5}{6}\\ \frac{5}{3} & \frac{5}{3} & \frac{1}{3}\end{bmatrix}$$

$\therefore\ X=\frac{1}{6}\begin{bmatrix}4 & 4 & 2\\ 11 & 8 & -5\\ 10 & 10 & 2\end{bmatrix}$.

**10. Find matrix X, if AX = B, where** $A=\begin{bmatrix}1 & 2 & 3\\ -1 & 1 & 2\\ 1 & 2 & 4\end{bmatrix}$ **and** $B=\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$.

**Solution :** AX = B

$\therefore\ \begin{bmatrix}1 & 2 & 3\\ -1 & 1 & 2\\ 1 & 2 & 4\end{bmatrix}X=\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$

By $R_2+R_1$ and $R_3-R_1$, we get

$\begin{bmatrix}1 & 2 & 3\\ 0 & 3 & 5\\ 0 & 0 & 1\end{bmatrix}X=\begin{bmatrix}1\\ 3\\ 2\end{bmatrix}$

By $\left(\frac{1}{3}\right)R_2$, we get

$\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & \frac{5}{3}\\ 0 & 0 & 1\end{bmatrix}X=\begin{bmatrix}1\\ 1\\ 2\end{bmatrix}$

By $R_1-2R_2$, we get

$\begin{bmatrix}1 & 0 & -\frac{1}{3}\\ 0 & 1 & \frac{5}{3}\\ 0 & 0 & 1\end{bmatrix}X=\begin{bmatrix}-1\\ 1\\ 2\end{bmatrix}$

By $R_1+\frac{1}{3}R_3$ and $R_2-\frac{5}{3}R_3$, we get

$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}X=\begin{bmatrix}-\frac{1}{3}\\ -\frac{7}{3}\\ 2\end{bmatrix}$

$\therefore\ X=\begin{bmatrix}-\frac{1}{3}\\ -\frac{7}{3}\\ 2\end{bmatrix}$.

## ADDITIONAL SOLVED PROBLEMS – 2 (A)

**1. If** $A=\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 3 & 3 & 1\end{bmatrix}$**, then reduce it to $I_3$ by using column transformations.**

**Solution :**

$|A|=\begin{vmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 3 & 3 & 1\end{vmatrix}$

$=1(1-0)-0+0=1\neq 0$

$\therefore$ A is a non-singular matrix.

Hence, the required transformation is possible.

Now, $A=\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 3 & 3 & 1\end{bmatrix}$

By $C_1-2C_2$, we get

$A\sim\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -3 & 3 & 1\end{bmatrix}$

By $C_1+3C_3$ and $C_2-3C_3$, we get

$A\sim\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=I_3$.

**2. If** $A=\begin{bmatrix}2 & 1 & 3\\ 1 & 0 & 1\\ 1 & 1 & 1\end{bmatrix}$**, then reduce it to $I_3$ by using row transformations.**

**Solution :**

$|A|=\begin{vmatrix}2 & 1 & 3\\ 1 & 0 & 1\\ 1 & 1 & 1\end{vmatrix}$

$=2(0-1)-1(1-1)+3(1-0)$

$=-2-0+3=1\neq 0$

$\therefore$ A is a non-singular matrix.

Hence, the required transformation is possible.

Now, $A=\begin{bmatrix}2 & 1 & 3\\ 1 & 0 & 1\\ 1 & 1 & 1\end{bmatrix}$

By $R_1-R_2$, we get

$A\sim\begin{bmatrix}1 & 1 & 2\\ 1 & 0 & 1\\ 1 & 1 & 1\end{bmatrix}$

By $R_2 - R_1$ and $R_3 - R_1$, we get

$$A \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

By $(-1)R_2$ and $(-1)R_3$, we get

$$A \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_1 - R_2$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_1 - R_3$ and $R_2 - R_3$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3.$$

**3. If $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$, then show that $A^2 - 4A + I = 0$. Hence find $A^{-1}$.**

**Solution :**

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2 & 2+6 \\ 1+3 & 2+9 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix}$$

$\therefore\ A^2 - 4A + I$

$$= \begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix} - \begin{bmatrix} 4 & 8 \\ 4 & 12 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3-4+1 & 8-8+0 \\ 4-4+0 & 11-12+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore\ A^2 - 4A + I = 0$ ... (1)

$\because\ |A| = \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$

$= 3 - 2 = 1 \neq 0$

$\therefore\ A^{-1}$ exists.

Premultiply (1) by $A^{-1}$, we get

$A^{-1}(A^2 - 4A + I) = A^{-1} \cdot 0$

$\therefore\ A^{-1}(A \cdot A) - 4A^{-1} \cdot A + A^{-1}I = 0$

$\therefore\ (A^{-1}A)A - 4I + A^{-1} = 0$

$\therefore\ A - 4I + A^{-1} = 0$

$\therefore\ A^{-1} = 4I - A$

$$= 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

$$\therefore\ A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}.$$

**4. If $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$, verify that $A(\text{adj } A) = (\text{adj } A)A = |A| \cdot I$.**

**Solution :** $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$

$$\therefore\ |A| = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{vmatrix}$$

$= 1(0+0) + 1(9+2) + 2(0-0)$

$= 0 + 11 + 0 = 11$

First we have to find the cofactor matrix $= [A_{ij}]_{3 \times 3}$, where $A_{ij} = (-1)^{i+j} M_{ij}$

Now, $A_{11} = (-1)^{1+1} M_{11} = \begin{vmatrix} 0 & -2 \\ 0 & 3 \end{vmatrix} = 0 + 0 = 0$

$A_{12} = (-1)^{1+2} M_{12} = -\begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix} = -(9+2) = -11$

$A_{13} = (-1)^{1+3} M_{13} = \begin{vmatrix} 3 & 0 \\ 1 & 0 \end{vmatrix} = 0 - 0 = 0$

$A_{21} = (-1)^{2+1} M_{21} = -\begin{vmatrix} -1 & 2 \\ 0 & 3 \end{vmatrix} = -(-3-0) = 3$

$A_{22} = (-1)^{2+2} M_{22} = \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 3 - 2 = 1$

$A_{23} = (-1)^{2+3} M_{23} = -\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = -(0+1) = -1$

$A_{31} = (-1)^{3+1} M_{31} = \begin{vmatrix} -1 & 2 \\ 0 & -2 \end{vmatrix} = 2 - 0 = 2$

$A_{32} = (-1)^{3+2} M_{32} = -\begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix} = -(-2-6) = 8$

$A_{33} = (-1)^{3+3} M_{33} = \begin{vmatrix} 1 & -1 \\ 3 & 0 \end{vmatrix} = 0 + 3 = 3$

Hence, the cofactor matrix,

$$= \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 0 & -11 & 0 \\ 3 & 1 & -1 \\ 2 & 8 & 3 \end{bmatrix}$$

$$\therefore \text{ adj } A = \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}$$

$\therefore$ A(adj A)

$$= \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0+11+0 & 3-1-2 & 2-8+6 \\ 0+0-0 & 9+0+2 & 6+0-6 \\ 0+0+0 & 3+0-3 & 2+0+9 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix} \quad \ldots (1)$$

(adj A) A

$$= \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0+9+2 & 0+0+0 & 0-6+6 \\ -11+3+8 & 11+0+0 & -22-2+24 \\ 0-3+3 & 0-0+0 & 0+2+9 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix} \quad \ldots (2)$$

$$|A| \cdot I = 11 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix} \quad \ldots (3)$$

From (1), (2) and (3), we get

$A(\text{adj } A) = (\text{adj } A)A = |A| \cdot I$.

**Note :** This relation is valid for any non-singular matrix A.

## EXAMPLES FOR PRACTICE 2.5

**1.** If $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$, then reduce it to $I_3$ by using row transformations.

**2.** If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 4 & 1 & 1 \end{bmatrix}$, then reduce it to $I_3$ by using column transformations.

**3.** Find the matrix X, such that

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix} X = \begin{bmatrix} 2 & 2 & -5 \\ -2 & -1 & 4 \\ 1 & 0 & -1 \end{bmatrix}$$

**4.** If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and X is a $2 \times 2$ matrix such that $XA = I$, then find X.

**5. Find which of the following matrices are invertible :**

(i) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix}$ (iv) $\begin{bmatrix} 2 & 3 \\ 10 & 15 \end{bmatrix}$

(v) $\begin{bmatrix} 3 & 4 & 3 \\ 1 & 1 & 0 \\ 1 & 4 & 5 \end{bmatrix}$ (vi) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 3 \\ 1 & 2 & 3 \end{bmatrix}$

(vii) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 6 & 8 \end{bmatrix}$.

**6.** Find AB, if $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 1 & 2 \\ 1 & -2 \end{bmatrix}$.

Hence, determine if AB has the inverse.

**7. Find the inverses of the following matrices :**

(i) $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$ (iii) $\begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ (v) $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$ (vi) $\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$.

**8.** If $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$, then find $(AB)^{-1}$.

**9. Find the inverse of each of the following matrices (if they exist) :**

(i) $\begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

(iii) $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ (iv) $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

(v) $\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$ (vi) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{bmatrix}$

(vii) $\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix}$ (viii) $\begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$.

**10.** Find the inverse of $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ by elementary column transformations.

**11.** If the matrix $A = \begin{bmatrix} 1 & 3 \\ 0 & 3 \end{bmatrix}$ satisfies the equation $A^2 - 4A + 3I = 0$, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and 0 is a zero matrix of order 2, find $A^{-1}$.

**12.** If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = 0$. Hence find $A^{-1}$.

**13. Find the adjoints of the following matrices :**

(i) $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & -3 & 1 \\ -1 & -1 & 3 \end{bmatrix}$ (iv) $\begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & 2 \\ -1 & 1 & 2 \end{bmatrix}$.

**14. Find the inverses of the following matrices by using adjoint method :**

(i) $\begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$

(iii) $\begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 7 & -6 & -2 \\ -18 & 16 & 5 \\ -10 & 9 & 3 \end{bmatrix}$

(v) $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & -1 \\ -4 & 5 & 2 \end{bmatrix}$ (vi) $\begin{bmatrix} 6 & 2 & -2 \\ -3 & 7 & 1 \\ 3 & 5 & -1 \end{bmatrix}$

(vii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ (viii) $\begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}$

(ix) $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ (x) $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$.

**15.** If $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$, verify that

$A(\text{adj } A) = (\text{adj } A)A = |A| \cdot I$.

**16.** If $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$, find AB and $(AB)^{-1}$. Verify that $(AB)^{-1} = B^{-1}A^{-1}$.

## Answers

**1.** Use $R_2 + R_1$, $R_1 + 2R_3$, $R_1 - 2R_2$ and $R_3 + 2R_2$, $R_1 + 2R_3$ to get $I_3$

**2.** Use $C_3 - C_1$, $C_3 - C_2$, $(-\frac{1}{4})C_3$, $C_1 - 4C_3$ and $C_2 - C_3$ to get $I_3$

**3.** $\begin{bmatrix} -5 & 2 & 3 \\ 2 & -3 & 2 \\ 1 & 2 & -4 \end{bmatrix}$

**4.** $\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

**5.** (i), (iii), (v) are invertible.

(ii), (iv), (vi), (vii) are not invertible.

**6.** $AB = \begin{bmatrix} 6 & -3 \\ -4 & 1 \end{bmatrix}$, $(AB)^{-1}$ exists.

**7.** (i) $\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$ (ii) $\begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$ (iii) $\frac{1}{7}\begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix}$

(iv) $\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$ (v) $\begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$ (vi) $\begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$.

**8.** $\begin{bmatrix} 1 & -1 \\ -\frac{4}{5} & 1 \end{bmatrix}$

**9.** (i) $\begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ (iv) $\frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$

(v) $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ (vi) $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -3 & 1 \end{bmatrix}$

(vii) $\frac{1}{5}\begin{bmatrix} 4 & -2 & -1 \\ 1 & 2 & -4 \\ -2 & 1 & 3 \end{bmatrix}$ (viii) $\frac{1}{5}\begin{bmatrix} -2 & 0 & 3 \\ -1 & 1 & 0 \\ 2 & 1 & -2 \end{bmatrix}$.

**10.** $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ **11.** $\begin{bmatrix} 1 & -1 \\ 0 & \frac{1}{3} \end{bmatrix}$

**12.** $\frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

**13.** (i) $\begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

(iii) $\begin{bmatrix} -8 & -3 & 1 \\ -1 & 6 & -2 \\ -3 & 1 & -6 \end{bmatrix}$ (iv) $\begin{bmatrix} 0 & -1 & 1 \\ -8 & 3 & -7 \\ 4 & -2 & 2 \end{bmatrix}$.

**14.** (i) $\frac{1}{13}\begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$ (ii) $\frac{1}{14}\begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$

(iii) $\frac{1}{2}\begin{bmatrix} -2 & 4 \\ 3 & -5 \end{bmatrix}$ (iv) $\begin{bmatrix} 3 & 0 & 2 \\ 4 & 1 & 1 \\ -2 & -3 & 4 \end{bmatrix}$

(v) $\frac{1}{25}\begin{bmatrix} 9 & 19 & -4 \\ 4 & 14 & 1 \\ 8 & 3 & 2 \end{bmatrix}$ (vi) $\frac{1}{12}\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 3 & 2 & -4 \end{bmatrix}$

(vii) $\frac{1}{18}\begin{bmatrix} -5 & 1 & 7 \\ 1 & 7 & -5 \\ 7 & -5 & 1 \end{bmatrix}$ (viii) $\begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}$

(ix) $\frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$ (x) $\frac{1}{3}\begin{bmatrix} 3 & 0 & 0 \\ -3 & 1 & 0 \\ 9 & 2 & -3 \end{bmatrix}$.

**16.** $AB = \begin{bmatrix} 11 & 3 \\ 7 & 2 \end{bmatrix}$, $(AB)^{-1} = \begin{bmatrix} 2 & -3 \\ -7 & 11 \end{bmatrix}$,

$A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$.

## 2.7 : APPLICATIONS OF MATRICES (Solution of System of Linear Equations)

You have already learnt the algebra of matrices and the inverse of a matrix. Now, we shall study the application of matrices for solving a system of linear equations. Consider the equations

$a_1x + b_1y = c_1$

$a_2x + b_2y = c_2$

These equations can be written as

$$\begin{bmatrix} a_1x + b_1y \\ a_2x + b_2y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

i.e. $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

i.e. $AX = B$

This is called the matrix form of the given equations, where A is the matrix of coefficients of $x$ and $y$ which is called **coefficient matrix** and it is of order $2 \times 2$, X is a matrix of variables $x$, $y$ and it is of order $2 \times 1$ and B is the matrix of the constants which is of the order $2 \times 1$.

Similarly, if there are three equations in three variables $x$, $y$, $z$, then as above they can be written as $AX = B$, where A is the coefficient matrix of order $3 \times 3$, X is a matrix of variables and it is of order $3 \times 1$ and B is a matrix of constants which is of order $3 \times 1$.

Using the matrix form $AX = B$, we can find the values of the variables which are the solution of the given equations.

There are two methods to find the solution of the given equations :

(1) Method of inversion

(2) Method of reduction.

### (1) Method of Inversion

First, we write the given linear equations in the matrix form as : $AX = B$.

If the solution of the given equations exists, then the matrix A is non-singular, i.e. $|A| \neq 0$

$\therefore$ $A^{-1}$ exists.

Now, premultiply $AX = B$ by $A^{-1}$, we get

$A^{-1}(AX) = A^{-1}B$

$\therefore$ $(A^{-1}A)X = A^{-1}B$

$\therefore$ $IX = A^{-1}B$

$\therefore$ $X = A^{-1}B$

By comparing both the sides, we can find $x$, $y$, $z$. This method is called method of inversion.

Let us illustrate this method in the following example :

**Ex. Solve the following equations by the method of inversion :**

**$3x - y = 5$, $2x - 3y = 8$.**

**Solution :** The given equations can be written in the matrix form as : $\begin{bmatrix} 3 & -1 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$

This is of the form $AX = B$, where

$A = \begin{bmatrix} 3 & -1 \\ 2 & -3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$

Let us find $A^{-1}$.

$|A| = \begin{vmatrix} 3 & -1 \\ 2 & -3 \end{vmatrix} = -9 + 2 = -7 \neq 0$

$\therefore$ $A^{-1}$ exists

Consider $AA^{-1} = I$

$$\therefore \begin{bmatrix} 3 & -1 \\ 2 & -3 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

By $R_1 - R_2$, we get

$$\begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

By $R_2 - 2R_1$, we get

$$\begin{bmatrix} 1 & 2 \\ 0 & -7 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

By $\left(-\frac{1}{7}\right) R_2$, we get

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & -1 \\ 2/7 & -3/7 \end{bmatrix}$$

By $R_1 - 2R_2$, we get

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 3/7 & -1/7 \\ 2/7 & -3/7 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} 3/7 & -1/7 \\ 2/7 & -3/7 \end{bmatrix}$$

Now, premultiply $AX = B$ by $A^{-1}$, we get

$$A^{-1}(AX) = A^{-1}B$$

$\therefore (A^{-1}A)X = A^{-1}B \quad \therefore IX = A^{-1}B$

$$\therefore X = \begin{bmatrix} 3/7 & -1/7 \\ 2/7 & -3/7 \end{bmatrix} \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{15}{7} - \frac{8}{7} \\ \frac{10}{7} - \frac{24}{7} \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

By equality of matrices,

$x = 1$, $y = -2$ is the required solution.

### (2) Method of Reduction

First, we write the given linear equations in the matrix form : $AX = B$.

In reduction method, we obtain $\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}$ in the first column of the coefficient matrix A by using elementary row transformations. Then we obtain another '0' in $R_2$ or $R_3$ by using $R_2$ and $R_3$ only.

The same row transformations are performed simultaneously on the matrix B.

Now, we can multiply the matrices in LHS and write the equivalent equations which can be easily solved.

Let us illustrate this in the following example :

**Ex. Solve the following equations by using reduction method :**

$x + 2y + 3z = 9$, $2x + 3y + z = 4$,

$4x + 5y + 4z = 15$.

**Solution :** The given equations can be written in the matrix form as :

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 4 & 5 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \\ 15 \end{bmatrix}$$

By $R_2 - 2R_1$ and $R_3 - 4R_1$, we get

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & -3 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -14 \\ -21 \end{bmatrix}$$

By $R_3 - 3R_2$, we get

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -14 \\ 21 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x + 2y + 3z \\ 0 - y - 5z \\ 0 + 0 + 7z \end{bmatrix} = \begin{bmatrix} 9 \\ -14 \\ 21 \end{bmatrix}$$

By equality of matrices,

$x + 2y + 3z = 9$ ... (1)

$-y - 5z = -14$ ... (2)

$7z = 21$ ... (3)

$\therefore z = 3$

$\therefore$ from (2),

$-y - 15 = -14$

$\therefore y = -1$

$\therefore$ from (1),

$x - 2 + 9 = 9 \quad \therefore x = 2$

$\therefore x = 2, y = -1, z = 3.$

**EXERCISE 2.6** | **Textbook pages 79 and 80**

**1. Solve the following equations by the method of inversion :**

**(i) $x + 2y = 2$, $2x + 3y = 3$.**

**Solution :** The given equations can be written in the matrix form as :

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

This is of the form $AX = B$, where

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Let us find $A^{-1}$.

$$|A| = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3 - 4 = -1 \neq 0$$

$\therefore$ $A^{-1}$ exists.

We write $AA^{-1} = I$

$$\therefore \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

By $R_2 - 2R_1$, we get

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

By $(-1)R_2$, we get

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$$

By $R_1 - 2R_2$, we get

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$$

Now, premultiply $AX = B$ by $A^{-1}$, we get

$A^{-1}(AX) = A^{-1}B$

$\therefore$ $(A^{-1}A)X = A^{-1}B$

$\therefore$ $IX = A^{-1}B$

$$\therefore X = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6+6 \\ 4-3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

By equality of matrices,

$x = 0$, $y = 1$ is the required solution.

**(ii) $2x + y = 5$, $3x + 5y = -3$.**

**Solution :** *Refer to the solution of Q. 1 (i).*

**Ans.** $x = 4$, $y = -3$.

**(iii) $2x - y + z = 1$, $x + 2y + 3z = 8$ and $3x + y - 4z = 1$.**

**Solution :** The given equations can be written in the matrix form as :

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 1 \end{bmatrix}$$

This is of the form $AX = B$, where

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & -4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 8 \\ 1 \end{bmatrix}$$

Let us find $A^{-1}$.

$$|A| = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & -4 \end{vmatrix}$$

$= 2(-8-3) + 1(-4-9) + 1(1-6)$

$= -22 - 13 - 5 = -40 \neq 0$

$\therefore$ $A^{-1}$ exists.

We write $AA^{-1} = I$

$$\therefore \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & -4 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_1 \leftrightarrow R_2$, we get

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & -4 \end{bmatrix} A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_2 - 2R_1$ and $R_3 - 3R_1$, we get

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 0 & -5 & -13 \end{bmatrix} A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

By $\left(-\frac{1}{5}\right) R_2$, we get

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -5 & -13 \end{bmatrix} A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{5} & \frac{2}{5} & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

By $R_1 - 2R_2$ and $R_3 + 5R_2$, we get

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -8 \end{bmatrix} A^{-1} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} & 0 \\ -\frac{1}{5} & \frac{2}{5} & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

By $\left(-\frac{1}{8}\right) R_3$, we get

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} & 0 \\ -\frac{1}{5} & \frac{2}{5} & 0 \\ \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \end{bmatrix}$$

By $R_1 - R_3$ and $R_2 - R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} \frac{11}{40} & \frac{3}{40} & \frac{1}{8} \\ -\frac{13}{40} & \frac{11}{40} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \end{bmatrix}$$

$$\therefore \ A^{-1} = \frac{1}{40} \begin{bmatrix} 11 & 3 & 5 \\ -13 & 11 & 5 \\ 5 & 5 & -5 \end{bmatrix}$$

Now, premultiply $AX = B$ by $A^{-1}$, we get

$A^{-1}(AX) = A^{-1}B$

$\therefore \ (A^{-1}A)X = A^{-1}B$

$\therefore \ IX = A^{-1}B$

$$\therefore \ X = \frac{1}{40} \begin{bmatrix} 11 & 3 & 5 \\ -13 & 11 & 5 \\ 5 & 5 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 8 \\ 1 \end{bmatrix}$$

$$= \frac{1}{40} \begin{bmatrix} 11 + 24 + 5 \\ -13 + 88 + 5 \\ 5 + 40 - 5 \end{bmatrix} = \frac{1}{40} \begin{bmatrix} 40 \\ 80 \\ 40 \end{bmatrix}$$

$$\therefore \ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

By equality of matrices,

$x = 1$, $y = 2$, $z = 1$ is the required solution.

**(iv) $x + y + z = 1$, $x - y + z = 2$ and $x + y - z = 3$.**

**Solution :** *Refer to the solution of Q. 1 (iii).*

**Ans.** $x = \frac{5}{2}, y = -\frac{1}{2}, z = -1.$

**2. Express the following equations in matrix form and solve them by method of reduction :**

**(i) $x + 3y = 2$, $3x + 5y = 4$.**

**Solution :** The given equations can be written in the matrix form as :

$$\begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

By $R_2 - 3R_1$, we get

$$\begin{bmatrix} 1 & 3 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$\therefore \ \begin{bmatrix} x + 3y \\ 0 - 4y \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

By equality of matrices,

$x + 3y = 2$ ... (1)

$-4y = -2$ ... (2)

From (2), $y = \frac{1}{2}$

Substituting $y = \frac{1}{2}$ in (1), we get

$x + \frac{3}{2} = 2$

$\therefore \ x = 2 - \frac{3}{2} = \frac{1}{2}$

Hence, $x = \frac{1}{2}, y = \frac{1}{2}$ is the required solution.

**(ii) $3x - y = 1$, $4x + y = 6$.**

**Solution :** The given equations can be written in the matrix form as :

$$\begin{bmatrix} 3 & -1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

By $4R_1$ and $3R_2$, we get

$$\begin{bmatrix} 12 & -4 \\ 12 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 18 \end{bmatrix}$$

By $R_2 - R_1$, we get

$$\begin{bmatrix} 12 & -4 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 14 \end{bmatrix}$$

$$\therefore \ \begin{bmatrix} 12x - 4y \\ 0 + 7y \end{bmatrix} = \begin{bmatrix} 4 \\ 14 \end{bmatrix}$$

By equality of matrices,

$12x - 4y = 4$ ... (1)

$7y = 14$ ... (2)

From (2), $y = 2$

Substituting $y = 2$ in (1), we get

$12x - 8 = 4$

$\therefore \ 12x = 12 \quad \therefore \ x = 1$

Hence, $x = 1$, $y = 2$ is the required solution.

**(iii) $x + 2y + z = 8$, $2x + 3y - z = 11$ and $3x - y - 2z = 5$.**

**Solution :** The given equations can be written in the matrix form as :

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \\ 3 & -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ 5 \end{bmatrix}$$

By $R_2 - 2R_1$ and $R_3 - 3R_1$, we get

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -3 \\ 0 & -7 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \\ -19 \end{bmatrix}$$

By $R_3 - 7R_2$, we get

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & 16 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \\ 16 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x+2y+z \\ 0-y-3z \\ 0+0+16z \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \\ 16 \end{bmatrix}$$

By equality of matrices,

$x + 2y + z = 8$ ... (1)

$-y - 3z = -5$ ... (2)

$16z = 16$ ... (3)

From (3), $z = 1$

Substituting $z = 1$ in (2), we get

$-y - 3 = -5$, $\therefore y = 2$

Substituting $y = 2$, $z = 1$ in (1), we get

$x + 4 + 1 = 8$ $\quad \therefore x = 3$

Hence, $x = 3$, $y = 2$, $z = 1$ is the required solution.

**(iv) $x + y + z = 1$, $2x + 3y + 2z = 2$ and $x + y + 2z = 4$.**

**Solution :** The given equations can be written in the matrix form as :

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

By $R_2 - 2R_1$ and $R_3 - R_1$, we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x+y+z \\ 0+y+0 \\ 0+0+z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

By equality of matrices,

$x + y + z = 1$ ... (1)

$y = 0$

$z = 3$

Substituting $y = 0$, $z = 3$ in (1), we get

$x + 0 + 3 = 1$

$\therefore x = -2$

Hence, $x = -2$, $y = 0$, $z = 3$ is the required solution.

**3. The total cost of 3 T.V. and 2 V.C.R. is ₹ 35000. The shopkeeper wants profit of ₹ 1000 per T.V. and ₹ 500 per V.C.R. He sell 2 T.V. and 1 V.C.R. and he gets total revenue of ₹ 21500. Find the cost and selling price of T.V. and V.C.R.**

**Solution :** Let the cost of each T.V. be ₹ $x$ and each V.C.R. be ₹ $y$.

Then the total cost of 3 T.V. and 2 V.C.R. is ₹ $(3x + 2y)$ which is given to be ₹ 35000.

$\therefore 3x + 2y = 35000$

The shopkeeper wants profit of ₹ 1000 per T.V. and ₹ 500 per V.C.R.

The selling price of each T.V. is ₹ $(x + 1000)$ and of each V.C.R. is ₹ $(y + 500)$.

$\therefore$ selling price of 2 T.V. and 1 V.C.R is

₹ $[2(x + 1000) + (y + 500)]$ which is given to be ₹ 21500.

$\therefore 2(x + 1000) + (y + 500) = 21500$

$\therefore 2x + 2000 + y + 500 = 21500$

$\therefore 2x + y = 19000$

Hence, the system of linear equations is

$3x + 2y = 35000$

$2x + y = 19000$

The equations can be written in matrix form as :

$$\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 35000 \\ 19000 \end{bmatrix}$$

By $R_1 - 2R_2$, we get

$$\begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3000 \\ 19000 \end{bmatrix}$$

$$\therefore \begin{bmatrix} -x+0 \\ 2x+y \end{bmatrix} = \begin{bmatrix} -3000 \\ 19000 \end{bmatrix}$$

By equality of matrices,

$-x = -3000$ ... (1)

$2x + y = 19000$ ... (2)

From (1), $x = 3000$

Substituting $x = 3000$ in (2), we get

$2(3000) + y = 19000$

$\therefore y = 19000 - 6000 = 13000$

Hence, the cost price of one T.V. is ₹ 3000 and of one V.C.R. is ₹ 13000 and the selling price of one T.V. is ₹ 4000 and of one V.C.R. is ₹ 13500.

**4. The sum of the cost of one Economics book, one Cooperation book and one Account book is ₹ 420. The total cost of an Economic book, 2 Cooperation books and an Account book is ₹ 480. Also the total cost of an Economic book, 3 Cooperation books and 2 Account books is ₹ 600. Find the cost of each book.**

**Solution :** Let the cost of 1 Economic book, 1 Cooperation book and 1 Account book be ₹ $x$, ₹ $y$ and ₹ $z$ respectively.

Then, from the given information

$x+y+z \quad = 420$

$x+2y+z \quad = 480$

$x+3y+2z = 600$

These equations can be written in matrix form as :

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 420 \\ 480 \\ 600 \end{bmatrix}$$

By $R_2 - R_1$ and $R_3 - R_1$, we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 420 \\ 60 \\ 180 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x+y+z \\ 0+y+0 \\ 0+2y+z \end{bmatrix} = \begin{bmatrix} 420 \\ 60 \\ 180 \end{bmatrix}$$

By equality of matrices,

$x+y+z = 420 \qquad \dots (1)$

$y = 60$

$2y+z = 180 \qquad \dots (2)$

Substituting $y = 60$ in (2), we get

$2(60)+z = 180$

$\therefore \ z = 180 - 120 = 60$

Substituting $y = 60$, $z = 60$ in (1), we get

$x + 60 + 60 = 420$

$\therefore \ x = 420 - 120 = 300$

Hence, the cost of each Economic book is ₹ 300, each Cooperation book is ₹ 60 and each Account book is ₹ 60.

## ADDITIONAL SOLVED PROBLEMS – 2 (B)

**1. Solve the following equations by method of inversion :**

**$5x - y + 4z = 5$, $2x + 3y + 5z = 2$ and $5x - 2y + 6z = -1$.**

**Solution :** The given equations can be written in the matrix form as :

$$\begin{bmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$$

This is of the form AX = B, where

$$A = \begin{bmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$$

Let us find $A^{-1}$.

$$|A| = \begin{vmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{vmatrix}$$

$= 5(18+10) + 1(12-25) + 4(-4-15)$

$= 140 - 13 - 76 = 51 \neq 0$

$\therefore \ A^{-1}$ exists.

We write $AA^{-1} = I$

$$\therefore \begin{bmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_1 - 2R_2$, we get

$$\begin{bmatrix} 1 & -7 & -6 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_2 - 2R_1$ and $R_3 - 5R_1$, we get

$$\begin{bmatrix} 1 & -7 & -6 \\ 0 & 17 & 17 \\ 0 & 33 & 36 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ -5 & 10 & 1 \end{bmatrix}$$

By $\left(\frac{1}{17}\right) R_2$, we get

$$\begin{bmatrix} 1 & -7 & -6 \\ 0 & 1 & 1 \\ 0 & 33 & 36 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & -2 & 0 \\ -2/17 & 5/17 & 0 \\ -5 & 10 & 1 \end{bmatrix}$$

By $R_1 + 7R_2$ and $R_3 - 33R_2$, we get

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix} A^{-1} = \begin{bmatrix} 3/17 & 1/17 & 0 \\ -2/17 & 5/17 & 0 \\ -19/17 & 5/17 & 1 \end{bmatrix}$$

By $\left(\frac{1}{3}\right)R_3$, we get

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 3/17 & 1/17 & 0 \\ -2/17 & 5/17 & 0 \\ -19/51 & 5/51 & 1/3 \end{bmatrix}$$

By $R_1 - R_3$ and $R_2 - R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 28/51 & -2/51 & -1/3 \\ 13/51 & 10/51 & -1/3 \\ -19/51 & 5/51 & 1/3 \end{bmatrix}$$

$$\therefore\ A^{-1} = \frac{1}{51}\begin{bmatrix} 28 & -2 & -17 \\ 13 & 10 & -17 \\ -19 & 5 & 17 \end{bmatrix}$$

Now, premultiply $AX = B$ by $A^{-1}$, we get

$A^{-1}(AX) = A^{-1}B$

$\therefore\ (A^{-1}A)X = A^{-1}B$

$\therefore\ IX = A^{-1}B$

$$\therefore\ X = \frac{1}{51}\begin{bmatrix} 28 & -2 & -17 \\ 13 & 10 & -17 \\ -19 & 5 & 17 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$$

$$= \frac{1}{51}\begin{bmatrix} 140-4+17 \\ 65+20+17 \\ -95+10-17 \end{bmatrix} = \frac{1}{51}\begin{bmatrix} 153 \\ 102 \\ -102 \end{bmatrix}$$

$$\therefore\ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}$$

By equality of matrices,

$x = 3,\ y = 2,\ z = -2$ is the required solution.

**2. Solve the following equation by the method of reduction :**

$$x + y = 1,\ y + z = \frac{5}{3},\ z + x = \frac{4}{3}.$$

**Solution :** The given equations can be written in the matrix form as :

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{5}{3} \\ \frac{4}{3} \end{bmatrix}$$

By $R_3 - R_1$, we get $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{5}{3} \\ \frac{1}{3} \end{bmatrix}$

By $R_3 + R_2$, we get

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{5}{3} \\ 2 \end{bmatrix}$$

$$\therefore\ \begin{bmatrix} x+y+0 \\ 0+y+z \\ 0+0+2z \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{5}{3} \\ 2 \end{bmatrix}$$

By equality of matrices,

$x + y = 1$ ... (1)

$y + z = \frac{5}{3}$ ... (2)

$2z = 2$ ... (3)

From (3), $z = 1$

Substituting $z = 1$ in (2), we get

$y + 1 = \frac{5}{3}$ $\quad \therefore\ y = \frac{2}{3}$

Substituting $y = \frac{2}{3}$ in (1), we get

$x + \frac{2}{3} = 1$ $\quad \therefore\ x = \frac{1}{3}$

Hence, $x = \frac{1}{3},\ y = \frac{2}{3},\ z = 1$ is the required solution.

**3. The sum of three numbers is 6. If we multiply third number by 3 and add it to the second number we get 11. By adding first and third numbers we get a number, which is double the second number. Use these informations, find a system of linear equations. Find these three numbers using matrices.**

**Solution :** Let the three numbers be $x$, $y$ and $z$.

$\therefore\ x + y + z = 6.$

According to the given conditions,

$3z + y = 11$, i.e. $y + 3z = 11$

and $x + z = 2y$, i.e. $x - 2y + z = 0$

Hence, the system of the linear equations is

$x + y + z = 6$

$y + 3z = 11$

$x - 2y + z = 0$

These equations can be written in the matrix form as :

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$$

By $R_3 - R_1$, we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \\ -6 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x + y + z \\ 0 + y + 3z \\ 0 - 3y + 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \\ -6 \end{bmatrix}$$

By equality of matrices,

$x + y + z = 6$ ... (1)

$y + 3z = 11$ ... (2)

$-3y = -6$ ... (3)

From (3), $y = 2$

Substituting $y = 2$ in (2), we get

$2 + 3z = 11$

$\therefore 3z = 9 \quad \therefore z = 3$

Put $y = 2$, $z = 3$ in (1), we get

$x + 2 + 3 = 6 \quad \therefore x = 1$

$\therefore x = 1, y = 2, z = 3$

Hence, the required numbers are 1, 2 and 3.

## EXAMPLES FOR PRACTICE 2.6

**1. Solve the following equations by the method of inversion :**

(i) $x + y = 5$, $3x - 2y = 5$

(ii) $4x + 3y = 1$, $2x + y = 1$

(iii) $2x + 3y = 5$, $6x - 2y = 4$

(iv) $x - y + z = 4$, $2x + y - 3z = 0$, $x + y + z = 2$

(v) $x + y + z = 2$, $2x - 3y + 2z = -6$, $x + y - 3z = 6$

(vi) $2x + y + z = 2$, $x + y + z = 0$, $4x - y - 3z = 20$

(vii) $x + z = -1$, $y + z = -4$, $4x + y + z = 0$

(viii) $x + 3y + 3z = 12$, $x + 4y + 4z = 15$, $x + 3y + 4z = 13.$

**2. Solve the following equations by the method of reduction :**

(i) $x - 2y = 3$, $3x + 4y = 19$

(ii) $x + y = 5$, $2x - y = 4$

(iii) $2x - y = -2$, $3x + 4y = 3$

(iv) $2x + 3y = 9$, $y - x = -2$

(v) $x + y + z = 1$, $2x + 3y + 2z = 2$, $x + y + 2z = 4$

(vi) $x + y + 2z = 3$, $x + 2y + 2z = 3$, $2x + 2y + 5z = 6$

(vii) $x + y + z = 3$, $3x - 2y + 3z = 4$, $5x + 5y + z = 11$

(viii) $x - y + z = 2$, $2x + y - z = 7$, $x + 2y + z = 8$

(ix) $x + 3y + 3z = 16$, $x + 4y + 4z = 21$, $x + 3y + 4z = 19$

(x) $x + y + z = 6$, $3x - y + 3z = 10$, $5x + y - 4z = 3.$

(xi) $x - y + z = 1$, $2x - y = 1$, $3x + 3y - 4z = 2$

(xii) $x + 2y + z = 8$, $2x + 3y - z = 11$, $3x - y - 2z = 5.$

**3.** Three chairs and two tables cost ₹ 1850. Five chairs and three tables cost ₹ 2850. Find the cost of four chairs and one table, by using matrices.

**4.** The cost of half a dozen of pencils, 2 erasers and 2 sharpeners is ₹ 14. The cost of 15 pencils, 5 erasers and 3 sharpeners is ₹ 35 and the cost of 4 pencils, 1 eraser and 1 sharpener is ₹ 9. Find the cost of each item, by using matrices.

**5.** If three numbers are added, their sum is 2. If 2 times the second number is subtracted from the sum of first and third numbers, we get 8 and if three times the first number is added to the sum of second and third numbers, we get 4. Find the numbers using matrices.

### Answers

**1.** (i) 3, 2 (ii) 1, $-1$ (iii) 1, 1 (iv) 2, $-1$, 1 (v) 1, 2, $-1$ (vi) 2, 3, $-5$ (vii) 1, $-2$, $-2$ (viii) 3, 2, 1.

**2.** (i) 5, 1 (ii) 3, 2 (iii) $\frac{-5}{11}, \frac{12}{11}$ (iv) 3, 1 (v) $-2$, 0, 3 (vi) 3, 0, 0 (vii) 1, 1, 1 (viii) 3, 2, 1 (ix) 1, 2, 3 (x) $\frac{17}{9}, 2, \frac{19}{9}$ (xi) 1, 1, 1 (xii) 3, 2, 1.

**3.** ₹ 1300

**4.** The price of pencil is ₹ 2, the price of eraser is ₹ 1 and the price of sharpener is 0, i.e. sharpener is offered free to each buyer.

**5.** 1, $-2$, 3.

## MISCELLANEOUS EXERCISE – 2

**(Textbook pages 81 to 86)**

**(I) Choose the correct alternative :**

1. If $AX = B$, where $A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then $X = \ldots\ldots\ldots$

(a) $\begin{bmatrix} 3 \\ 5 \\ 3 \\ 7 \end{bmatrix}$ (b) $\begin{bmatrix} 7 \\ 3 \\ 5 \\ 3 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

2. The matrix $\begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$ is

(a) identity matrix (b) scalar matrix
(c) null matrix (d) diagonal matrix

3. The matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is

(a) identity matrix (b) diagonal matrix
(c) scalar matix (d) null matrix

4. If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$, then $|\text{adj } A| = \ldots\ldots\ldots$

(a) $a^{12}$ (b) $a^9$ (c) $a^6$ (d) $a^{-3}$

5. Adjoint of $\begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix}$ is

(a) $\begin{bmatrix} -6 & 3 \\ -4 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 6 & 3 \\ -4 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} -6 & 3 \\ 4 & -2 \end{bmatrix}$

6. If $A = \text{diag. } [d_1, d_2, d_3, \ldots, d_n]$, where $d_i \neq 0$, for $i = 1, 2, 3, \ldots, n$, then $A^{-1} = \ldots\ldots\ldots$

(a) diag $[1/d_1, 1/d_2, 1/d_3, \ldots, 1/d_n]$ (b) D
(c) 1 (d) 0

7. If $A^2 + mA + nI = 0$ and $n \neq 0$, $|A| \neq 0$, then $A^{-1} = \ldots\ldots\ldots$

(a) $\frac{-1}{m}(A + nI)$ (b) $\frac{-1}{n}(A + mI)$

(c) $\frac{-1}{n}(I + mA)$ (d) $(A + mnI)$

8. If a $3 \times 3$ matrix B has its inverse equal to B, then $B^2 = \ldots\ldots\ldots$

(a) $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

9. If $A = \begin{bmatrix} \alpha & 4 \\ 4 & \alpha \end{bmatrix}$ and $|A^3| = 729$, then $\alpha = \ldots\ldots\ldots$

(a) $\pm 3$ (b) $\pm 4$ (c) $\pm 5$ (d) $\pm 6$

10. If A and B square matrices of order $n \times n$ such that $A^2 - B^2 = (A - B)(A + B)$, then which of the following will be always true?

(a) $AB = BA$
(b) either of A or B is a zero matrix
(c) either of A and B is an identity matrix
(d) $A = B$

11. If $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$, then $A^{-1} = \ldots\ldots\ldots$

(a) $\begin{bmatrix} 3 & -5 \\ 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & 5 \\ -1 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 3 & -5 \\ 1 & -2 \end{bmatrix}$

12. If A is a $2 \times 2$ matrix such that $A(\text{adj } A) = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$, then $|A| = \ldots\ldots\ldots$

(a) 0 (b) 5 (c) 10 (d) 25

13. If A is a non-singular matrix, then $\det(A^{-1}) = \ldots\ldots\ldots$

(a) 1 (b) 0 (c) $\det(A)$ (d) $1/\det(A)$

14. If $A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 1 & 5 \end{bmatrix}$, then $AB = \ldots\ldots\ldots$

(a) $\begin{bmatrix} 1 & -10 \\ 1 & 20 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 10 \\ -1 & 20 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 10 \\ 1 & 20 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 10 \\ -1 & -20 \end{bmatrix}$

[**Note** : *Question is modified.*]

15. If $x + y + z = 3$, $x + 2y + 3z = 4$, $x + 4y + 9z = 6$, then $(y, z) = \ldots\ldots\ldots$

(a) $(-1, 0)$ (b) $(1, 0)$ (c) $(1, -1)$ (d) $(-1, 1)$

## Answers

1. (c) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ 2. (b) scalar matrix 3. (d) null matrix

4. (c) $a^6$

[**Hint** : $\text{adj } A = \begin{bmatrix} a^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2 \end{bmatrix}$

$\therefore |\text{adj } A| = a^2(a^4 - 0) = a^6$.]

5. (a) $\begin{bmatrix} -6 & 3 \\ -4 & 2 \end{bmatrix}$

6. (a) diag $[1/d_1, 1/d_2, 1/d_3, \ldots, 1/d_n]$

7. (b) $\frac{-1}{n}(A+mI)$

[Hint : $A^2+mA+nI=0$

$\therefore (A^2+mA+nI)\cdot A^{-1}=0\cdot A^{-1}$

$\therefore A(AA^{-1})+m(AA^{-1})+nIA^{-1}=0$

$\therefore AI+mI+nA^{-1}=0$

$\therefore nA^{-1}=-A-mI$

$\therefore A^{-1}=-\frac{1}{n}(A+mI)$]

8. (d) $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

[Hint : $B^{-1}=B$ $\therefore B^2=B\cdot B^{-1}=I.$]

9. (c) $\pm 5$

[Hint : $|A|=\begin{vmatrix}\alpha&4\\4&\alpha\end{vmatrix}=\alpha^2-16$

$\therefore |A^3|=|A|^3=(\alpha^2-16)^3=729$

$\therefore \alpha^2-16=9$ $\therefore \alpha^2=25$

$\therefore \alpha=\pm 5.$]

10. (a) $AB=BA$

[Hint : $A^2-B^2=(A-B)(A+B)$

$=A^2+AB-BA-B^2$

$\therefore 0=AB-BA$ $\therefore AB=BA.$]

11. (b) $\begin{bmatrix}3&-5\\-1&2\end{bmatrix}$

12. (b) 5 [Hint : $A(\text{adj } A)=|A|\cdot I.$]

13. (d) 1 / det (A)

[Hint : $AA^{-1}=I$ $\therefore |A|\cdot|A^{-1}|=1$

$\therefore |A^{-1}|=\frac{1}{|A|}.$]

14. (c) $\begin{bmatrix}1&10\\1&20\end{bmatrix}$

15. (b) (1, 0).

**(II) Fill in the blanks :**

1. $A=\begin{bmatrix}3\\1\end{bmatrix}$ is ......... matrix.

2. Order of matrix $\begin{bmatrix}2&1&1\\5&1&8\end{bmatrix}$ is .........

3. If $A=\begin{bmatrix}4&x\\6&3\end{bmatrix}$ is a singular matrix, then $x$ is .........

4. Matrix $B=\begin{bmatrix}0&3&1\\-3&0&-4\\p&4&0\end{bmatrix}$ is a skew-symmetric, then value of $p$ is .........

5. If $A=[a_{ij}]_{2\times 3}$ and $B=[b_{ij}]_{m\times 1}$ and AB is defined, then $m=$ .........

6. If $A=\begin{bmatrix}3&-5\\2&5\end{bmatrix}$, then cofactor of $a_{12}$ is .........

7. If $A=[a_{ij}]_{m\times m}$ is non-singular matrix, then $A^{-1}=\frac{1}{\dots}$ adj (A).

8. $(A^T)^T=$ .........

9. If $A=\begin{bmatrix}2&1\\1&1\end{bmatrix}$ and $A^{-1}=\begin{bmatrix}1&-1\\x&2\end{bmatrix}$, then $x=$ ......

10. If $a_1x+b_1y=c_1$ and $a_2x+b_2y=c_2$, then matrix form is $\begin{bmatrix}\dots&\dots\\\dots&\dots\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\dots\\\dots\end{bmatrix}$

**Answers**

| | | |
|---|---|---|
| 1. column | 2. $2\times 3$ | 3. 2 |
| 4. $-1$ | 5. 3 | 6. $-2$ |
| 7. $\lvert A\rvert$ | 8. A | 9. $-1$ |

10. $\begin{bmatrix}a_1&b_1\\a_2&b_2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}c_1\\c_2\end{bmatrix}$.

**(III) State whether each of the following is *True* or *False* :**

1. Single element matrix is row as well as column matrix.
2. Every scalar matrix is unit matrix.
3. $A=\begin{bmatrix}4&5\\6&1\end{bmatrix}$ is non-singular matrix.
4. If A is symmetric, then $A=-A^T$.
5. If AB and BA both exist, then $AB=BA$.
6. If A and B are square matrices of same order, then $(A+B)^2=A^2+2AB+B^2$.
7. If A and B are conformable for the product AB, then $(AB)^T=A^TB^T$.
8. Singleton matrix is only row matrix.
9. $A=\begin{bmatrix}2&1\\10&5\end{bmatrix}$ is invertible matrix.
10. $A(\text{adj } A)=|A|I$, where I is the unit matrix.

## Answers

1. True 2. False 3. True 4. False
5. False 6. False 7. False 8. False
9. False 10. True.

### (IV) Solve the following :

**1. Find $k$, if $\begin{bmatrix} 7 & 3 \\ 5 & k \end{bmatrix}$ is a singular matrix.**

**Solution :** Let $A = \begin{bmatrix} 7 & 3 \\ 5 & k \end{bmatrix}$

Since, A is singular matrix, $|A| = 0$

$\therefore \begin{vmatrix} 7 & 3 \\ 5 & k \end{vmatrix} = 0$

$\therefore 7k - 15 = 0 \qquad \therefore k = \frac{15}{7}.$

**2. Find $x, y, z$ if $\begin{bmatrix} 2 & x & 5 \\ 3 & 1 & z \\ y & 5 & 8 \end{bmatrix}$ is a symmetric matrix.**

**Solution :** Let $A = \begin{bmatrix} 2 & x & 5 \\ 3 & 1 & z \\ y & 5 & 8 \end{bmatrix}$

Then $A^T = \begin{bmatrix} 2 & 3 & y \\ x & 1 & 5 \\ 5 & z & 8 \end{bmatrix}$

Since, A is symmetric matrix, $A = A^T$

$\therefore \begin{bmatrix} 2 & x & 5 \\ 3 & 1 & z \\ y & 5 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 3 & y \\ x & 1 & 5 \\ 5 & z & 8 \end{bmatrix}$

By equality of matrices,

$x = 3$, $y = 5$ and $z = 5$.

**3. If $A = \begin{bmatrix} 1 & 5 \\ 7 & 8 \\ 9 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 \\ 1 & 5 \\ -8 & 6 \end{bmatrix}$, $C = \begin{bmatrix} -2 & 3 \\ 1 & -5 \\ 7 & 8 \end{bmatrix}$, then show that $(A+B)+C = A+(B+C)$.**

**Solution :**

$A + B = \begin{bmatrix} 1 & 5 \\ 7 & 8 \\ 9 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 1 & 5 \\ -8 & 6 \end{bmatrix}$

$= \begin{bmatrix} 1+2 & 5+4 \\ 7+1 & 8+5 \\ 9-8 & 5+6 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 8 & 13 \\ 1 & 11 \end{bmatrix}$

$\therefore (A+B)+C = \begin{bmatrix} 3 & 9 \\ 8 & 13 \\ 1 & 11 \end{bmatrix} + \begin{bmatrix} -2 & 3 \\ 1 & -5 \\ 7 & 8 \end{bmatrix}$

$= \begin{bmatrix} 3-2 & 9+3 \\ 8+1 & 13-5 \\ 1+7 & 11+8 \end{bmatrix} = \begin{bmatrix} 1 & 12 \\ 9 & 8 \\ 8 & 19 \end{bmatrix}$ ... (1)

Also, $B + C = \begin{bmatrix} 2 & 4 \\ 1 & 5 \\ -8 & 6 \end{bmatrix} + \begin{bmatrix} -2 & 3 \\ 1 & -5 \\ 7 & 8 \end{bmatrix}$

$= \begin{bmatrix} 2-2 & 4+3 \\ 1+1 & 5-5 \\ -8+7 & 6+8 \end{bmatrix} = \begin{bmatrix} 0 & 7 \\ 2 & 0 \\ -1 & 14 \end{bmatrix}$

$\therefore A + (B+C) = \begin{bmatrix} 1 & 5 \\ 7 & 8 \\ 9 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 7 \\ 2 & 0 \\ -1 & 14 \end{bmatrix}$

$= \begin{bmatrix} 1+0 & 5+7 \\ 7+2 & 8+0 \\ 9-1 & 5+14 \end{bmatrix} = \begin{bmatrix} 1 & 12 \\ 9 & 8 \\ 8 & 19 \end{bmatrix}$ ... (2)

From (1) and (2),

$(A+B)+C = A+(B+C)$.

**4. If $A = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 7 \\ -3 & 0 \end{bmatrix}$, find the matrix $A - 4B + 7I$, where I is the unit matrix of order 2.**

**Solution :**

$A - 4B + 7I = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix} - 4\begin{bmatrix} 1 & 7 \\ -3 & 0 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix} - \begin{bmatrix} 4 & 28 \\ -12 & 0 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

$= \begin{bmatrix} 2-4+7 & 5-28+0 \\ 3-(-12)+0 & 7-0+7 \end{bmatrix}$

$= \begin{bmatrix} 5 & -23 \\ 15 & 14 \end{bmatrix}.$

**5. If $A = \begin{bmatrix} 2 & -3 \\ 3 & -2 \\ -1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 4 & 1 \\ 2 & -1 & -3 \end{bmatrix}$, verify**

**(i) $(A + 2B^T)^T = A^T + 2B$**

**(ii) $(3A - 5B^T)^T = 3A^T - 5B$.**

**Solution :**

$A = \begin{bmatrix} 2 & -3 \\ 3 & -2 \\ -1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 4 & 1 \\ 2 & -1 & -3 \end{bmatrix}$

$\therefore\ A^T = \begin{bmatrix} 2 & 3 & -1 \\ -3 & -2 & 4 \end{bmatrix},\ B^T = \begin{bmatrix} -3 & 2 \\ 4 & -1 \\ 1 & -3 \end{bmatrix}$

**(i)** $A + 2B^T = \begin{bmatrix} 2 & -3 \\ 3 & -2 \\ -1 & 4 \end{bmatrix} + 2\begin{bmatrix} -3 & 2 \\ 4 & -1 \\ 1 & -3 \end{bmatrix}$

$= \begin{bmatrix} 2 & -3 \\ 3 & -2 \\ -1 & 4 \end{bmatrix} + \begin{bmatrix} -6 & 4 \\ 8 & -2 \\ 2 & -6 \end{bmatrix}$

$= \begin{bmatrix} 2-6 & -3+4 \\ 3+8 & -2-2 \\ -1+2 & 4-6 \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 11 & -4 \\ 1 & -2 \end{bmatrix}$

$\therefore\ (A + 2B^T)^T = \begin{bmatrix} -4 & 11 & 1 \\ 1 & -4 & -2 \end{bmatrix}$ ... (1)

Also, $A^T + 2B$

$= \begin{bmatrix} 2 & 3 & -1 \\ -3 & -2 & 4 \end{bmatrix} + 2\begin{bmatrix} -3 & 4 & 1 \\ 2 & -1 & -3 \end{bmatrix}$

$= \begin{bmatrix} 2 & 3 & -1 \\ -3 & -2 & 4 \end{bmatrix} + \begin{bmatrix} -6 & 8 & 2 \\ 4 & -2 & -6 \end{bmatrix}$

$= \begin{bmatrix} 2-6 & 3+8 & -1+2 \\ -3+4 & -2-2 & 4-6 \end{bmatrix}$

$= \begin{bmatrix} -4 & 11 & 1 \\ 1 & -4 & -2 \end{bmatrix}$ ... (2)

From (1) and (2),

$(A + 2B^T)^T = A^T + 2B.$

**(ii)** $3A - 5B^T = 3\begin{bmatrix} 2 & -3 \\ 3 & -2 \\ -1 & 4 \end{bmatrix} - 5\begin{bmatrix} -3 & 2 \\ 4 & -1 \\ 1 & -3 \end{bmatrix}$

$= \begin{bmatrix} 6 & -9 \\ 9 & -6 \\ -3 & 12 \end{bmatrix} - \begin{bmatrix} -15 & 10 \\ 20 & -5 \\ 5 & -15 \end{bmatrix}$

$= \begin{bmatrix} 6-(-15) & -9-10 \\ 9-20 & -6-(-5) \\ -3-5 & 12-(-15) \end{bmatrix}$

$= \begin{bmatrix} 21 & -19 \\ -11 & -1 \\ -8 & 27 \end{bmatrix}$

$\therefore\ (3A - 5B^T)^T = \begin{bmatrix} 21 & -11 & -8 \\ -19 & -1 & 27 \end{bmatrix}$ ... (1)

Also, $3A^T - 5B$

$= 3\begin{bmatrix} 2 & 3 & -1 \\ -3 & -2 & 4 \end{bmatrix} - 5\begin{bmatrix} -3 & 4 & 1 \\ 2 & -1 & -3 \end{bmatrix}$

$= \begin{bmatrix} 6 & 9 & -3 \\ -9 & -6 & 12 \end{bmatrix} - \begin{bmatrix} -15 & 20 & 5 \\ 10 & -5 & -15 \end{bmatrix}$

$= \begin{bmatrix} 6-(-15) & 9-20 & -3-5 \\ -9-10 & -6-(-5) & 12-(-15) \end{bmatrix}$

$= \begin{bmatrix} 21 & -11 & -8 \\ -19 & -1 & 27 \end{bmatrix}$ ... (2)

From (1) and (2),

$(3A - 5B^T)^T = 3A^T = 3A^T - 5B.$

**6. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix}$, then show that AB and BA are both singular matrices.**

**Solution :**

$AB = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1-6-6 & -1+4+3 & 1-2+0 \\ 2-12-12 & -2+8+6 & 2-4+0 \\ 1-6-6 & -1+4+3 & 1-2+0 \end{bmatrix}$

$= \begin{bmatrix} -11 & 6 & -1 \\ -22 & 12 & -2 \\ -11 & 6 & -1 \end{bmatrix}$

$\therefore\ |AB| = \begin{bmatrix} -11 & 6 & -1 \\ -22 & 12 & -2 \\ -11 & 6 & -1 \end{bmatrix}$

By taking $-6$ common from $C_2$, we get

$\therefore\ |AB| = -6\begin{vmatrix} -11 & -1 & -1 \\ -22 & -2 & -2 \\ -11 & -1 & -1 \end{vmatrix}$

$= -6 \times 0$ ... $[\because C_2 \equiv C_3]$

$= 0$

$\therefore$ AB is a singular matrix.

Also, $BA = \begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1-2+1 & 2-4+2 & 3-6+3 \\ -3+4-1 & -6+8-2 & -9+12-3 \\ -2+2+0 & -4+4+0 & -6+6+0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0 \qquad \therefore\ |BA| = 0$

$\therefore$ BA is also a singular matrix.

Hence, AB and BA are both singular matrices.

**7. If $A = \begin{bmatrix} 3 & 1 \\ 1 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 5 & -2 \end{bmatrix}$, verify $|AB| = |A||B|$.**

**Solution :**

$$AB = \begin{bmatrix} 3 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 5 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3+5 & 6-2 \\ 1+25 & 2-10 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 26 & -8 \end{bmatrix}$$

$$\therefore \ |AB| = \begin{vmatrix} 8 & 4 \\ 26 & -8 \end{vmatrix}$$

$$= -64 - 104 = -168 \quad \dots (1)$$

$$|A| = \begin{vmatrix} 3 & 1 \\ 1 & 5 \end{vmatrix} = 15 - 1 = 14$$

$$|B| = \begin{vmatrix} 1 & 2 \\ 5 & -2 \end{vmatrix} = -2 - 10 = -12$$

$$\therefore \ |A||B| = 14(-12) = -168 \quad \dots (2)$$

From (1) and (2), $|AB| = |A||B|$.

**8. If $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, then show that $A^2 - 4A + 3I = 0$.**

**Solution :**

$$A^2 = A \cdot A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1 & -2-2 \\ -2-2 & 1+4 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$$

$\therefore \ A^2 - 4A + 3I$

$$= \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix} - 4\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix} - \begin{bmatrix} 8 & -4 \\ -4 & 8 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5-8+3 & -4-(-4)+0 \\ -4-(-4)+0 & 5-8+3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore \ A^2 - 4A + 3I = 0.$

**9. If $A = \begin{bmatrix} -3 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & a \\ b & 0 \end{bmatrix}$ and $(A+B)(A-B) = A^2 - B^2$, find $a$ and $b$.**

**Solution :** $(A+B)(A-B) = A^2 - B^2$

$\therefore \ A^2 - AB + BA - B^2 = A^2 - B^2$

$\therefore \ -AB + BA = 0$

$\therefore \ AB = BA$

$$\therefore \ \begin{bmatrix} -3 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & a \\ b & 0 \end{bmatrix} = \begin{bmatrix} 1 & a \\ b & 0 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 2 & 4 \end{bmatrix}$$

$$\therefore \ \begin{bmatrix} -3+2b & -3a+0 \\ 2+4b & 2a+0 \end{bmatrix} = \begin{bmatrix} -3+2a & 2+4a \\ -3b+0 & 2b+0 \end{bmatrix}$$

$$\therefore \ \begin{bmatrix} -3+2b & -3a \\ 2+4b & 2a \end{bmatrix} = \begin{bmatrix} -3+2a & 2+4a \\ -3b & 2b \end{bmatrix}$$

By equality of matrices,

$-3 + 2b = -3 + 2a \quad \dots (1)$

$-3a = 2 + 4a \quad \dots (2)$

$2 + 4b = -3b \quad \dots (3)$

$2a = 2b \quad \dots (4)$

From (2), $7a = -2 \quad \therefore \ a = \frac{-2}{7}$

From (3), $7b = -2 \quad \therefore \ b = -\frac{2}{7}$

These values of $a$ and $b$ also satisfy equations (1) and (4).

Hence, $a = \frac{-2}{7}$ and $b = -\frac{2}{7}$.

**10. If $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$, then find $A^3$.**

**Solution :**

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1-2 & 2+6 \\ -1-3 & -2+9 \end{bmatrix} = \begin{bmatrix} -1 & 8 \\ -4 & 7 \end{bmatrix}$$

$$\therefore \ A^3 = A^2 \cdot A = \begin{bmatrix} -1 & 8 \\ -4 & 7 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1-8 & -2+24 \\ -4-7 & -8+21 \end{bmatrix}$$

$$\therefore \ A^3 = \begin{bmatrix} -9 & 22 \\ -11 & 13 \end{bmatrix}.$$

**11. Find $x$, $y$, $z$ if**

$$\left\{ 5\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 3 & -2 \\ 1 & 3 \end{bmatrix} \right\} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} x-1 \\ y+1 \\ 2z \end{bmatrix}$$

**Solution :** *Refer to the solution of Q. 14 of Exercise 2.3.*

**Ans.** $x = 1$, $y = 5$, $z = 5$.

[**Note :** *Answer in the textbook is incorrect.*]

**12. If $A = \begin{bmatrix} 2 & -4 \\ 3 & -2 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & 0 \end{bmatrix}$, then show that $(AB)^T = B^T A^T$.**

**Solution :** *Refer to the solution of Q. 12 (i) of Exercise 2.4.*

**13.** If $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{bmatrix}$, then reduce it to unit matrix by row tranformation.

**Solution :** *Refer to the solution of Q. 1 of Additional Solved Problems 2 (A).*

**14.** Two farmers Shantaram and Kantaram cultivate three crops rice, wheet and groundnut. The sale (in ₹) of these crops by both the farmers for the month of April and May 2016 is given below :

**April 2016 (in ₹)**

| | Rice | Wheat | Groundnut |
|---|---|---|---|
| Shantaram | 15000 | 13000 | 12000 |
| Kantaram | 18000 | 15000 | 8000 |

**May 2016 (in ₹)**

| | Rice | Wheat | Groundnut |
|---|---|---|---|
| Shantaram | 18000 | 15000 | 12000 |
| Kantaram | 21000 | 16500 | 16000 |

Find (i) the total sale in rupees for two months of each farmer for each crop.

(ii) the increase in sale from April to May for every crop of each farmer.

**Solution :** The given information can be written in matrix form as :

**April 2016 (in ₹)**

$$\begin{array}{cc} & \text{Rice} \quad \text{Wheat} \quad \text{Groundnut} \\ A = & \begin{bmatrix} 15000 & 13000 & 12000 \\ 18000 & 15000 & 8000 \end{bmatrix} \begin{matrix} \text{Shantaram} \\ \text{Kantaram} \end{matrix} \end{array}$$

**May 2016 (in ₹)**

$$\begin{array}{cc} & \text{Rice} \quad \text{Wheat} \quad \text{Groundnut} \\ B = & \begin{bmatrix} 18000 & 15000 & 12000 \\ 21000 & 16500 & 16000 \end{bmatrix} \begin{matrix} \text{Shantaram} \\ \text{Kantaram} \end{matrix} \end{array}$$

(i) The total sale in ₹ for two months of each farmer for each crop can be obtained by the addition A + B.

Now, A + B

$$= \begin{bmatrix} 15000 & 13000 & 12000 \\ 18000 & 15000 & 8000 \end{bmatrix} + \begin{bmatrix} 18000 & 15000 & 12000 \\ 21000 & 16500 & 16000 \end{bmatrix}$$

$$= \begin{bmatrix} 15000+18000 & 13000+15000 & 12000+12000 \\ 18000+21000 & 15000+16500 & 8000+16000 \end{bmatrix}$$

$$= \begin{bmatrix} 33000 & 28000 & 24000 \\ 39000 & 31500 & 24000 \end{bmatrix}$$

∴ total sale in ₹ for two months of each farmer for each crop is given by

$$\begin{array}{c} \text{Rice} \quad \text{Wheat} \quad \text{Groundnut} \\ \begin{bmatrix} 33000 & 28000 & 24000 \\ 39000 & 31500 & 24000 \end{bmatrix} \begin{matrix} \text{Shantaram} \\ \text{Kantaram} \end{matrix} \end{array}$$

Hence, the total sale for Shantaram are ₹ 33000 for Rice, ₹ 28000 for Wheat, ₹ 24000 for Groundnut and for Kantaram are ₹ 39000 for Rice, ₹ 31500 for Wheat, ₹ 24000 for Groundnut.

(ii) The increase in sale from April to May for every crop of each farmer can be obtained the subtraction of A from B.

Now, B − A

$$= \begin{bmatrix} 18000 & 15000 & 12000 \\ 21000 & 16500 & 16000 \end{bmatrix} - \begin{bmatrix} 15000 & 13000 & 12000 \\ 18000 & 15000 & 8000 \end{bmatrix}$$

$$= \begin{bmatrix} 18000-15000 & 15000-13000 & 12000-12000 \\ 21000-18000 & 16500-15000 & 16000-8000 \end{bmatrix}$$

$$= \begin{bmatrix} 3000 & 2000 & 0 \\ 3000 & 1500 & 8000 \end{bmatrix}$$

∴ the increase in sale from April to May is given by

$$\begin{array}{c} \text{Rice} \quad \text{Wheat} \quad \text{Groundnut} \\ \begin{bmatrix} 3000 & 2000 & 0 \\ 3000 & 1500 & 8000 \end{bmatrix} \begin{matrix} \text{Shantaram} \\ \text{Kantaram} \end{matrix} \end{array}$$

Hence, the increase in sale from April to May of Shantam are ₹ 3000 in Rice, ₹ 2000 in Wheat, nothing in Groundnut and of Kantaram are ₹ 3000 in Rice, ₹ 1500 in Wheat, ₹ 8000 in Groundnut.

**15.** Check whether following matrices are invertible or not :

(i) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 3 & 4 & 3 \\ 1 & 1 & 0 \\ 1 & 4 & 5 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 2 & 4 & 6 \end{bmatrix}$.

**Solution :**

(i) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Then $|A| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1 \neq 0$

∴ A is a non-singular matrix.

Hence, $A^{-1}$ exists.

**(ii)** Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Then $|A| = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 1 - 1 = 0$

$\therefore$ A is a singular matrix.

Hence, $A^{-1}$ does not exist.

**(iii)** Let $A = \begin{bmatrix} 3 & 4 & 3 \\ 1 & 1 & 0 \\ 1 & 4 & 5 \end{bmatrix}$

Then $|A| = \begin{vmatrix} 3 & 4 & 3 \\ 1 & 1 & 0 \\ 1 & 4 & 5 \end{vmatrix}$

$= 3(5-0) - 4(5-0) + 3(4-1)$

$= 15 - 20 + 9 = 4 \neq 0$

$\therefore$ A is a non-singular matrix.

Hence, $A^{-1}$ exists.

**(iv)** Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

Then $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 2 & 4 & 6 \end{vmatrix}$

$= 1(24-20) - 2(12-10) + 3(8-8)$

$= 4 - 4 + 0 = 0$

$\therefore$ A is a singular matrix.

Hence, $A^{-1}$ does not exist.

**16. Find inverse of the following matrices (if they exist) by elementary transformation :**

**(i)** $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

**Solution :** Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

Then $|A| = \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 3 - (-2) = 5 \neq 0$

$\therefore$ $A^{-1}$ exists.

We write, $AA^{-1} = I$

$\therefore \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

By $R_2 - 2R_1$, we get

$\begin{bmatrix} 1 & -1 \\ 0 & 5 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

By $\left(\frac{1}{5}\right) R_2$, we get

$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}$

By $R_1 + R_2$, we get

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}$

$\therefore A^{-1} = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}$

**Note :** $A^{-1}$ can also be obtained by using elementary column transformation taking $A^{-1}A = I$.

**(ii)** $\begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix}$

**Solution :** Let $A = \begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix}$

Then $|A| = \begin{vmatrix} 2 & 1 \\ 7 & 4 \end{vmatrix} = 8 - 7 = 1 \neq 0$

$\therefore$ $A^{-1}$ exist.

We write $A^{-1}A = I$

$\therefore A^{-1} \begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

By $C_1 - C_2$, we get

$A^{-1} \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

By $C_2 - C_1$, we get

$A^{-1} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

By $C_1 - 3C_2$, we get

$A^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}$

$\therefore A^{-1} = \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}$.

**Note :** $A^{-1}$ can also be obtained by using elementary row transformation taking $AA^{-1} = I$.

**(iii)** $\begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$

**Solution :** Let $A = \begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$

Then $|A| = \begin{vmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{vmatrix}$

$= 2(4+6)+3(4-9)+3(-4-6)$

$= 20-15-30 = -25 \neq 0$

$\therefore$ $A^{-1}$ exists.

We write $A^{-1}A = I$

$$\therefore\ A^{-1}\begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $C_1 \leftrightarrow C_3$, we get

$$A^{-1}\begin{bmatrix} 3 & -3 & 2 \\ 3 & 2 & 2 \\ 2 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

By $C_1 - C_3$, we get

$$A^{-1}\begin{bmatrix} 1 & -3 & 2 \\ 1 & 2 & 2 \\ -1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

By $C_2 + 3C_1$ and $C_3 - 2C_1$, we get

$$A^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 0 \\ -1 & -5 & 5 \end{bmatrix} = \begin{bmatrix} -1 & -3 & 3 \\ 0 & 1 & 0 \\ 1 & 3 & -2 \end{bmatrix}$$

By $\left(\frac{1}{5}\right)C_2$, we get

$$A^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -1 & 5 \end{bmatrix} = \begin{bmatrix} -1 & -\frac{3}{5} & 3 \\ 0 & \frac{1}{5} & 0 \\ 1 & \frac{3}{5} & -2 \end{bmatrix}$$

By $C_1 - C_2$, we get

$$A^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 5 \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} & -\frac{3}{5} & 3 \\ -\frac{1}{5} & \frac{1}{5} & 0 \\ \frac{2}{5} & \frac{3}{5} & -2 \end{bmatrix}$$

By $\left(\frac{1}{5}\right)C_3$, we get

$$A^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} & -\frac{3}{5} & \frac{3}{5} \\ -\frac{1}{5} & \frac{1}{5} & 0 \\ \frac{2}{5} & \frac{3}{5} & -\frac{2}{5} \end{bmatrix}$$

By $C_2 + C_3$, we get

$$A^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} & 0 & \frac{3}{5} \\ -\frac{1}{5} & \frac{1}{5} & 0 \\ \frac{2}{5} & \frac{1}{5} & -\frac{2}{5} \end{bmatrix}$$

$$\therefore\ A^{-1} = \frac{1}{5}\begin{bmatrix} -2 & 0 & 3 \\ -1 & 1 & 0 \\ 2 & 1 & -2 \end{bmatrix}.$$

**Note :** $A^{-1}$ can also be obtained by using elementary row transformation taking $AA^{-1} = I$.

**(iv)** $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

**Solution :** Let $A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

$\therefore\ |A| = \begin{vmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{vmatrix}$

$= 2(3-0)-0-1(5-0)$

$= 6-0-5$

$= 1 \neq 0$

$\therefore$ $A^{-1}$ exists.

We write $AA^{-1} = 1$

$$\therefore\ \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_1 \leftrightarrow R_2$, we get

$$\begin{bmatrix} 5 & 1 & 0 \\ 2 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix} A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_1 - 2R_2$, we get

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix} A^{-1} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_2 - 2R_1$, we get

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & -5 \\ 0 & 1 & 3 \end{bmatrix} A^{-1} = \begin{bmatrix} -2 & 1 & 0 \\ 5 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_2 + 3R_3$, we get

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 4 \\ 0 & 1 & 3 \end{bmatrix} A^{-1} = \begin{bmatrix} -2 & 1 & 0 \\ 5 & -2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_1 - R_2$ and $R_3 - R_2$, we get

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & -1 \end{bmatrix} A^{-1} = \begin{bmatrix} -7 & 3 & -3 \\ 5 & -2 & 3 \\ -5 & 2 & -2 \end{bmatrix}$$

By $(-1)R_3$, we get

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} -7 & 3 & -3 \\ 5 & -2 & 3 \\ 5 & -2 & 2 \end{bmatrix}$$

By $R_1 + 2R_3$ and $R_2 - 4R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$\therefore\ A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}.$$

**Note :** $A^{-1}$ can also be obtained by using elementary column transformation taking $A^{-1}A = I$.

**17. Find the inverse of $\begin{bmatrix} 3 & 1 & 5 \\ 2 & 7 & 8 \\ 1 & 2 & 5 \end{bmatrix}$ by adjoint method.**

**Solution :** Let $A = \begin{bmatrix} 3 & 1 & 5 \\ 2 & 7 & 8 \\ 1 & 2 & 5 \end{bmatrix}$

Then $|A| = \begin{vmatrix} 3 & 1 & 5 \\ 2 & 7 & 8 \\ 1 & 2 & 5 \end{vmatrix}$

$= 3(35 - 16) - 1(10 - 8) + 5(4 - 7)$

$= 57 - 2 - 15 = 40 \neq 0$

$\therefore\ A^{-1}$ exists.

First, we have to find the cofactor matrix $= [A_{ij}]_{3 \times 3}$, where $A_{ij} = (-1)^{i+j} M_{ij}$

Now, $A_{11} = (-1)^{1+1} M_{11} = \begin{vmatrix} 7 & 8 \\ 2 & 5 \end{vmatrix}$

$= 35 - 16 = 19$

$A_{12} = (-1)^{1+2} M_{12} = -\begin{vmatrix} 2 & 8 \\ 1 & 5 \end{vmatrix}$

$= -(10 - 8) = -2$

$A_{13} = (-1)^{1+3} M_{13} = \begin{vmatrix} 2 & 7 \\ 1 & 2 \end{vmatrix}$

$= 4 - 7 = -3$

$A_{21} = (-1)^{2+1} M_{21} = -\begin{vmatrix} 1 & 5 \\ 2 & 5 \end{vmatrix}$

$= -(5 - 10) = 5$

$A_{22} = (-1)^{2+2} M_{22} = \begin{vmatrix} 3 & 5 \\ 1 & 5 \end{vmatrix}$

$= 15 - 5 = 10$

$A_{23} = (-1)^{2+3} M_{23} = -\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix}$

$= -(6 - 1) = -5$

$A_{31} = (-1)^{3+1} M_{31} = \begin{vmatrix} 1 & 5 \\ 7 & 8 \end{vmatrix}$

$= 8 - 35 = -27$

$A_{32} = (-1)^{3+2} M_{32} = -\begin{vmatrix} 3 & 5 \\ 2 & 8 \end{vmatrix}$

$= -(24 - 10) = -14$

$A_{33} = (-1)^{3+3} M_{33} = \begin{vmatrix} 3 & 1 \\ 2 & 7 \end{vmatrix}$

$= 21 - 2 = 19$

$\therefore$ the cofactor matrix

$$= \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 19 & -2 & -3 \\ 5 & 10 & -5 \\ -27 & -14 & 19 \end{bmatrix}$$

$$\therefore\ \text{adj } A = \begin{bmatrix} 19 & 5 & -27 \\ -2 & 10 & -14 \\ -3 & -5 & 19 \end{bmatrix}$$

$$\therefore\ A^{-1} = \frac{1}{|A|}(\text{adj } A)$$

$$\therefore\ A^{-1} = \frac{1}{40}\begin{bmatrix} 19 & 5 & -27 \\ -2 & 10 & -14 \\ -3 & -5 & 19 \end{bmatrix}.$$

**18. Solve the following equations by method of inversion :**

**(i) $4x - 3y - 2 = 0$, $3x - 4y + 6 = 0$**

**Solution :** The given equations are

$4x - 3y = 2$

$3x - 4y = -6$

These equations can be written in matrix form as :

$$\begin{bmatrix}4 & -3\\3 & -4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}2\\-6\end{bmatrix}$$

This is of the form AX = B, where

$$A=\begin{bmatrix}4 & -3\\3 & -4\end{bmatrix}, X=\begin{bmatrix}x\\y\end{bmatrix} \text{ and } B=\begin{bmatrix}2\\-6\end{bmatrix}$$

Let us find $A^{-1}$.

$$|A|=\begin{vmatrix}4 & -3\\3 & -4\end{vmatrix}=-16-(-9)=-7\neq 0$$

$\therefore$ $A^{-1}$ exists.

We write $AA^{-1}=I$

$$\therefore \begin{bmatrix}4 & -3\\3 & -4\end{bmatrix}A^{-1}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

By $R_1-R_2$, we get

$$\begin{bmatrix}1 & 1\\3 & -4\end{bmatrix}A^{-1}=\begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}$$

By $R_2-3R_1$, we get

$$\begin{bmatrix}1 & 1\\0 & -7\end{bmatrix}A^{-1}=\begin{bmatrix}1 & -1\\-3 & 4\end{bmatrix}$$

By $\left(-\frac{1}{7}\right)R_2$, we get

$$\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}A^{-1}=\begin{bmatrix}1 & -1\\\frac{3}{7} & -\frac{4}{7}\end{bmatrix}$$

By $R_1-R_2$, we get

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}A^{-1}=\begin{bmatrix}\frac{4}{7} & -\frac{3}{7}\\\frac{3}{7} & -\frac{4}{7}\end{bmatrix}$$

$$\therefore A^{-1}=\begin{bmatrix}\frac{4}{7} & -\frac{3}{7}\\\frac{3}{7} & -\frac{4}{7}\end{bmatrix}$$

Now, premultiply AX = B by $A^{-1}$, we get

$A^{-1}(AX)=A^{-1}B$

$\therefore$ $(A^{-1}A)X=A^{-1}B$

$\therefore$ $IX=A^{-1}B$

$$\therefore X=\begin{bmatrix}\frac{4}{7} & -\frac{3}{7}\\\frac{3}{7} & -\frac{4}{7}\end{bmatrix}\begin{bmatrix}2\\-6\end{bmatrix}$$

$$\therefore \begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\frac{8}{7}+\frac{18}{7}\\\frac{6}{7}+\frac{24}{7}\end{bmatrix}=\begin{bmatrix}\frac{26}{7}\\\frac{30}{7}\end{bmatrix}$$

By equality of matrices,

$x=\frac{26}{7}, y=\frac{30}{7}$ is the required solution.

**(ii) $x+y-z=2$, $x-2y+z=3$ and $2x-y-3z=-1$**

**Solution :** The given equations can be written in matrix form as :

$$\begin{bmatrix}1 & 1 & -1\\1 & -2 & 1\\2 & -1 & -3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2\\3\\-1\end{bmatrix}$$

This is of the form AX = B, where

$$A=\begin{bmatrix}1 & 1 & -1\\1 & -2 & 1\\2 & -1 & -3\end{bmatrix} X=\begin{bmatrix}x\\y\\z\end{bmatrix} \text{ and } B=\begin{bmatrix}2\\3\\-1\end{bmatrix}$$

Let us find $A^{-1}$.

$$|A|=\begin{vmatrix}1 & 1 & -1\\1 & -2 & 1\\2 & -1 & -3\end{vmatrix}$$

$=1(6+1)-1(-3-2)-1(-1+4)$

$=7+5-3=9\neq 0$

$\therefore$ $A^{-1}$ exists.

We write $AA^{-1}=I$

$$\therefore \begin{bmatrix}1 & 1 & -1\\1 & -2 & 1\\2 & -1 & -3\end{bmatrix}A^{-1}=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

By $R_2-R_1$ and $R_3-2R_1$, we get

$$\begin{bmatrix}1 & 1 & -1\\0 & -3 & 2\\0 & -3 & -1\end{bmatrix}A^{-1}=\begin{bmatrix}1 & 0 & 0\\-1 & 1 & 0\\-2 & 0 & 1\end{bmatrix}$$

By $\left(-\frac{1}{3}\right)R_2$, we get

$$\begin{bmatrix}1 & 1 & -1\\0 & 1 & -\frac{2}{3}\\0 & -3 & -1\end{bmatrix}A^{-1}=\begin{bmatrix}1 & 0 & 0\\\frac{1}{3} & -\frac{1}{3} & 0\\-2 & 0 & 1\end{bmatrix}$$

By $R_1-R_2$ and $R_3+3R_2$, we get

$$\begin{bmatrix}1 & 0 & -\frac{1}{3}\\0 & 1 & -\frac{2}{3}\\0 & 0 & -3\end{bmatrix}A^{-1}=\begin{bmatrix}\frac{2}{3} & \frac{1}{3} & 0\\\frac{1}{3} & -\frac{1}{3} & 0\\-1 & -1 & 1\end{bmatrix}$$

By $\left(-\frac{1}{3}\right)R_3$, we get

$$\begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & -\frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

By $R_1 + \frac{1}{3}R_3$ and $R_2 + \frac{2}{3}R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} \frac{7}{9} & \frac{4}{9} & -\frac{1}{9} \\ \frac{5}{9} & -\frac{1}{9} & -\frac{2}{9} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

$$\therefore\ A^{-1} = \frac{1}{9}\begin{bmatrix} 7 & 4 & -1 \\ 5 & -1 & -2 \\ 3 & 3 & -3 \end{bmatrix}$$

Now, premultiply $AX = B$ by $A^{-1}$, we get

$A^{-1}(AX) = A^{-1}B$

$\therefore\ (A^{-1}A)X = A^{-1}B$

$\therefore\ IX = A^{-1}B$

$$\therefore\ X = \frac{1}{9}\begin{bmatrix} 7 & 4 & -1 \\ 5 & -1 & -2 \\ 3 & 3 & -3 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

$$\therefore\ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 14+12+1 \\ 10-3+2 \\ 6+9+3 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 27 \\ 9 \\ 18 \end{bmatrix}$$

$$\therefore\ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

By equality of matrices,

$x = 3,\ y = 1,\ z = 2$ is the required solution.

**(iii) $x - y + z = 4$, $2x + y - 3z = 0$ and $x + y + z = 2$**

**Solution :** *Refer to the solution of Q. 18 (ii).*

**Ans.** $x = 2,\ y = -1,\ z = 1$.

**19. Solve the following equations by method of reduction :**

**(i) $2x + y = 5$, $3x - 5y = -3$**

**Solution :** The given equation can be written in matrix form as :

$$\begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

By $R_2 - 5R_1$, we get

$$\begin{bmatrix} 2 & 1 \\ -7 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -28 \end{bmatrix}$$

$$\therefore\ \begin{bmatrix} 2x + y \\ -7x + 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -28 \end{bmatrix}$$

By equality of matrices,

$2x + y = 5$ ... (1)

$-7x = -28$ ... (2)

From (2), $x = 4$

Substituting $x = 4$ in (1), we get

$2(4) + y = 5 \quad \therefore\ y = -3$

Hence, $x = 4$ and $y = -3$ is the required solution.

**(ii) $x + 2y + z = 3$, $3x - y + 2z = 1$ and $2x - 3y + 3z = 2$**

**Solution :** The given equations can be written in matrix form as :

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & 2 \\ 2 & -3 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

By $R_2 - 3R_1$ and $R_3 - 2R_1$, we get

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -7 & -1 \\ 0 & -7 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -8 \\ -4 \end{bmatrix}$$

By $R_3 - R_2$, we get

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -7 & -1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -8 \\ 4 \end{bmatrix}$$

$$\therefore\ \begin{bmatrix} x + 2y + z \\ 0 - 7y - z \\ 0 + 0 + 2z \end{bmatrix} = \begin{bmatrix} 3 \\ -8 \\ 4 \end{bmatrix}$$

By equality of matrices,

$x + 2y + z = 3$ ... (1)

$-7y - z = -8$ ... (2)

$2z = 4$ ... (3)

From (3), $z = 2$

Substituting $z = 2$ in (2), we get

$-7y - 2 = -8$

$\therefore\ -7y = -6 \quad \therefore\ y = \frac{6}{7}$

Substituting $y = \frac{6}{7}$, $z = 2$ in (1), we get

$x + 2\left(\frac{6}{7}\right) + 2 = 3$

$\therefore\ x = 3 - 2 - \frac{12}{7} = -\frac{5}{7}$

Hence, $x = -\frac{5}{7}$, $y = \frac{6}{7}$ and $z = 2$ is the required solution.

[**Note :** *First equation is modified.*]

**(iii)** $x - 3y + z = 2$, $3x + y + z = 1$ **and** $5x + y + 3z = 3$.

**Solution :** *Refer to the solution of Q. 19 (ii).*

**Ans.** $x = \frac{1}{6}$, $y = -\frac{1}{3}$, $z = \frac{5}{6}$.

**20. The sum of three numbers is 6. If we multiply third number by 3 and add it to the second number, we get 11. By adding first and third numbers we get a number which is double the second number. Use this information and find a system of linear equations. Find the three numbers using matrices.**

**Solution :** Let the three numbers be $x$, $y$ and $z$.

According to the given condition,

$x + y + z = 6$

$3z + y = 11$, i.e. $y + 3z = 11$

and $x + z = 2y$, i.e. $x - 2y + z = 0$

Hence, the system of linear equations is

$x + y + z = 6$

$y + 3z = 11$

$x - 2y + z = 0$

These equations can be written in matrix form as :

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$$

By $R_3 - R_1$, we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \\ -6 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x + y + z \\ 0 + y + 3z \\ 0 - 3y + 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \\ -6 \end{bmatrix}$$

By equality of matrices,

$x + y + z = 6$ ... (1)

$y + 3z = 11$ ... (2)

$-3y = -6$ ... (3)

From (3), $y = 2$

Substituting $y = 2$ in (2), we get

$2 + 3z = 11$

$3z = 9 \quad \therefore\ z = 3$

Substituting $y = 2$, $z = 3$ in (1), we get

$x + 2 + 3 = 6 \quad \therefore\ x = 1$

$\therefore\ x = 1$, $y = 2$, $z = 3$

Hence, the required numbers are 1, 2 and 3.

## ACTIVITIES | Textbook pages 86 to 88

**1. Construct a matrix of order $2 \times 2$ where the $(ij)^{th}$ element given by $a_{ij} = \frac{(i+j)^2}{2+i}$.**

**Solution :** Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2 \times 2}$ be the required matrix.

Given that $a_{ij} = \frac{(i+j)^2}{2+i}$, $a_{11} = \frac{(1+1)^2}{2+1} = \frac{4}{3}$,

$a_{12} = \frac{(1+2)^2}{2+1} = \frac{9}{3} = 3$, $a_{21} = \frac{(2+1)^2}{2+2} = \frac{9}{4}$,

$a_{22} = \frac{(2+2)^2}{2+2} = \frac{16}{4} = 4$

$$\therefore\ A = \begin{bmatrix} \frac{4}{3} & 3 \\ \frac{9}{4} & 4 \end{bmatrix}$$

**2. If $A = \begin{bmatrix} 1 & -5 \\ 6 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 \\ 4 & 8 \end{bmatrix}$, find AB − 2I, where I is unit matrix of order 2.**

**Solution :** Given $A = \begin{bmatrix} 1 & -5 \\ 6 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 \\ 4 & 8 \end{bmatrix}$

Consider AB − 2I

$$= \begin{bmatrix} 1 & -5 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 4 & 8 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore\ AB - 2I = \begin{bmatrix} 2 - 20 & -3 - 40 \\ 12 + 28 & -18 + 56 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -18 & -43 \\ 40 & 38 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\therefore\ AB - 2I = \begin{bmatrix} -20 & -43 \\ 40 & 36 \end{bmatrix}.$$

**3. If $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$, then find $A^{-1}$ by the adjoint method.**

**Solution :** Given $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 3 + 2 = 5 \neq 0$

$\therefore$ $A^{-1}$ is exist.

$M_{11} = 3$ $\therefore$ $A_{11} = (-1)^{1+1} M_{11} = 1(3) = 3$

$M_{12} = 2$ $\therefore$ $A_{12} = (-1)^{1+2} M_{12} = (-1)(2) = -2$

$M_{21} = -1$ $\therefore$ $A_{21} = (-1)^{2+1} M_{21} = (-1)(-1) = 1$

$M_{22} = 1$ $\therefore$ $A_{22} = (-1)^{2+2} M_{22} = 1(1) = 1$

$\therefore$ $[A_{ij}]_{2 \times 2} = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$

$\text{Adj}(A) = [A_{ij}]^T = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} \text{adj}(A)$

$A^{-1} = \frac{1}{5} \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$

**4. Solve the following equations by inversion method :**

$x + 2y = 1$

$2x - 3y = 4$

**Solution :** $A = \begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Given equations can be written as $AX = B$

Premultiplying by $A^{-1}$, we get

$A^{-1}(AX) = A^{-1}B$

$(A^{-1}A)X = A^{-1}B$

$IX = A^{-1}B$

$X = A^{-1}B$

First, we find the inverse of A by row transformation.

We write $AA^{-1} = I$

Using $R_2 \to R_2 - 2R_1$ $\begin{bmatrix} 1 & 2 \\ 0 & -7 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

Using $\left(\frac{-1}{7}\right)R_2$ $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ \frac{2}{7} & \frac{-1}{7} \end{bmatrix}$

Using $R_1 \to R_1 - 2R_2$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{-1}{7} \end{bmatrix}$

$A^{-1} = \frac{1}{7} \begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix}$

$X = A^{-1}B = \frac{1}{7} \begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$= \frac{1}{7} \begin{bmatrix} 3 + 8 \\ 2 - 4 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 11 \\ -2 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{11}{7} \\ -\frac{2}{7} \end{bmatrix}$

$\therefore$ $x = \frac{11}{7}$, $y = \frac{-2}{7}$

Hence, the solution of given linear equation is $x = \frac{11}{7}$, $y = \frac{-2}{7}$.

**5. Express the following equations in matrix form and solve them by the method of reduction :**

$x + 3y + 3z = 12$, $x + 4y + 4z = 15$, $x + 3y + 4z = 13$.

**Solution :** The given equations can be write as

$x + 3y + 3z = 12$, $x + 4y + 4z = 15$, $x + 3y + 4z = 13$

Hence, the matrix equation is $AX = B$

$\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 4 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 15 \\ 13 \end{bmatrix}$ (i.e. $AX = B$)

By $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$,

$\begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 3 \\ 1 \end{bmatrix}$

We write equation as

$x + 3y + 3z = 12$ ... (1)

$y + z = 3$ ... (2)

$z = 1$ ... (3)

from (3), $z = 1$

Put $z = 1$ in equation (2)

$y + 1 = 3$

$\therefore$ $y = 2$

Put $y = 2$, $z = 1$ in equation (1)

$x + 3(2) + 3(1) = 12$, $x = 3$

$\therefore$ $x = 3$, $y = 2$, $z = 1$.

## ACTIVITIES FOR PRACTICE

**1. Construct a matrix of order 2 × 2 where $a_{ij} = \frac{1}{2}(3i - 2j)$.**

**Solution :** Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be the required matrix.

Given : $a_{ij} = \frac{1}{2}(3i - 2j)$

$\therefore a_{11} = \frac{1}{2}(3\square - 2\square) = \square$

$a_{12} = \frac{1}{2}(3\square - 2\square) = \square$

$a_{21} = \frac{1}{2}(3\square - 2\square) = \square$

$a_{22} = \frac{1}{2}(3\square - 2\square) = \square$

$\therefore A = \begin{bmatrix} \frac{1}{2} & \square \\ \square & 1 \end{bmatrix}$.

**2. Find the cofactors of the elements of the matrix $\begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}$.**

**Solution :**

$M_{11} = \ldots\ldots,\ A_{11} = \ldots\ldots$

$M_{12} = \ldots\ldots,\ A_{12} = \ldots\ldots$

$M_{21} = \ldots\ldots,\ A_{21} = \ldots\ldots$

$M_{22} = \ldots\ldots,\ A_{22} = \ldots\ldots$

**3. If $A = \begin{bmatrix} 4 & -3 \\ 5 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 3 \\ 4 & -2 \end{bmatrix}$, find BA.**

**Solution :** $BA = \begin{bmatrix} -1 & 3 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ 5 & 2 \end{bmatrix}$

$= \begin{bmatrix} -4 + \ldots & \ldots + 6 \\ \ldots + (-10) & -12 + \ldots \end{bmatrix}$

$= \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix}$.

**4. Find AB, if $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 1 & 2 \\ 1 & -2 \end{bmatrix}$.**

**Examine whether AB has inverse or not.**

**Solution :** $AB = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 2 \\ 1 & -2 \end{bmatrix}$

$= \begin{bmatrix} 1 + \ldots + 3 & \ldots + 4 + \ldots \\ \ldots + \ldots + (-3) & \ldots + \ldots + \ldots \end{bmatrix}$

$= \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix}$

$\therefore |AB| = \begin{vmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{vmatrix} = \square - \square$

$= \square\square 0$

$\therefore (AB)^{-1}$ ………

**5. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and X is a 2 × 2 matrix such that AX = I, find X.**

**Solution :** We have AX = I

$\therefore \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

By $R_2 - 3R_1$, we get

$\begin{bmatrix} 1 & 2 \\ 0 & \ldots \end{bmatrix} X = \begin{bmatrix} 1 & 0 \\ \ldots & 1 \end{bmatrix}$

By $\left(-\frac{1}{2}\right)R_2$, we get

$\begin{bmatrix} 1 & 2 \\ 0 & \ldots \end{bmatrix} X = \begin{bmatrix} 1 & 0 \\ \ldots & \ldots \end{bmatrix}$

By $R_1 - 2R_2$, we get

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} X = \begin{bmatrix} \ldots & \ldots \\ \ldots & \ldots \end{bmatrix}$

$\therefore X = \begin{bmatrix} \ldots & \ldots \\ \ldots & \ldots \end{bmatrix}$

**6. Find the inverse of the matrix $A = \begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$ by adjoint method.**

**Solution :**

$|A| = \begin{vmatrix} -1 & 5 \\ -3 & 2 \end{vmatrix} = \ldots\ldots + \ldots\ldots = \ldots\ldots \neq 0$

$\therefore A^{-1}$ exists.

Now, $M_{11} = \ldots\ldots,\ A_{11} = \ldots\ldots$

$M_{12} = \ldots\ldots,\ A_{12} = \ldots\ldots$

$M_{21} = \ldots\ldots,\ A_{21} = \ldots\ldots$

$M_{22} = \ldots\ldots,\ A_{22} = \ldots\ldots$

$\therefore \text{adj } A = \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{\ldots}(\text{adj } A)$

$\therefore A^{-1} = \frac{1}{\ldots}\begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix}$.

**7. Express the following equations in matrix form and solve them by method of reduction :**

**$x-y+z=1$, $2x-y=1$, $3x+3y-4z=2$.**

**Solution :** The given equations can be written in matrix form as :

$$\begin{bmatrix} \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} \ldots \\ \ldots \\ \ldots \end{bmatrix}$$

By $R_2-2R_1$ and $R_3-3R_1$ we get

$$\begin{bmatrix} 1 & -1 & 1 \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 1 \\ \ldots \\ \ldots \end{bmatrix}$$

By $R_3-6R_2$, we get

$$\begin{bmatrix} 1 & -1 & 1 \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 1 \\ \ldots \\ \ldots \end{bmatrix}$$

$$\therefore \begin{bmatrix} x+y+z \\ \ldots+\ldots+\ldots \\ \ldots+\ldots+\ldots \end{bmatrix}=\begin{bmatrix} 1 \\ \ldots \\ \ldots \end{bmatrix}$$

By equality of matrices,

$x+y+z=1$ ... (1)

$y-2z=\ldots$ ... (2)

$5z=\ldots$ ... (3)

From (3), $z=\ldots$

Put $z=\ldots$ in (2), we get

$y-2(\ldots)=\ldots$

$\therefore\ y=\ldots$

Put $y=\ldots$, $z=\ldots$ in (1), we get

$x+\ldots+\ldots=1$

$\therefore\ x=\ldots$

Hence, $x=\ldots$, $y=\ldots$, $z=\ldots$

**8. If the matrix $A=\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ satisfies $A^2-4A+I=0$, find $A^{-1}$.**

**Solution :** $A^2-4A+I=0$

Premultiply by $A^{-1}$, we get

$A^{-1}(A^2-4A+I)=A^{-1}\cdot 0$

$\therefore\ A^{-1}(A\cdot A)-4A^{-1}\cdot A+A^{-1}I=0$

$\therefore\ (A^{-1}\square)A-4\square+A^{-1}=0$

$\therefore\ \square A-4\square+A^{-1}=0$

$\therefore\ A-4\square+A^{-1}=0$

$\therefore\ A^{-1}=4\square-\square$

$$\therefore\ A^{-1}=4\begin{bmatrix} \ldots & \ldots \\ \ldots & \ldots \end{bmatrix}-\begin{bmatrix} \ldots & \ldots \\ \ldots & \ldots \end{bmatrix}$$

$$=\begin{bmatrix} \ldots & \ldots \\ \ldots & \ldots \end{bmatrix}-\begin{bmatrix} \ldots & \ldots \\ \ldots & \ldots \end{bmatrix}$$

$$\therefore\ A^{-1}=\begin{bmatrix} \ldots & \ldots \\ \ldots & \ldots \end{bmatrix}.$$

## OBJECTIVE SECTION

### MULTIPLE CHOICE QUESTIONS

**Select and write the correct answer from the given alternatives in each of the following questions :**

**1.** If A is a square matrix and $|A|\neq 0$, then A is called

(a) a singular matrix (b) a non-singular matrix

(c) a diagonal matrix (d) a unit matrix

**2.** If $A=\begin{bmatrix} 0 & -5i & x \\ y & 0 & z \\ \frac{3}{2} & -\sqrt{2} & 0 \end{bmatrix}$ is a skew-symmetric matrix, then

(a) $x=-\frac{3}{2}$, $y=5i$, $z=\sqrt{2}$

(b) $x=\frac{3}{2}$, $y=5i$, $z=\sqrt{2}$

(c) $x=\frac{3}{2}$, $y=-5i$, $z=\sqrt{2}$

(d) $x=-\frac{3}{2}$, $y=-5i$, $z=\sqrt{2}$

**3.** If A is a $3\times 4$ matrix and B is a matrix such that $A^TB$ and $BA^T$ are both defined. Then order of B is

(a) $3\times 4$ (b) $3\times 3$ (c) $4\times 4$ (d) $4\times 3$

**4.** If A is a diagonal matrix diag$(d_1, d_2, d_3, \ldots, d_n)$, then $A^n$, $n\in N$ is

(a) diag $(nd_1, nd_2, nd_3, \ldots, nd_n)$

(b) diag $(d_1^n, d_2^n, d_3^n, \ldots, d_n^n)$

(c) diag $(d_1^{n-1}, d_2^{n-1}, d_3^{n-1}, \ldots, d_n^{n-1})$

(d) diag $(d_1^{n+1}, d_2^{n+1}, d_3^{n+1}, \ldots, d_n^{n+1})$

5. A and B are two matrices such that A − B and AB are both defined, then

(a) number of columns of A = number of rows of B

(b) A, B are square matrices of the same order

(c) A, B are of same order

(d) A = B

6. If $A-2B=\begin{bmatrix}1 & 5\\ 3 & 7\end{bmatrix}$ and $2A-3B=\begin{bmatrix}-2 & 5\\ 0 & 7\end{bmatrix}$, then matrix B is equal to

(a) $\begin{bmatrix}-4 & -5\\ -6 & -7\end{bmatrix}$ (b) $\begin{bmatrix}0 & 6\\ -3 & 7\end{bmatrix}$

(c) $\begin{bmatrix}2 & -1\\ 3 & 2\end{bmatrix}$ (d) $\begin{bmatrix}6 & -1\\ 0 & 1\end{bmatrix}$

7. If $A=\begin{bmatrix}-1 & 3 & 5\\ 1 & -3 & 5\\ -1 & 3 & 5\end{bmatrix}$, then $A^2$ is

(a) A (b) 2A (c) 3A (d) I

8. The value of $x$ if

$[1\ x\ 1]\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 3 & 2 & 5\end{bmatrix}\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}=0$ is

(a) $\frac{9}{8}$ (b) $-\frac{8}{9}$ (c) $-\frac{9}{8}$ (d) $\frac{8}{9}$

9. If $A=\begin{bmatrix}6 & 9\\ -4 & -6\end{bmatrix}$, then $A^2$ is

(a) null matrix (b) unit matrix

(c) 2A (d) 3A

10. The matrix A such that

$\begin{bmatrix}1\\ -2\\ 3\end{bmatrix}A=\begin{bmatrix}2 & -1 & 4\\ -4 & 2 & -8\\ 6 & -3 & 12\end{bmatrix}$ is

(a) $[2\ \ -1\ \ 4]$ (b) $\begin{bmatrix}2\\ -1\\ 4\end{bmatrix}$

(c) $[-4\ \ 2\ \ -9]$ (d) $[6\ \ -3\ \ 12]$

11. If $[-2\ 0\ 3]\left\{3\begin{bmatrix}2 & 8\\ -1 & 6\\ 4 & -2\end{bmatrix}-2\begin{bmatrix}-4 & -2\\ 3 & 0\\ 1 & 4\end{bmatrix}\right\}$ $=[x\ y]$, then

(a) $x=2, y=1$ (b) $x=2, y=-98$

(c) $x=2, y=98$ (d) $x=-2, y=-98$

12. If $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ and $A(\text{adj } A)=kI$, then the value of $k$ is

(a) 2 (b) −2 (c) 10 (d) −10

13. If $A^{-1}=-\frac{1}{2}\begin{bmatrix}1 & -4\\ -1 & 2\end{bmatrix}$, then A is

(a) $\begin{bmatrix}1 & -4\\ -1 & 2\end{bmatrix}$ (b) $\begin{bmatrix}2 & 4\\ -1 & 1\end{bmatrix}$

(c) $\begin{bmatrix}2 & 4\\ 1 & 1\end{bmatrix}$ (d) $\begin{bmatrix}2 & -4\\ 1 & 1\end{bmatrix}$

14. The inverse of matrix $\begin{bmatrix}1 & -1\\ 2 & 3\end{bmatrix}$ is

(a) $\frac{1}{5}\begin{bmatrix}3 & -1\\ -2 & 1\end{bmatrix}$ (b) $\frac{1}{5}\begin{bmatrix}3 & 1\\ -2 & 1\end{bmatrix}$

(c) $\frac{1}{5}\begin{bmatrix}-3 & 1\\ -2 & 1\end{bmatrix}$ (d) $\frac{1}{5}\begin{bmatrix}3 & -1\\ 2 & -1\end{bmatrix}$

15. If $A=\begin{bmatrix}5 & -1 & 3\\ 0 & 1 & 2\end{bmatrix}$, $B=\begin{bmatrix}0 & 2 & 3\\ 1 & -1 & 4\end{bmatrix}$, then $(AB^T)^T$ is equal to

(a) $\begin{bmatrix}7 & 8\\ 7 & 0\end{bmatrix}$ (b) $\begin{bmatrix}-7 & 8\\ 0 & 7\end{bmatrix}$

(c) $\begin{bmatrix}7 & 8\\ 18 & 7\end{bmatrix}$ (d) $\begin{bmatrix}7 & 18\\ 8 & 7\end{bmatrix}$

16. The element in third row and first column of the inverse of the matrix $\begin{bmatrix}1 & -3 & 2\\ -3 & 3 & -1\\ 2 & -1 & 0\end{bmatrix}$ is

(a) −3 (b) 2 (c) 4 (d) 3

17. If a 3 × 3 matrix A has its inverse equal to A, then $A^2=$ .........

(a) $\begin{bmatrix}0 & 1 & 0\\ 1 & 1 & 1\\ 0 & 1 & 0\end{bmatrix}$ (b) $\begin{bmatrix}1 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 1\end{bmatrix}$

(c) $\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ (d) $\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$

**18.** The inverse of the matrix $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ is

(a) $\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$ (d) $-\frac{1}{2}\begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

**19.** If $A^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 4 & -2 \\ -2 & -5 & 4 \\ 1 & -2 & 1 \end{bmatrix}$ and $|A| = 3$, then $(\text{adj } A) = \ldots\ldots$

(a) $\frac{1}{9}\begin{bmatrix} 1 & 4 & -2 \\ -2 & -5 & 4 \\ 1 & -2 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -2 & 1 \\ 4 & -5 & -2 \\ -2 & 4 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 4 & -2 \\ -2 & -5 & 4 \\ 1 & -2 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & -4 & 2 \\ 2 & 5 & -4 \\ 1 & -2 & 1 \end{bmatrix}$

**20.** If $A = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix}$, then adjoint of matrix A is

(a) $\begin{bmatrix} 1 & 3 \\ -4 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -3 \\ -4 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 3 \\ 4 & -2 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & -3 \\ -4 & 2 \end{bmatrix}$

### Answers

**1.** (b) a non-singular matrix

**2.** (a) $x = -\frac{3}{2}$, $y = 5i$, $z = \sqrt{2}$

**3.** (a) $3 \times 4$ **4.** (b) $\text{diag}(d_1^n, d_2^n, d_3^n, \ldots, d_n^n)$

**5.** (b) A, B are square matrices of the same order

**6.** (a) $\begin{bmatrix} -4 & -5 \\ -6 & -7 \end{bmatrix}$ **7.** (a) A **8.** (c) $-\frac{9}{8}$

**9.** (a) null matrix **10.** (a) $[2 \ \ -1 \ \ 4]$

**11.** (b) $x = 2$, $y = -98$ **12.** (b) $-2$

**13.** (c) $\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$ **14.** (b) $\frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$

**15.** (c) $\begin{bmatrix} 7 & 8 \\ 18 & 7 \end{bmatrix}$ **16.** (d) 3 **17.** (c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

**18.** (a) $\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ **19.** (c) $\begin{bmatrix} 1 & 4 & -2 \\ -2 & -5 & 4 \\ 1 & -2 & 1 \end{bmatrix}$

**20.** (a) $\begin{bmatrix} 1 & 3 \\ -4 & 2 \end{bmatrix}$.

### TRUE OR FALSE

**State whether the following statements are *True* or *False* :**

1. Every identity matrix is a scalar matrix but every scalar matrix need not be identity matrix.
2. If $A = \begin{bmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{bmatrix}$, then A is a non-singular matrix.
3. If B is a square matrix and $B^T = -B$, then B is called a skew-symmetric matrix.
4. If AB exists, then BA may or may not exist.
5. If AB and BA both exist, then AB = BA.
6. If $A = \begin{bmatrix} 5 & 0 & -1 \\ 0 & 7 & 0 \\ 0 & 0 & 8 \end{bmatrix}$, then A is a lower triangular matrix.
7. The matrix $A = \begin{bmatrix} x+y & y+z & z+x \\ 1 & 1 & 1 \\ z & x & y \end{bmatrix}$ is a singular matrix.
8. If the order of A is $2 \times 3$, order of B is $3 \times 2$ and order of C is $3 \times 3$, then $C(A + B^T)$ is not defined.
9. If $\begin{bmatrix} i & 0 \\ 3 & -i \end{bmatrix} + X = \begin{bmatrix} i & 2 \\ 3 & 4+i \end{bmatrix} - X$, then $X = \begin{bmatrix} 0 & -1 \\ 0 & 2+i \end{bmatrix}$.
10. The diagonal matrices are symmetric.
11. If $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, then $A^2 - 7A - 2I$ is a null matrix.
12. If A and B are two matrices, then $(A+B)^2 = A^2 + 2AB + B^2$.
13. If $A = \begin{bmatrix} 3 & \alpha \\ -1 & 2 \end{bmatrix}$ and $A^2 - 5A + 7I = 0$, then $\alpha = 1$.

**14.** If $A = \begin{bmatrix} 5 & 4 \\ -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 3 \\ 4 & -1 \end{bmatrix}$ and $3A - 2B + C = I$, where I is the unit matrix of order 2, then $C^T = \begin{bmatrix} -7 & 8 \\ 0 & 7 \end{bmatrix}$.

**15.** If A and B are square matrices of the same order, then $(A+B)^2 = A^2 + B^2$ if and only if $AB = BA$.

**16.** If $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 1 \\ 3 & -1 \\ 0 & 2 \end{bmatrix}$, then AB is a singular matrix.

**17.** If $A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$, then $A^{-1}$ does not exist.

**18.** $\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 13 & 9 \\ -3 & 4 & -1 \end{bmatrix}$.

**19.** If $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & a & 1 \end{bmatrix}$ and $A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ -8 & 6 & 2c \\ 5 & -3 & 1 \end{bmatrix}$, then $a = 1$, $c = 1$.

**20.** If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 6 & 8 \end{bmatrix}$, then $A^{-1}$ does not exist.

### Answers

| | | | | |
|---|---|---|---|---|
| **1.** True | **2.** False | **3.** True | **4.** True | **5.** False |
| **6.** False | **7.** True | **8.** True | **9.** False | **10.** True |
| **11.** True | **12.** False | **13.** True | **14.** False | **15.** False |
| **16.** False | **17.** True | **18.** False | **19.** False | **20.** True. |

## FILL IN THE BLANKS

**Fill in the following blanks with an appropriate words/numbers :**

**1.** If $A = [a_{ij}]$ is a square matrix of even order such that $a_{ij} = i^2 - j^2$, then A is a .........

**2.** If $A = \begin{bmatrix} 5 & -3 \\ 1 & 0 \\ -4 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 7 \\ -3 & 1 \\ 2 & -2 \end{bmatrix}$, then $2A - 3B = $ .........

**3.** If $A = [a_{ij}]_{3 \times 2}$ such that $a_{ij} = \frac{i + 4j}{2}$, then A = .........

**4.** If the order of matrix A is $3 \times 4$ and AB, $AB^T$ both exist, then the order of B is .........

**5.** If A and B are any two matrices, then AB exists if and only if .........

**6.** If $A = \begin{bmatrix} 2 & a & 3 \\ -7 & 4 & 5 \\ c & b & 6 \end{bmatrix}$ is a symmetric matrix, then $a + b + c = $ .........

**7.** If $A = \begin{bmatrix} 2 & 6 \\ 3 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 3 & x \\ y & 2 \end{bmatrix}$ and AB is a null matrix, then $x = $ ........., $y = $ .........

**8.** If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$ and $A^2 = kA - 2I$, then $k = $ ......... .

**9.** If $A = [1\ 3\ 2]$ and $B = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, then AB = .........

**10.** The value of $[1\ 2]\begin{bmatrix} 3 \\ 4 \end{bmatrix} + [1\ 2\ 3\ 4]\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ is .........

**11.** If A, B, C are the matrices of orders $2 \times 3$, $2 \times 3$, $3 \times 1$ respectively and $(3A - B)C = X$, then order of X is ......... .

**12.** If the matrix $A = \begin{bmatrix} 4 & a \\ 3 & 4 \end{bmatrix}$ satisfies the equation $AA^T = 25I$, then $a = $ ......... .

**13.** If A and B are two square matrices of same order, then $(BA)^T = $ .........

**14.** If $A = \begin{bmatrix} 1 & 2 & 4 \\ 4 & 3 & -2 \\ 1 & 0 & -3 \end{bmatrix}$, then $A_{22} = $ ......... .

**15.** If A is a square matrix such that $A(\text{adj } A) = \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix}$, then $|A| = $ .........

**16.** $B = \begin{bmatrix} 4 & a \\ -2 & 5 \end{bmatrix}$ is the adjoint of a matrix A and $|A| = 6$, then $a = $ ......... .

**17.** If the matrix $\begin{bmatrix} 6 & -5 & 1 \\ 4 & 2 & -1 \\ 14 & -1 & k \end{bmatrix}$ has no inverse, then $k = $ ......... .

**18.** The inverse of $\begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$ is ......... .

**19.** If $A = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $A^{-1}X = B$, then X = ......... .

**20.** If $A = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix}$, then $A^T \cdot B^T = $ ......... .

## Answers

1. skew-symmetric matrix

2. $\begin{bmatrix} 4 & -27 \\ 11 & -3 \\ -14 & 2 \end{bmatrix}$

3. $\begin{bmatrix} \frac{5}{2} & \frac{9}{2} \\ 3 & 5 \\ \frac{7}{2} & \frac{11}{2} \end{bmatrix}$

4. $4 \times 4$

5. the number of columns of A = the number of rows of B.

6. 1

7. $-6, -1$

8. 1

9. [11]

10. [13]

11. $2 \times 1$

12. $-3$

13. $A^T \cdot B^T$

14. $-7$

15. 20

16. $-7$

17. $-1$

18. $\frac{1}{13}\begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$

19. $\begin{bmatrix} 6 \\ 4 \end{bmatrix}$

20. $\begin{bmatrix} 66 & 55 \\ 55 & 66 \end{bmatrix}$.

❋❋❋

# 3 DIFFERENTIATION

## CHAPTER OUTLINE



## IMPORTANT FORMULAE

**1.** If $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ exists, then this limit is called the derivative of $f$ and it is denoted by $f'(x)$.

Finding the derivative of a given function by using the above definition is called finding its derivatives from *first principle*.

**2.** $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$, if it exists, is called derivative of $f$ at $x = a$ and is denoted by $f'(a)$.

**3. Derivatives of Standard Functions :**

| Functions | Derivatives $\frac{dy}{dx}$ or $f'(x)$ |
|---|---|
| $x^n$ | $nx^{n-1}$ |
| $k$ | $0$ |
| $\sqrt{x}$ | $\frac{1}{2\sqrt{x}}$ |
| $\frac{1}{x}$ | $-\frac{1}{x^2}$ |
| $a^x$ | $a^x \cdot \log a$ |
| $e^x$ | $e^x$ |
| $\log x$ | $\frac{1}{x}$ |

In derivatives, if there is a function of $x$ in place of $x$, use the same formula and multiply the result by the derivative of the function.

e.g. (1) $\frac{d}{dx}[f(x)]^n = n \cdot [f(x)]^{n-1} \cdot f'(x)$

(2) $\frac{d}{dx}[a^{f(x)}] = a^{f(x)} \cdot \log a \cdot f'(x)$

(3) $\frac{d}{dx}\{\log [f(x)]\} = \frac{1}{f(x)} \times f'(x)$.

**4. Rules of Differentiation :**

(i) If $y = u \pm v$, then $\frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$

(ii) If $y = ku$, then $\frac{dy}{dx} = k\frac{du}{dx}$

(iii) If $y = uv$, then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

(iv) If $y = \frac{u}{v}$, then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$, $(v \neq 0)$

**5.** If $y = f(u)$ is differentiable function of $u$ and $u = g(x)$ is a differentiable function of $x$, then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

**6.** If $y = f(x)$ is a differentiable function of $x$ such that the inverse function $x = f^{-1}(y)$ exists, then $x$ is a differentiable function of $y$ and $\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)}$, $\frac{dy}{dx} \neq 0$.

7. If $y=[f(x)]^{g(x)}$, then it is convenient to find the derivative of the logarithm of the function, and $\frac{dy}{dx}=[f(x)]^{g(x)}\left[\frac{g(x)\cdot f'(x)}{f(x)}+g'(x)\cdot \log f(x)\right]$

*[This formula cannot be used directly.]*

8. When a given relation expresses $y$ as an implicit function of $x$ and we want to find $\frac{dy}{dx}$, then we differentiate every term of the relation w.r.t. $x$, remembering that a term in $y$ is first differentiated w.r.t. $y$ and then multiplied by $\frac{dy}{dx}$.

9. If $x$ and $y$ are differentiable functions of $t$, then $\frac{dy}{dx}=\left(\frac{dy}{dt}\right)\Big/\left(\frac{dx}{dt}\right)$, $\frac{dx}{dt}\neq 0$.

10. The derivative of first order derivative, i.e. $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is called second order derivative and it is denoted by $\frac{d^2y}{dx^2}$ or $y_2$ or $f''(x)$ or $y''$.

## INTRODUCTION

The concept of derivative of a function and the method of obtaining the derivative of some functions has been introduced in Std. XI. We have used this concept in calculating the marginal cost, the marginal demand of a commodity.

Let us recall the definition and the rules of differentiation that you have studied.

**Definition :** If $x$ and $x+\delta x$ belong to the domain of $f$ and $\lim_{\delta x \to 0}\frac{f(x+\delta x)-f(x)}{\delta x}$ exists, then the function $f$ is said to be **differentiable** or **derivable** at $x$ and this limit is called the **derivative** of $f$ at $x$, which is denoted by $f'(x)$ or $\frac{d}{dx}[f(x)]$.

Thus, $f'(x)=\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}$.

**Derivative at a point :**

If $f$ is a function and $a$, $a+h$ belong to the domain of $f$, then the limit given by $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$, if it exists, is called the **derivative** of $f$ at $x=a$, and is denoted by $f'(a)$.

Thus, $f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$.

**Leibnitz Notation of Derivatives :**

The ratio $\frac{\delta y}{\delta x}=\frac{f(x+\delta x)-f(x)}{\delta x}$ is called incrementry ratio and $f'(x)=\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}$

$=\lim_{\delta x\to 0}\frac{\delta y}{\delta x}$

This limit is denoted by $\frac{d}{dx}(y)$ or simply as $\frac{dy}{dx}$. This is Leibnitz notation for the derivative.

Thus, $\frac{dy}{dx}=\lim_{\delta x\to 0}\frac{\delta y}{\delta x}$, if it exists.

**Note :** The derivative of $y=f(x)$ is also denoted by $y_1$ or $y'$.

**Derivatives of Standard Functions :**

The derivatives of some standard functions are given in tabular form.

| Sr. No. | $y$ or $f(x)$ | $\frac{dy}{dx}$ or $f'(x)$ |
|---|---|---|
| 1. | $k$ (constant) | 0 |
| 2. | $x^n$ | $nx^{n-1}$ |
| 3. | $\frac{1}{x}$ | $\frac{-1}{x^2}$ |
| 4. | $\sqrt{x}$ | $\frac{1}{2\sqrt{x}}$ |
| 5. | $a^x$ $(a>0, a\neq 1)$ | $a^x \log a$ |
| 6. | $e^x$ | $e^x$ |
| 7. | $\log x$ | $\frac{1}{x}$ |

**Rules of Differentiation :**

If $u$ and $v$ are differentiable functions of $x$, $k$ is a constant and

(i) $y=u+v$, then $\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$

(ii) $y=u-v$, then $\frac{dy}{dx}=\frac{du}{dx}-\frac{dv}{dx}$

(iii) $y=ku$, then $\frac{dy}{dx}=k\frac{du}{dx}$

(iv) $y = uv$, then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

(v) $y = uvw$, then $\frac{dy}{dx} = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$

(vi) $y = \frac{u}{v}$, then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$, $(v \neq 0)$.

## 3.1 : DERIVATIVE OF A COMPOSITE FUNCTION

Consider, the function given as $y = (3x^2 - 2x + 5)^{\frac{7}{2}}$. This function cannot be expressed either as a sum or a difference or product or quotient of function.

Now, put $u = 3x^2 - 2x + 5$. Then $y = u^{\frac{7}{2}}$. Thus, $y$ is a function of $u$ where $u$ is a function of $x$ and in turn $y$ becomes a function of $x$. In this case, $y$ is a function of a function or a **composite function.**

We state (without proof) the rule for differentiating such a function in the following theorem which is called **chain rule.**

**Theorem 1 :**

If $y = f(u)$ is a differentiable function of $u$ and $u = g(x)$ is a differentiable function of $x$, then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

**Note :** If $y$ is a differentiable function of $u_1$, $u_i$ is a differentiable function of $u_{i+1}$, $i = 1, 2, 3, ..., n-1$ and $u_n$ is a differentiable function of $x$, then

$$\frac{dy}{dx} = \frac{dy}{du_1} \times \frac{du_1}{du_2} \times \frac{du_2}{du_3} \times ... \times \frac{du_n}{dx}.$$

**Result 1 : If $y = [f(x)]^n$, where $f(x)$ is a differentiable functions of $x$, then $\frac{dy}{dx} = n[f(x)]^{n-1} \cdot f'(x)$.**

**Proof :** $y = [f(x)]^n$

Put $u = f(x)$. Then $y = u^n$

$\therefore \frac{dy}{du} = \frac{d}{du}(u^n) = nu^{n-1} = n[f(x)]^{n-1}$

and $\frac{du}{dx} = \frac{d}{dx}[f(x)] = f'(x)$

$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = n[f(x)]^{n-1} \cdot f'(x)$.

**Result 2 : If $y = \log [f(x)]$, where $f(x)$ is differentiable function of $x$, then $\frac{dy}{dx} = \frac{1}{f(x)} \times f'(x)$.**

**Proof :** $y = \log [f(x)]$

Put $u = f(x)$. Then $y = \log u$

$\therefore \frac{dy}{du} = \frac{d}{du}(\log u) = \frac{1}{u} = \frac{1}{f(x)}$

and $\frac{du}{dx} = \frac{d}{dx}[f(x)] = f'(x)$

$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{f(x)} \times f'(x)$.

**Note :** We can prove the other formulae in the similar way. They are given in the following table :

| Sr. No. | $y$ | $\frac{dy}{dx}$ |
|---|---|---|
| 1. | $[f(x)]^n$ | $n[f(x)]^{n-1} \times f'(x)$ |
| 2. | $\sqrt{f(x)}$ | $\frac{1}{2\sqrt{f(x)}} \times f'(x)$ |
| 3. | $\frac{1}{f(x)}$ | $\frac{-1}{[f(x)]^2} \times f'(x)$ |
| 4. | $a^{f(x)}$ | $a^{f(x)} \cdot \log a \times f'(x)$ |
| 5. | $e^{f(x)}$ | $e^{f(x)} \times f'(x)$ |
| 6. | $\log [f(x)]$ | $\frac{1}{f(x)} \times f'(x)$ |
| 7. | $\log_a [f(x)]$ | $\frac{1}{f(x) \cdot \log_e a} \times f'(x)$ |

**EXERCISE 3.1** **Textbook page 91**

**1. Find $\frac{dy}{dx}$ if :**

**(1)** $y = \sqrt{x + \frac{1}{x}}$ **(2)** $y = \sqrt[3]{a^2 + x^2}$

**(3)** $y = (5x^3 - 4x^2 - 8x)^9$.

**Solution :**

**(1)** Given : $y = \sqrt{x + \frac{1}{x}}$

Let $u = x + \frac{1}{x}$

Then $y = \sqrt{u}$

$\therefore \frac{dy}{du} = \frac{d}{du}(u^{\frac{1}{2}}) = \frac{1}{2}u^{-\frac{1}{2}}$

$$= \frac{1}{2\sqrt{u}} = \frac{1}{2\sqrt{x+\frac{1}{x}}}$$

and $\frac{du}{dx} = \frac{d}{dx}\left(x + \frac{1}{x}\right)$

$$= \frac{d}{dx}(x) + \frac{d}{dx}(x^{-1})$$

$$= 1 + (-1)x^{-2} = 1 - \frac{1}{x^2}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2\sqrt{x+\frac{1}{x}}} \cdot \left(1 - \frac{1}{x^2}\right)$$

$$= \frac{1}{2}\left(x + \frac{1}{x}\right)^{-\frac{1}{2}}\left(1 - \frac{1}{x^2}\right).$$

**(2)** Given : $y = \sqrt[3]{a^2 + x^2}$

Let $u = a^2 + x^2$

Then $y = \sqrt[3]{u}$

$$\therefore \frac{dy}{du} = \frac{d}{du}(u^{\frac{1}{3}}) = \frac{1}{3}u^{-\frac{2}{3}}$$

$$= \frac{1}{3}(a^2 + x^2)^{-\frac{2}{3}}$$

and $\frac{du}{dx} = \frac{d}{dx}(a^2 + x^2)$

$$= 0 + 2x = 2x$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{3}(a^2 + x^2)^{-\frac{2}{3}} \cdot 2x$$

$$= \frac{2x}{3}(a^2 + x^2)^{-\frac{2}{3}}.$$

**(3)** Given : $y = (5x^3 - 4x^2 - 8x)^9$

Let $u = 5x^3 - 4x^2 - 8x$

Then $y = u^9$

$$\therefore \frac{dy}{du} = \frac{d}{du}(u^9) = 9u^8$$

$$= 9(5x^3 - 4x^2 - 8x)^8$$

and $\frac{du}{dx} = \frac{d}{dx}(5x^3 - 4x^2 - 8x)$

$$= 5\frac{d}{dx}(x^3) - 4\frac{d}{dx}(x^2) - 8\frac{d}{dx}(x)$$

$$= 5 \times 3x^2 - 4 \times 2x - 8 \times 1$$

$$= 15x^2 - 8x - 8$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= 9(5x^3 - 4x^2 - 8x)^8(15x^2 - 8x - 8).$$

**2. Find $\frac{dy}{dx}$ if :**

**(1)** $y = \log(\log x)$ **(2)** $y = \log(10x^4 + 5x^3 - 3x^2 + 2)$

**(3)** $y = \log(ax^2 + bx + c)$.

**Solution :**

**(1)** Given : $y = \log(\log x)$

Let $u = \log x$

Then $y = \log u$

$$\therefore \frac{dy}{du} = \frac{d}{du}(\log u)$$

$$= \frac{1}{u} = \frac{1}{\log x}$$

and $\frac{du}{dx} = \frac{d}{dx}(\log x) = \frac{1}{x}$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{\log x} \times \frac{1}{x}$$

$$= \frac{1}{x \log x}.$$

**(2)** Given : $y = \log(10x^4 + 5x^3 - 3x^2 + 2)$

Let $u = 10x^4 + 5x^3 - 3x^2 + 2$

Then $y = \log u$

$$\therefore \frac{dy}{du} = \frac{d}{du}(\log u) = \frac{1}{u}$$

$$= \frac{1}{10x^4 + 5x^3 - 3x^2 + 2}$$

and $\frac{du}{dx} = \frac{d}{dx}(10x^4 + 5x^3 - 3x^2 + 2)$

$$= 10\frac{d}{dx}(x^4) + 5\frac{d}{dx}(x^3) - 3\frac{d}{dx}(x^2) + \frac{d}{dx}(2)$$

$$= 10 \times 4x^3 + 5 \times 3x^2 - 3 \times 2x + 0$$

$$= 40x^3 + 15x^2 - 6x$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{10x^4 + 5x^3 - 3x^2 + 2} \times (40x^3 + 15x^2 - 6x)$$

$$= \frac{40x^3 + 15x^2 - 6x}{10x^4 + 5x^3 - 3x^2 + 2}.$$

(3) Given : $y = \log(ax^2 + bx + c)$

Let $u = ax^2 + bx + c$

Then $y = \log u$

$\therefore \frac{dy}{du} = \frac{d}{du}(\log u) = \frac{1}{u}$

$= \frac{1}{ax^2 + bx + c}$

and $\frac{du}{dx} = \frac{d}{dx}(ax + bx + c)$

$= a\frac{d}{dx}(x^2) + b\frac{d}{dx}(x) + \frac{d}{dx}(c)$

$= a \times 2x + b \times 1 \times 0 = 2ax + b$

$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{ax^2 + bx + c} \times (2ax + b)$

$= \frac{2ax + b}{ax^2 + bx + c}$.

**3. Find $\frac{dy}{dx}$ if :**

**(1) $y = e^{5x^2 - 2x + 4}$** **(2) $y = a^{(1 + \log x)}$**

**(3) $y = 5^{(x + \log x)}$**

**Solution :**

(1) Given : $y = e^{5x^2 - 2x + 4}$

Let $u = 5x^2 - 2x + 4$

Then $y = e^u$

$\therefore \frac{dy}{du} = \frac{d}{du}(e^u) = e^u$

$= e^{5x^2 - 2x + 4}$

and $\frac{du}{dx} = \frac{d}{dx}(5x^2 - 2x + 4)$

$= 5\frac{d}{dx}(x^2) - 2\frac{d}{dx}(x) + \frac{d}{dx}(4)$

$= 5 \times 2x - 2 \times 1 + 0 = 10x - 2$

$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= e^{5x^2 - 2x + 4} \times (10x - 2)$

$= (10x - 2)e^{5x^2 - 2x + 4}$.

(2) Given : $y = a^{(1 + \log x)}$

Let $u = 1 + \log x$

Then $y = a^u$

$\therefore \frac{dy}{du} = \frac{d}{du}(a^u) = a^u \cdot \log a$

$= a^{(1 + \log x)} \cdot \log a$

and $\frac{du}{dx} = \frac{d}{dx}(1 + \log x)$

$= 0 + \frac{1}{x} = \frac{1}{x}$

$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= a^{(1 + \log x)} \cdot \log a \cdot \frac{1}{x}$.

(3) Given : $y = 5^{(x + \log x)}$

Let $u = x + \log x$

Then $y = 5^u$

$\therefore \frac{dy}{du} = \frac{d}{du}(5^u) = 5^u \cdot \log 5$

$= 5^{(x + \log x)} \cdot \log 5$

and $\frac{du}{dx} = \frac{d}{dx}(x + \log x)$

$= 1 + \frac{1}{x}$

$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= 5^{(x + \log x)} \cdot \log 5 \cdot \left(1 + \frac{1}{x}\right)$.

## EXAMPLES FOR PRACTICE 3.1

**1. Find $\frac{dy}{dx}$ if :**

(1) $y = \sqrt{2x^2 + 5}$ (2) $y = \frac{1}{(x^2 + 3)^2}$

(3) $y = (4x^3 + 3x^2 - 2x)^6$ (4) $y = (3x^4 - x^3 + 4)^{\frac{5}{2}}$

(5) $y = \sqrt[3]{\left(2x^2 + 3 + \frac{1}{x}\right)^5}$ (6) $y = \sqrt[3]{(3x^2 + 8x - 7)^5}$.

**2. Find $\frac{dy}{dx}$ if :**

(1) $y = \log(2x^2 + 1)$ (2) $y = \log(4x^2 + 3x - 1)$

(3) $y = \log\sqrt{x^2 + 4}$ (4) $y = \log(\log x^2)$.

**3. Find $\frac{dy}{dx}$ if :**

(1) $y = e^{\log(\log x)}$ (2) $y = e^{x + 2\log x}$

(3) $y = e^{x^3} \cdot e^x$ (4) $y = e^{ax^2 + bx + c}$

(5) $y = e^{(\log x + 6)}$.

4. **Find $\frac{dy}{dx}$ if :**

(1) $y = 10^{x^3 - 2x^2 + 1}$ (2) $y = 7^{x + \frac{1}{x}}$

(3) $y = a^{x + \log x}$ (4) $y = 3^{\sqrt{x} + 2}$.

## Answers

1. (1) $\frac{2x}{\sqrt{2x^2+5}}$ (2) $\frac{-4x}{(x^2+3)^3}$

(3) $6(4x^3 + 3x^2 - 2x)^5(12x^2 + 6x - 2)$

(4) $\frac{15}{2}x^2(4x-1)(3x^4 - x^3 + 4)^{\frac{3}{2}}$

(5) $\frac{5}{3}\left(2x^2 + 3 + \frac{1}{x}\right)^{\frac{2}{3}} \cdot \left(4x - \frac{1}{x^2}\right)$

(6) $\frac{5}{3}(3x^2 + 8x - 7)^{\frac{2}{3}} \cdot (6x + 8)$.

2. (1) $\frac{4x}{2x^2+1}$ (2) $\frac{8x+3}{4x^2+3x-1}$

(3) $\frac{x}{x^2+4}$ (4) $\frac{1}{2x \log x}$.

3. (1) $\frac{1}{x}$ (2) $(x^2 + 2x)e^x$ (3) $e^{x^5 + x} \cdot (5x^4 + 1)$

(4) $(2ax + b)e^{ax^2 + bx + c}$ (5) $\frac{e^{\log x + 6}}{x}$.

4. (1) $(3x^2 - 4x) \cdot 10^{x^3 - 2x^2 + 1} \cdot \log 10$

(2) $\left(\frac{x^2 - 1}{x^2}\right) \cdot 7^{x + \frac{1}{x}} \cdot \log 7$

(3) $\left(\frac{x+1}{x}\right) \cdot a^{x + \log x} \cdot \log a$ (4) $\frac{3^{\sqrt{x}+2} \cdot \log 3}{2\sqrt{x}}$.

## 3.2 : DERIVATIVE OF INVERSE FUNCTION

We know that if a function $f : A \to B$ is one-one and onto, then its inverse function $f^{-1} : B \to A$ exists. If we exhibit the function $f$ by $y = f(x)$, then its inverse function $f^{-1}$ is exhibited by $x = f^{-1}(y)$. We state (without proof) the rule for finding the derivative of $f^{-1}$ in terms of the derivative of $f$.

**Theorem 2 :**

If $y = f(x)$ is a derivable function of $x$ such that the inverse function $x = f^{-1}(y)$ is defined, then $x$ is differentiable function of $y$ and

$\frac{dx}{dy} = \frac{1}{(dy/dx)}$, where $\frac{dy}{dx} \neq 0$. OR

If $y = f(x)$ and $x = g(y)$, where $g$ is the inverse of $f$, i.e. $g = f^{-1}$ and if $\frac{dy}{dx}$ and $\frac{dx}{dy}$ both exist and $\frac{dx}{dy} \neq 0$, then $\frac{dy}{dx} = \frac{1}{(dx/dy)}$.

| EXERCISE 3.2 | Textbook page 92 |
|---|---|

1. **Find the rate of change of demand ($x$) of a commodity with respect to price ($y$) if :**

**(1) $y = 12 + 10x + 25x^2$**

**(2) $y = 18x + \log(x - 4)$**

**(3) $y = 25x + \log(1 + x^2)$.**

**Solution :**

**(1)** Given : $y = 12 + 10x + 25x^2$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(12 + 10x + 25x^2)$$

$$= \frac{d}{dx}(12) + 10\frac{d}{dx}(x) + 25\frac{d}{dx}(x^2)$$

$$= 0 + 10 \times 1 + 25 \times 2x = 10 + 50x$$

By derivative of inverse function,

$$\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)} = \frac{1}{10 + 50x}$$

Hence, the rate of change of demand ($x$) with respect to price ($y$)

$$= \frac{dx}{dy} = \frac{1}{10 + 50x}.$$

**(2)** Given : $y = 18x + \log(x - 4)$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}[18x + \log(x - 4)]$$

$$= 18\frac{d}{dx}(x) + \frac{d}{dx}[\log(x - 4)]$$

$$= 18 \times 1 + \frac{1}{x-4} \cdot \frac{d}{dx}(x - 4)$$

$$= 18 + \frac{1}{x-4} \times (1 - 0)$$

$$= 18 + \frac{1}{x-4} = \frac{18x - 72 + 1}{x - 4}$$

$$= \frac{18x - 71}{x - 4}$$

By derivative of inverse function

$$\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)} = \frac{x - 4}{18x - 71}$$

Hence, the rate of change of demand ($x$) with respect to price ($y$)

$$= \frac{dx}{dy} = \frac{x-4}{18x-71}.$$

**(3)** Given : $y = 25x + \log(1+x^2)$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}[25x + \log(1+x^2)]$$

$$= 25\frac{d}{dx}(x) + \frac{d}{dx}[\log(1+x^2)]$$

$$= 25 \times 1 + \frac{1}{1+x^2}\cdot\frac{d}{dx}(1+x^2)$$

$$= 25 + \frac{1}{1+x^2} \times (0+2x)$$

$$= 25 + \frac{2x}{1+x^2} = \frac{25+25x^2+2x}{1+x^2}$$

$$= \frac{25x^2+2x+25}{1+x^2}$$

By derivative of inverse function,

$$\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)} = \frac{1+x^2}{25x^2+2x+25}$$

Hence, the rate of change of demand ($x$) with respect to price ($y$)

$$= \frac{dx}{dy} = \frac{1+x^2}{25x^2+2x+25}.$$

**2. Find the marginal demand of a commodity where demand is $x$ and price is $y$ and**

**(1)** $y = xe^{-x} + 7$ **(2)** $y = \frac{x+2}{x^2+1}$

**(3)** $y = \frac{5x+9}{2x-10}$.

**Solution :**

**(1)** Given : $y = xe^{-x} + 7$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(xe^{-x}+7)$$

$$= \frac{d}{dx}(xe^{-x}) + \frac{d}{dx}(7)$$

$$= x\cdot\frac{d}{dx}(e^{-x}) + e^{-x}\cdot\frac{d}{dx}(x) + 0$$

$$= x \times e^{-x}\cdot\frac{d}{dx}(-x) + e^{-x} \times 1$$

$$= xe^{-x}(-1) + e^{-x}$$

$$= e^{-x}(-x+1) = \frac{1-x}{e^x}$$

By the derivative of inverse function,

$$\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)} = \frac{e^x}{1-x}$$

Hence, marginal demand $= \frac{dx}{dy} = \frac{e^x}{1-x}$.

**(2)** Given : $y = \frac{x+2}{x^2+1}$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(\frac{x+2}{x^2+1}\right)$$

$$= \frac{(x^2+1)\cdot\frac{d}{dx}(x+2) - (x+2)\cdot\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$= \frac{(x^2+1)(1+0)-(x+2)(2x+0)}{(x^2+1)^2}$$

$$= \frac{x^2+1-2x^2-4x}{(x^2+1)^2} = \frac{1-4x-x^2}{(x^2+1)^2}$$

By the derivative of inverse function,

$$\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)} = \frac{(x^2+1)^2}{1-4x-x^2}$$

Hence, marginal demand $= \frac{dx}{dy}$

$$= \frac{(x^2+1)^2}{1-4x-x^2}.$$

**(3)** Given : $y = \frac{5x+9}{2x-10}$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(\frac{5x+9}{2x-10}\right)$$

$$= \frac{(2x-10)\cdot\frac{d}{dx}(5x+9) - (5x+9)\cdot\frac{d}{dx}(2x-10)}{(2x-10)^2}$$

$$= \frac{(2x-10)(5\times1+0)-(5x+9)(2\times1-0)}{(2x-10)^2}$$

$$= \frac{10x-50-10x-18}{(2x-10)^2} = \frac{-68}{(2x-10)^2}$$

By the derivative of inverse function,

$$\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)} = -\frac{(2x-10)^2}{68}$$

Hence, marginal demand $= \frac{dx}{dy} = \frac{-(2x-10)^2}{68}$.

## EXAMPLES FOR PRACTICE 3.2

**1. Find the rate of change of demand ($x$) of a commodity with respect to its price ($y$), if :**

(1) $y = 15 + 17x + 35x^2$

(2) $y = 5 + x^2e^{-x} + 2x$

(3) $y = 29x + \log(1 + x^2)$

(4) $y = e^{2x-3} + 5.$

**2. Find the marginal demand of a commodity where demand is $x$ and price is $y$, if :**

(1) $y = (x^2 - 1)^2$ (2) $y = \sqrt[3]{x-2}$

(3) $y = \dfrac{2x+7}{7x-13}$ (4) $y = \dfrac{x}{e^x} + x \log x.$

### Answers

**1.** (1) $\dfrac{1}{17+70x}$ (2) $\dfrac{1}{xe^{-x}(2-x)+2}$

(3) $\dfrac{1+x^2}{29+2x+29x^2}$ (4) $\dfrac{1}{2e^{2x-3}}.$

**2.** (1) $\dfrac{1}{4x(x^2-1)}$ (2) $3(x-2)^{\frac{2}{3}}$

(3) $-\dfrac{(7x-13)^2}{75}$ (4) $\dfrac{e^x}{(1-x)+e^x(1+\log x)}.$

## 3.3 : LOGARITHMIC DIFFERENTIATION

When we want to find the derivative of a function which is expressed as :

(i) a product of a number of functions or

(ii) a quotient of functions or

(iii) of the form $[f(x)]^{g(x)}$,

then it is convenient to find the derivative of the logarithm of the function. Hence, this method of finding the derivative of a function is known as **logarithmic differentiation**. Here we note that by chain rule,

$$\frac{d}{dx}(\log y) = \frac{d}{dy}(\log y)\cdot\frac{dy}{dx} = \frac{1}{y}\frac{dy}{dx}.$$

- **Some basic laws of logarithms :**

**1.** $\log_a(mn) = \log_a m + \log_a n$ **2.** $\log_a\left(\dfrac{m}{n}\right) = \log_a m - \log_a n$

**3.** $\log_a m^n = n\log_a m$ **4.** $\log_n m = \dfrac{\log_a m}{\log_a n}$

**5.** $\log_a a = 1$, $\log_a 1 = 0$ **6.** $\log_a a^x = x.$

**Remark :** The logarithmic function to the base '$e$' i.e. $\log_e x$ is called natural logarithm and the logarithmic function to the base '10' i.e. $\log_{10} x$ is called common logarithm.

If the base of the logarithmic function is not given, then it is considered as '$e$'. i.e. $\log x$ means $\log_e x$.

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**1. Find $\dfrac{dy}{dx}$, if :**

**(1)** $y = x^{x^{2x}}$ **(2)** $y = x^{e^x}$ **(3)** $y = e^{x^x}$.

**Solution :**

**(1)** $y = x^{x^{2x}}$

$\therefore \log y = \log x^{x^{2x}} = x^{2x}\cdot\log x$

Differentiating both sides w.r.t. $x$, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{d}{dx}(x^{2x}\cdot\log x)$$

$$= x^{2x}\cdot\frac{d}{dx}(\log x) + (\log x)\cdot\frac{d}{dx}(x^{2x})$$

$$= x^{2x}\times\frac{1}{x} + (\log x)\cdot\frac{d}{dx}(x^{2x})$$

$$\therefore \frac{dy}{dx} = y\left[\frac{x^{2x}}{x} + (\log x)\cdot\frac{d}{dx}(x^{2x})\right]$$

$$= x^{x^{2x}}\left[\frac{x^{2x}}{x} + (\log x)\cdot\frac{d}{dx}(x^{2x})\right] \quad \ldots (1)$$

Let $u = x^{2x}$

Then $\log u = \log x^{2x} = 2x \log x$

Differentiating both sides w.r.t. $x$, we get

$$\frac{1}{u}\cdot\frac{du}{dx} = 2\frac{d}{dx}(x\log x)$$

$$= 2\left[x\frac{d}{dx}(\log x) + (\log x)\cdot\frac{d}{dx}(x)\right]$$

$$= 2\left[x\times\frac{1}{x} + (\log x)\times 1\right]$$

$$\therefore \frac{du}{dx} = 2u(1 + \log x)$$

$$\therefore \frac{d}{dx}(x^{2x}) = 2x^{2x}(1 + \log x)$$

$\therefore$ from (1),

$$\frac{dy}{dx}=x^{x^{2x}}\left[\frac{x^{2x}}{x}+(\log x)\times 2x^{2x}(1+\log x)\right]$$

$$=x^{x^{2x}}\cdot x^{2x}\cdot\log x\left[2(1+\log x)+\frac{1}{x\ \log x}\right].$$

**(2)** $y=x^{e^x}$

$\therefore\ \log y=\log x^{e^x}=e^x\cdot\log x$

Differentiating both sides w.r.t. $x$, we get

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{d}{dx}(e^x\cdot\log x)$$

$$=e^x\cdot\frac{d}{dx}(\log x)+(\log x)\cdot\frac{d}{dx}(e^x)$$

$$=e^x\cdot\frac{1}{x}+(\log x)(e^x)$$

$$\therefore\ \frac{dy}{dx}=y\left[\frac{e^x}{x}+e^x\cdot\log x\right]$$

$$=x^{e^x}\cdot e^x\left[\frac{1}{x}+\log x\right].$$

**(3)** $y=e^{x^x}$

$\therefore\ \log y=\log e^{x^x}=x^x\log e$

$\therefore\ \log y=x^x$ ... $[\because\ \log e=1]$

Differentiating both sides w.r.t. $x$, we get

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{d}{dx}(x^x)$$

$$\therefore\ \frac{dy}{dx}=y\cdot\frac{d}{dx}(x^x)=e^{x^x}\cdot\frac{d}{dx}(x^x) \quad \text{... (1)}$$

Let $u=x^x$

Then $\log u=\log x^x=x\log x$

Differentiating both sides w.r.t. $x$, we get

$$\frac{1}{u}\cdot\frac{du}{dx}=\frac{d}{dx}(x\log x)$$

$$=x\cdot\frac{d}{dx}(\log x)+(\log x)\cdot\frac{d}{dx}(x)$$

$$=x\times\frac{1}{x}+(\log x)\times 1$$

$$\therefore\ \frac{du}{dx}=u(1+\log x)$$

$$\therefore\ \frac{d}{dx}(x^x)=x^x(1+\log x)$$

$\therefore$ from (1),

$$\frac{dy}{dx}=e^{x^x}\cdot x^x(1+\log x).$$

**2. Find $\frac{dy}{dx}$ if :**

**(1)** $y=\left(1+\frac{1}{x}\right)^x$ **(2)** $y=(2x+5)^x$

**(3)** $y=\sqrt[3]{\dfrac{3x-1}{(2x+3)(5-x)^2}}$.

**Solution :**

**(1)** $y=\left(1+\frac{1}{x}\right)^x$

$$\therefore\ \log y=\log\left(1+\frac{1}{x}\right)^x=x\log\left(1+\frac{1}{x}\right)$$

Differentiating both sides w.r.t. $x$, we get

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{d}{dx}\left[x\log\left(1+\frac{1}{x}\right)\right]$$

$$=x\frac{d}{dx}\left[\log\left(1+\frac{1}{x}\right)\right]+\left[\log\left(1+\frac{1}{x}\right)\right]\cdot\frac{d}{dx}(x)$$

$$=x\times\frac{1}{1+\frac{1}{x}}\cdot\frac{d}{dx}\left(1+\frac{1}{x}\right)+\left[\log\left(1+\frac{1}{x}\right)\right]\times 1$$

$$=x\times\frac{x}{x+1}\times\left(0-\frac{1}{x^2}\right)+\log\left(1+\frac{1}{x}\right)$$

$$\therefore\ \frac{dy}{dx}=y\left[\frac{-1}{x+1}+\log\left(1+\frac{1}{x}\right)\right]$$

$$=\left(1+\frac{1}{x}\right)^x\left[\log\left(1+\frac{1}{x}\right)-\frac{1}{1+x}\right].$$

**(2)** $y=(2x+5)^x$

$\therefore\ \log y=\log(2x+5)^x=x\log(2x+5)$

Differentiating both sides w.r.t. $x$, we get

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{d}{dx}[x\log(2x+5)]$$

$$=x\frac{d}{dx}[\log(2x+5)]+[\log(2x+5)]\cdot\frac{d}{dx}(x)$$

$$=x\times\frac{1}{2x+5}\cdot\frac{d}{dx}(2x+5)+[\log(2x+5)]\times 1$$

$$=\frac{x}{2x+5}\times(2\times 1+0)+\log(2x+5)$$

$$\therefore\ \frac{dy}{dx}=y\left[\frac{2x}{2x+5}+\log(2x+5)\right]$$

$$=(2x+5)^x\left[\log(2x+5)+\frac{2x}{2x+5}\right].$$

**(3)** $y=\sqrt[3]{\dfrac{3x-1}{(2x+3)(5-x)^2}}$

$$\therefore\ \log y=\log\left[\frac{3x-1}{(2x+3)(5-x)^2}\right]^{\frac{1}{3}}$$

$$=\frac{1}{3}\log\left[\frac{3x-1}{(2x+3)(5-x)^2}\right]$$

$$=\frac{1}{3}[\log(3x-1)-\log(2x+3)-\log(5-x)^2]$$

$$=\frac{1}{3}\log(3x-1)-\frac{1}{3}\log(2x+3)-\frac{2}{3}\log(5-x)$$

Differentiating both sides w.r.t. $x$, we get

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{1}{3}\frac{d}{dx}[\log(3x-1)]-\frac{1}{3}\frac{d}{dx}[\log(2x+3)]-\frac{2}{3}\frac{d}{dx}[\log(5-x)]$$

$$=\frac{1}{3}\times\frac{1}{3x-1}\cdot\frac{d}{dx}(3x-1)-\frac{1}{3}\times\frac{1}{2x+3}\cdot\frac{d}{dx}(2x+3)-\frac{2}{3}\times\frac{1}{5-x}\cdot\frac{d}{dx}(5-x)$$

$$=\frac{1}{3(3x-1)}\times(3\times1-0)-\frac{1}{3(2x+3)}\times(2\times1+0)-\frac{2}{3(5-x)}\times(0-1)$$

$$\therefore\ \frac{dy}{dx}=y\left[\frac{3}{3(3x-1)}-\frac{2}{3(2x+3)}+\frac{2}{3(5-x)}\right]$$

$$=\frac{1}{3}\cdot\sqrt[3]{\frac{3x-1}{(2x+3)(5-x)^2}}\left[\frac{3}{3x-1}-\frac{2}{2x+3}+\frac{2}{5-x}\right].$$

**3. Find $\frac{dy}{dx}$, if :**

**(1)** $y=(\log x)^x+x^{\log x}$ **(2)** $y=x^x+a^x$

**(3)** $y=10^{x^x}+10^{x^{10}}+10^{10^x}$.

**Solution :**

**(1)** $y=(\log x)^x+x^{\log x}$

Let $u=(\log x)^x$ and $v=x^{\log x}$

Then $y=u+v$

$$\therefore\ \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \quad \dots (1)$$

Take $u=(\log x)^x$

$\therefore\ \log u=\log(\log x)^x=x\log(\log x)$

Differentiating both sides w.r.t. $x$, we get

$$\frac{1}{u}\cdot\frac{du}{dx}=\frac{d}{dx}[x\log(\log x)]$$

$$=x\cdot\frac{d}{dx}[\log(\log x)]+[\log(\log x)]\cdot\frac{d}{dx}(x)$$

$$=x\times\frac{1}{\log x}\cdot\frac{d}{dx}(\log x)+[\log(\log x)]\times1$$

$$=x\times\frac{1}{\log x}\times\frac{1}{x}+\log(\log x)$$

$$\therefore\ \frac{du}{dx}=u\left[\frac{1}{\log x}+\log(\log x)\right]$$

$$=(\log x)^x\left[\frac{1}{\log x}+\log(\log x)\right] \quad \dots (2)$$

Also, $v=x^{\log x}$

$\therefore\ \log v=\log x^{\log x}=\log x\cdot\log x=(\log x)^2$

Differentiating both sides w.r.t. $x$, we get

$$\frac{1}{v}\cdot\frac{dv}{dx}=\frac{d}{dx}(\log x)^2$$

$$=2(\log x)\cdot\frac{d}{dx}(\log x)$$

$$=2\log x\times\frac{1}{x}$$

$$\therefore\ \frac{dv}{dx}=v\left[\frac{2\log x}{x}\right]$$

$$=x^{\log x}\left[\frac{2\log x}{x}\right] \quad \dots (3)$$

From (1), (2) and (3), we get

$$\frac{dy}{dx}=(\log x)^x\left[\frac{1}{\log x}+\log(\log x)\right]+x^{\log x}\left[\frac{2\log x}{x}\right].$$

[**Note :** *Answer in the textbook is incorrect.*]

**(2)** $y=x^x+a^x$

Let $u=x^x$

Then $\log u=\log x^x=x\cdot\log x$

Differentiating both sides w.r.t. $x$, we get

$$\frac{1}{u}\cdot\frac{du}{dx}=\frac{d}{dx}(x\cdot\log x)$$

$$=x\cdot\frac{d}{dx}(\log x)+(\log x)\cdot\frac{d}{dx}(x)$$

$$=x\times\frac{1}{x}+(\log x)\times1$$

$$\therefore\ \frac{du}{dx}=u(1+\log x)=x^x(1+\log x) \quad \dots (1)$$

Now, $y=u+a^x$

$$\therefore\ \frac{dy}{dx}=\frac{du}{dx}+\frac{d}{dx}(a^x)$$

$$=x^x(1+\log x)+a^x\cdot\log a \quad \dots \text{[By (1)]}$$

**(3)** $y=10^{x^x}+10^{x^{10}}+10^{10^x}$

Let $u=x^x$

Then $\log u=\log x^x=x\log x$

Differentiating both sides w.r.t. $x$, we get

$$\frac{1}{u}\cdot\frac{du}{dx}=\frac{d}{dx}(x\log x)$$

$$=x\frac{d}{dx}(\log x)+(\log x)\cdot\frac{d}{dx}(x)$$

$$=x\times\frac{1}{x}+(\log x)\times1$$

$$\therefore \frac{du}{dx} = u(1 + \log x) = x^x(1 + \log x) \quad \ldots (1)$$

Now, $y = 10^u + 10^{x^{10}} + 10^{10^x}$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(10^u) + \frac{d}{dx}(10^{x^{10}}) + \frac{d}{dx}(10^{10^x})$$

$$= 10^u \cdot \log 10 \cdot \frac{du}{dx} + 10^{x^{10}} \cdot \log 10 \cdot \frac{d}{dx}(x^{10}) + 10^{10^x} \cdot \log 10 \cdot \frac{d}{dx}(10^x)$$

$$= 10^{x^x} \cdot \log 10 \cdot x^x(1 + \log x) + 10^{x^{10}} \cdot \log 10 \times 10x^9 + 10^{10^x} \cdot \log 10 \times 10^x \cdot \log 10 \quad \ldots \text{[By (1)]}$$

$$\therefore \frac{dy}{dx} = 10^{x^x} \cdot x^x \cdot (\log 10)(1 + \log x) + 10^{x^{10}}(10x^9) \log 10 + 10^{10^x} \cdot 10^x \cdot (\log 10)^2.$$

## ADDITIONAL SOLVED PROBLEMS – 3 (A)

**1. Find $\frac{dy}{dx}$ if $y = x^x + a^x + x^a + a^a$.**

**Solution :** $y = x^x + a^x + x^a + a^a$

Let $u = x^x$

Then $\log u = \log x^x = x \log x$

Differentiating both sides w.r.t. $x$, we get

$$\frac{1}{u} \cdot \frac{du}{dx} = \frac{d}{dx}(x \log x)$$

$$= x \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(x)$$

$$= x \times \frac{1}{x} + (\log x) \times 1$$

$$\therefore \frac{du}{dx} = u(1 + \log x) = x^x(1 + \log x) \quad \ldots (1)$$

Now, $y = u + a^x + x^a + a^a$

$$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{d}{dx}(a^x) + \frac{d}{dx}(x^a) + \frac{d}{dx}(a^a)$$

$$= x^x(1 + \log x) + a^x \cdot \log a + a x^{a-1} + 0 \quad \ldots \text{[By (1)]}$$

$$\therefore \frac{dy}{dx} = x^x(1 + \log x) + a^x \cdot \log a + a x^{a-1}.$$

**2. Find $\frac{dy}{dx}$ if $y = \left(\frac{x^2}{x+1}\right)^x$.**

**Solution :** $y = \left(\frac{x^2}{x+1}\right)^x$

$$\therefore \log y = \log\left(\frac{x^2}{x+1}\right)^x$$

$$= x \cdot \log\left(\frac{x^2}{x+1}\right)$$

$$= x[\log x^2 - \log(x+1)]$$

$$= x[2 \log x - \log(x+1)]$$

$$= 2x \log x - x \log(x+1)$$

Differentiating both sides w.r.t. $x$, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = 2\frac{d}{dx}(x \log x) - \frac{d}{dx}[x \log(x+1)]$$

$$= 2\left[x \frac{d}{dx}(\log x) + (\log x)\frac{d}{dx}(x)\right] - \left\{x \frac{d}{dx}[\log(x+1)] + \log(x+1) \cdot \frac{d}{dx}(x)\right\}$$

$$= 2\left[x \times \frac{1}{x} + (\log x) \times 1\right] - \left[x \cdot \frac{1}{x+1} \cdot \frac{d}{dx}(x+1) + \log(x+1) \cdot 1\right]$$

$$= 2 + 2\log x - \frac{x}{x+1}(1+0) - \log(x+1)$$

$$\therefore \frac{dy}{dx} = y\left[2 - \frac{x}{x+1} + \log x^2 - \log(x+1)\right]$$

$$= \left(\frac{x^2}{x+1}\right)^x \left[2 - \frac{x}{x+1} + \log\left(\frac{x^2}{x+1}\right)\right].$$

## EXAMPLES FOR PRACTICE 3.3

**1. Find $\frac{dy}{dx}$, if :**

(1) $y = (1+x)^x$ (2) $y = (2x-3)^{x^2+4}$

(3) $y = x^{4^x}$ (4) $y = 2^{x^x}$

(5) $y = (1 + \log x)^x$ (6) $y = x^{x^x}$

(7) $y = \left(\frac{x}{x+1}\right)^x$.

**2.** Find $\frac{dy}{dx}$, if :

(1) $y=(x+5)^{\frac{3}{2}}\cdot(2x-1)^{\frac{1}{4}}\cdot(3x-2)^6$

(2) $y=\sqrt{\frac{2+x}{2-x}}\cdot(x^2+4)$  (3) $y=\sqrt{\frac{(x-2)^3}{(x+1)^5(2x+5)}}$

(4) $y=\frac{(3x^2-1)\sqrt{1+x^2}}{x^3}$  (5) $y=\sqrt{\frac{(2x+3)^5}{(3x-1)^3(5x-2)}}$.

**3.** Find $\frac{dy}{dx}$, if :

(1) $y=x^x+x^a$  (2) $y=x^4+(\log x)^x$

(3) $y=5^x+x^x$.

**4.** Find $\frac{dy}{dx}$, if :

(1) $y=x^{x^2}+(x^2)^x$  (2) $y=x^x+(x+1)^x$

(3) $y=x^x+(\log x)^x$  (4) $y=x^{\sqrt{x}}+(\sqrt{x})^x$

(5) $y=x^{e^x}+(1+2x)^{2x}$.

## Answers

**1.** (1) $(1+x)^x\left[\frac{x}{1+x}+\log(1+x)\right]$

(2) $(2x-3)^{x^2+4}\left[\frac{2(x^2+4)}{2x-3}+2x\log(2x-3)\right]$

(3) $x^{4^x}\cdot 4^x\left[\frac{1}{x}+(\log x)(\log 4)\right]$

(4) $2^{x^x}\cdot\log 2\cdot x^x(1+\log x)$

(5) $(1+\log x)^x\left[\frac{1}{1+\log x}+\log(1+\log x)\right]$

(6) $x^{x^x}\cdot x^x\left[\frac{1}{x}+(\log x)(1+\log x)\right]$

(7) $\left(\frac{x}{x+1}\right)^x\left[\frac{1}{x+1}+\log\left(\frac{x}{x+1}\right)\right]$.

**2.** (1) $(x+5)^{\frac{3}{2}}\cdot(2x-1)^{\frac{1}{4}}\cdot(3x-2)^6\times$
$$\left[\frac{3}{2(x+5)}+\frac{1}{2(2x-1)}+\frac{18}{3x-2}\right]$$

(2) $\sqrt{\frac{2+x}{2-x}}\cdot(x^2+4)\left[\frac{1}{2(2+x)}+\frac{1}{2(2-x)}+\frac{2x}{x^2+4}\right]$

(3) $\sqrt{\frac{(x-2)^3}{(x+1)^5(2x+5)}}\left[\frac{3}{2(x-2)}+\frac{5}{2(x+1)}-\frac{1}{2x+5}\right]$

(4) $\frac{(3x^2-1)\sqrt{1+x^2}}{x^3}\left[\frac{6x}{3x^2-1}+\frac{x}{1+x^2}-\frac{3}{x}\right]$

(5) $\sqrt{\frac{(2x+3)^5}{(3x-1)^3(5x-2)}}\left[\frac{5}{2x+3}-\frac{9}{2(3x-1)}-\frac{5}{2(5x-2)}\right]$.

**3.** (1) $x^x(1+\log x)+ax^{a-1}$

(2) $4x^3+(\log x)^x\left[\frac{1}{\log x}+\log(\log x)\right]$

(3) $5^x\cdot\log 5+x^x(1+\log x)$

**4.** (1) $x^{x^2}\cdot x(1+2\log x)+2\cdot(x^2)^x(1+\log x)$

(2) $x^x(1+\log x)+(x+1)^x\left[\frac{x}{x+1}+\log(x+1)\right]$

(3) $x^x(1+\log x)+(\log x)^x\left[\frac{1}{\log x}+\log(\log x)\right]$

(4) $x^{\sqrt{x}}\left[\frac{2+\log x}{2\sqrt{x}}\right]+(\sqrt{x})^x\left[\frac{1+\log x}{2}\right]$

(5) $x^{e^x}\cdot e^x\left[\frac{1}{x}+\log x\right]+(1+2x)^x\left[\frac{2x}{1+2x}+\log(1+2x)\right]$.

## 3.4 : DERIVATIVE OF AN IMPLICIT FUNCTION

Suppose a function $f$ is exhibited by $y=f(x)=x^2-3x+2$. Here the value $y$ of the function $f$ is expressed entirely in terms of $x$. In this case we say that $y$ is expressed explicitly as a function of $x$ or $y$ is an **explicit function** of $x$.

Quite often, we come across equations like

$x^3+y^3-3xy=0$,  $x^2+y^2-9=0$,

that do not give $y$ explicitly in terms of $x$. However, each of these equations defines a relation between $x$ and $y$.

By assigning a particular value to $x$, we can find the corresponding value of $y$. For example, in the first equation if $x=0$, then $y=0$. In this case, we say that $y$ is expressed implicitly as a function of $x$ or $y$ is an **implicit function** of $x$, which is expressed by the relation $f(x, y)=0$.

e.g. (i) $f(x, y)=x^3+y^3-3xy=0$

(ii) $g(x, y)=x^2+y^2-9=0$

When a given relation expresses $y$ as an implicit function of $x$ and we want to find $\frac{dy}{dx}$, then we differentiate every term of the relation w.r.t. $x$, remembering that a term in $y$ is first differentiated w.r.t. $y$ and then multiplied by $\frac{dy}{dx}$.

For example :

$$\frac{d}{dx}(xy^2) = x\frac{d}{dx}(y^2) + y^2\frac{d}{dx}(x)$$

$$= x\cdot\frac{d}{dy}(y^2)\cdot\frac{dy}{dx} + y^2\cdot\frac{d}{dx}(x)$$

$$= x(2y)\frac{dy}{dx} + y^2(1) = 2xy\frac{dy}{dx} + y^2.$$

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|---|---|

**1. Find $\frac{dy}{dx}$, if :**

**(1)** $\sqrt{x} + \sqrt{y} = \sqrt{a}$ **(2)** $x^3 + y^3 + 4x^3y = 0$

**(3)** $x^3 + x^2y + xy^2 + y^3 = 81.$

**Solution :**

**(1)** $\sqrt{x} + \sqrt{y} = \sqrt{a}$

Differentiating both sides w.r.t. $x$, we get

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\cdot\frac{dy}{dx} = 0$$

$$\therefore \frac{1}{2\sqrt{y}}\cdot\frac{dy}{dx} = -\frac{1}{2\sqrt{x}}$$

$$\therefore \frac{dy}{dx} = -\frac{2\sqrt{y}}{2\sqrt{x}} = -\sqrt{\frac{y}{x}}.$$

**(2)** $x^3 + y^3 + 4x^3y = 0$

Differentiating both sides w.r.t. $x$, we get

$$3x^2 + 3y^2\frac{dy}{dx} + 4\left[x^3\frac{dy}{dx} + y\frac{d}{dx}(x^3)\right] = 0$$

$$\therefore 3x^2 + 3y^2\frac{dy}{dx} + 4x^3\frac{dy}{dx} + 4y \times 3x^2 = 0$$

$$\therefore 3y^2\frac{dy}{dx} + 4x^3\frac{dy}{dx} = -3x^2 - 12x^2y$$

$$\therefore (3y^2 + 4x^3)\frac{dy}{dx} = -3x^2(1 + 4y)$$

$$\therefore \frac{dy}{dx} = -\frac{3x^2(1 + 4y)}{3y^2 + 4x^3}.$$

**(3)** $x^3 + x^2y + xy^2 + y^3 = 81$

Differentiating both sides w.r.t. $x$, we get

$$3x^2 + \left[x^2\frac{dy}{dx} + y\cdot\frac{d}{dy}(x^2)\right] + \left[x\cdot\frac{d}{dx}(y^2) + y^2\cdot\frac{d}{dx}(x)\right] + 3y^2\frac{dy}{dx} = 0$$

$$\therefore 3x^2 + x^2\frac{dy}{dx} + y \times 2x + x \times 2y\cdot\frac{dy}{dx} + y^2 \times 1 + 3y^2\frac{dy}{dx} = 0$$

$$\therefore x^2\frac{dy}{dx} + 2xy\frac{dy}{dx} + 3y^2\frac{dy}{dx} = -3x^2 - 2xy - y^2$$

$$\therefore (x^2 + 2xy + 3y^2)\frac{dy}{dx} = -(3x^2 + 2xy + y^2)$$

$$\therefore \frac{dy}{dx} = -\left(\frac{3x^2 + 2xy + y^2}{x^2 + 2xy + 3y^2}\right).$$

**2. Find $\frac{dy}{dx}$, if :**

**(1)** $y\cdot e^x + x\cdot e^y = 1$ **(2)** $x^y = e^{(x-y)}$

**(3)** $xy = \log(xy)$.

**Solution :**

**(1)** $y\cdot e^x + x\cdot e^y = 1$

Differentiating both sides w.r.t. $x$, we get

$$\frac{d}{dx}(ye^x) + \frac{d}{dx}(xe^y) = 0$$

$$\therefore y\cdot\frac{d}{dx}(e^x) + e^x\cdot\frac{dy}{dx} + x\cdot\frac{d}{dx}(e^y) + e^y\cdot\frac{d}{dx}(x) = 0$$

$$\therefore y\cdot e^x + e^x\cdot\frac{dy}{dx} + x\cdot e^y\cdot\frac{dy}{dx} + e^y \times 1 = 0$$

$$\therefore (e^x + xe^y)\frac{dy}{dx} = -e^y - ye^x$$

$$\therefore \frac{dy}{dx} = -\left(\frac{e^y + ye^x}{e^x + xe^y}\right).$$

**(2)** $x^y = e^{(x-y)}$

$\therefore \log x^y = \log e^{(x-y)}$

$\therefore y\log x = (x - y)\log e$

$\therefore y\log x = x - y$ ... [$\because \log e = 1$]

$\therefore y + y\log x = x$

$\therefore y(1 + \log x) = x$

$$\therefore y = \frac{x}{1 + \log x}$$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}\left(\frac{x}{1+\log x}\right)$$

$$=\frac{(1+\log x)\cdot\frac{d}{dx}(x)-x\frac{d}{dx}(1+\log x)}{(1+\log x)^2}$$

$$=\frac{(1+\log x)\cdot 1-x\left(0+\frac{1}{x}\right)}{(1+\log x)^2}$$

$$=\frac{1+\log x-1}{(1+\log x)^2}$$

$$=\frac{\log x}{(1+\log x)^2}.$$

**(3)** $xy=\log(xy)$

$\therefore xy=\log x+\log y$

Differentiating both sides w.r.t. $x$, we get

$$x\cdot\frac{dy}{dx}+y\cdot\frac{d}{dx}(x)=\frac{1}{x}+\frac{1}{y}\cdot\frac{dy}{dx}$$

$$\therefore x\cdot\frac{dy}{dx}+y\times 1=\frac{1}{x}+\frac{1}{y}\cdot\frac{dy}{dx}$$

$$\therefore \left(x-\frac{1}{y}\right)\frac{dy}{dx}=\frac{1}{x}-y$$

$$\therefore \left(\frac{xy-1}{y}\right)\frac{dy}{dx}=\frac{1-xy}{x}=\frac{-(xy-1)}{x}$$

$$\therefore \frac{1}{y}\frac{dy}{dx}=-\frac{1}{x}$$

$$\therefore \frac{dy}{dx}=-\frac{y}{x}.$$

**3. Solve the following :**

**(1) If $x^5\cdot y^7=(x+y)^{12}$, then show that $\frac{dy}{dx}=\frac{y}{x}$.**

**Solution :** $x^5\cdot y^7=(x+y)^{12}$

$\therefore \log(x^5\cdot y^7)=\log(x+y)^{12}$

$\therefore \log x^5+\log y^7=\log(x+y)^{12}$

$\therefore 5\log x+7\log y=12\log(x+y)$

Differentiating both sides w.r.t. $x$, we get

$$5\times\frac{1}{x}+7\times\frac{1}{y}\cdot\frac{dy}{dx}=12\times\frac{1}{x+y}\cdot\frac{d}{dx}(x+y)$$

$$\therefore \frac{5}{x}+\frac{7}{y}\cdot\frac{dy}{dx}=\frac{12}{x+y}\left(1+\frac{dy}{dx}\right)$$

$$\therefore \frac{5}{x}+\frac{7}{y}\cdot\frac{dy}{dx}=\frac{12}{x+y}+\frac{12}{x+y}\cdot\frac{dy}{dx}$$

$$\therefore \left(\frac{7}{y}-\frac{12}{x+y}\right)\frac{dy}{dx}=\frac{12}{x+y}-\frac{5}{x}$$

$$\therefore \left[\frac{7x+7y-12y}{y(x+y)}\right]\frac{dy}{dx}=\frac{12x-5x-5y}{x(x+y)}$$

$$\therefore \left[\frac{7x-5y}{y(x+y)}\right]\frac{dy}{dx}=\frac{7x-5y}{x(x+y)}$$

$$\therefore \frac{1}{y}\cdot\frac{dy}{dx}=\frac{1}{x}$$

$$\therefore \frac{dy}{dx}=\frac{y}{x}.$$

**(2) If $\log(x+y)=\log(xy)+a$, then show that $\frac{dy}{dx}=-\frac{y^2}{x^2}$.**

**Solution :** $\log(x+y)=\log(xy)+a$

$\therefore \log(x+y)=\log x+\log y+a$

Differentiating both sides w.r.t. $x$, we get

$$\frac{1}{x+y}\cdot\frac{d}{dx}(x+y)=\frac{1}{x}+\frac{1}{y}\cdot\frac{dy}{dx}+0$$

$$\therefore \frac{1}{x+y}\cdot\left(1+\frac{dy}{dx}\right)=\frac{1}{x}+\frac{1}{y}\cdot\frac{dy}{dx}$$

$$\therefore \frac{1}{x+y}+\frac{1}{x+y}\cdot\frac{dy}{dx}=\frac{1}{x}+\frac{1}{y}\frac{dy}{dx}$$

$$\therefore \left(\frac{1}{x+y}-\frac{1}{y}\right)\frac{dy}{dx}=\frac{1}{x}-\frac{1}{x+y}$$

$$\therefore \left[\frac{y-x-y}{y(x+y)}\right]\frac{dy}{dx}=\frac{x+y-x}{x(x+y)}$$

$$\therefore \left[\frac{-x}{y(x+y)}\right]\frac{dy}{dx}=\frac{y}{x(x+y)}$$

$$\therefore -\frac{x}{y}\cdot\frac{dy}{dx}=\frac{y}{x}$$

$$\therefore \frac{dy}{dx}=-\frac{y^2}{x^2}.$$

**(3) If $e^x+e^y=e^{(x+y)}$, then show that $\frac{dy}{dx}=-e^{y-x}$.**

**Solution :** $e^x+e^y=e^{(x+y)}$ ... (1)

Differentiating both sides w.r.t. $x$, we get

$$e^x+e^y\cdot\frac{dy}{dx}=e^{(x+y)}\cdot\frac{d}{dx}(x+y)$$

$$\therefore e^x+e^y\cdot\frac{dy}{dx}=e^{(x+y)}\cdot\left(1+\frac{dy}{dx}\right)$$

$$\therefore e^x+e^y\cdot\frac{dy}{dx}=e^{(x+y)}+e^{(x+y)}\cdot\frac{dy}{dx}$$

$\therefore\ [e^y - e^{(x+y)}]\dfrac{dy}{dx} = e^{(x+y)} - e^x$

$\therefore\ (e^y - e^x - e^y)\dfrac{dy}{dx} = e^x + e^y - e^x$ ... [By (1)]

$\therefore\ -e^x \cdot \dfrac{dy}{dx} = e^y$

$\therefore\ \dfrac{dy}{dx} = -\dfrac{e^y}{e^x} = -e^{y-x}.$

## ADDITIONAL SOLVED PROBLEMS – 3 (B)

**1. If $\sqrt{\dfrac{x}{y}} + \sqrt{\dfrac{y}{x}} = 5$, find $\dfrac{dy}{dx}$.**

**Solution :** $\sqrt{\dfrac{x}{y}} + \sqrt{\dfrac{y}{x}} = 5$

$\therefore\ \dfrac{x+y}{\sqrt{xy}} = 5$

$\therefore\ x + y = 5\sqrt{xy}$

Differentiating both sides w.r.t. $x$, we get

$1 + \dfrac{dy}{dx} = 5 \times \dfrac{1}{2\sqrt{xy}} \cdot \dfrac{d}{dx}(xy)$

$\therefore\ 1 + \dfrac{dy}{dx} = \dfrac{5}{2\sqrt{xy}} \cdot \left[x\dfrac{dy}{dx} + y \times 1\right]$

$\therefore\ 1 + \dfrac{dy}{dx} = \dfrac{5x}{2\sqrt{xy}}\dfrac{dy}{dx} + \dfrac{5y}{2\sqrt{xy}}$

$\therefore\ 1 + \dfrac{dy}{dx} = \dfrac{5\sqrt{x}}{2\sqrt{y}}\dfrac{dy}{dx} + \dfrac{5\sqrt{y}}{2\sqrt{x}}$

$\therefore\ \left(1 - \dfrac{5\sqrt{x}}{2\sqrt{y}}\right)\dfrac{dy}{dx} = \dfrac{5\sqrt{y}}{2\sqrt{x}} - 1$

$\therefore\ \left(\dfrac{2\sqrt{y} - 5\sqrt{x}}{2\sqrt{y}}\right)\dfrac{dy}{dx} = \dfrac{5\sqrt{y} - 2\sqrt{x}}{2\sqrt{x}}$

$\therefore\ \dfrac{dy}{dx} = \dfrac{5\sqrt{y} - 2\sqrt{x}}{2\sqrt{x}} \times \dfrac{2\sqrt{y}}{2\sqrt{y} - 5\sqrt{x}}$

$= \left(\dfrac{5\sqrt{y} - 2\sqrt{x}}{2\sqrt{y} - 5\sqrt{x}}\right) \cdot \sqrt{\dfrac{y}{x}}.$

**2. If $\log\left(\dfrac{x^4 - y^4}{x^4 + y^4}\right) = k$, then show that $\dfrac{dy}{dx} = \dfrac{y}{x}$.**

**Solution :** $\log\left(\dfrac{x^4 - y^4}{x^4 + y^4}\right) = k$

$\therefore\ \dfrac{x^4 - y^4}{x^4 + y^4} = e^k = p$ ... (Say)

$\therefore\ x^4 - y^4 = px^4 + py^4$

$\therefore\ y^4 + py^4 = x^4 - px^4$

$\therefore\ (1+p)y^4 = (1-p)x^4$

$\therefore\ \dfrac{y^4}{x^4} = \dfrac{1-p}{1+p}$

$\therefore\ \dfrac{y}{x} = \sqrt[4]{\dfrac{1-p}{1+p}}$ ... (A constant)

Differentiating both sides w.r.t. $x$, we get

$\dfrac{d}{dx}\left(\dfrac{y}{x}\right) = 0$

$\therefore\ \dfrac{x\dfrac{dy}{dx} - y\dfrac{d}{dx}(x)}{x^2} = 0$

$\therefore\ x\dfrac{dy}{dx} - y \times 1 = 0$

$\therefore\ x\dfrac{dy}{dx} = y \qquad \therefore\ \dfrac{dy}{dx} = \dfrac{y}{x}.$

*Alternative Method :*

$\log\left(\dfrac{x^4 - y^4}{x^4 + y^4}\right) = k$

$\therefore\ \log(x^4 - y^4) - \log(x^4 + y^4) = k$

Differentiating both sides w.r.t. $x$, we get

$\dfrac{1}{x^4 - y^4} \cdot \dfrac{d}{dx}(x^4 - y^4) - \dfrac{1}{x^4 + y^4} \cdot \dfrac{d}{dx}(x^4 + y^4) = 0$

$\therefore\ \dfrac{1}{x^4 - y^4}\left(4x^3 - 4y^3\dfrac{dy}{dx}\right) - \dfrac{1}{x^4 + y^4}\left(4x^3 + 4y^3\dfrac{dy}{dx}\right) = 0$

$\therefore\ \dfrac{4x^3}{x^4 - y^4} - \dfrac{4y^3}{x^4 - y^4}\dfrac{dy}{dx} - \dfrac{4x^3}{x^4 + y^4} - \dfrac{4y^3}{x^4 + y^4}\dfrac{dy}{dx} = 0$

$\therefore\ \dfrac{4y^3}{x^4 - y^4}\dfrac{dy}{dx} + \dfrac{4y^3}{x^4 + y^4}\dfrac{dy}{dx} = \dfrac{4x^3}{x^4 - y^4} - \dfrac{4x^3}{x^4 + y^4}$

$\therefore\ 4y^3\left(\dfrac{1}{x^4 - y^4} + \dfrac{1}{x^4 + y^4}\right)\dfrac{dy}{dx} = 4x^3\left(\dfrac{1}{x^4 - y^4} - \dfrac{1}{x^4 + y^4}\right)$

$\therefore\ 4y^3\left[\dfrac{x^4 + y^4 + x^4 - y^4}{(x^4 - y^4)(x^4 + y^4)}\right]\dfrac{dy}{dx} = 4x^3\left[\dfrac{x^4 + y^4 - x^4 + y^4}{(x^4 - y^4)(x^4 + y^4)}\right]$

$\therefore\ 4y^3(2x^4)\dfrac{dy}{dx} = 4x^3(2y^4)$

$\therefore\ 8x^4y^3\dfrac{dy}{dx} = 8x^3y^4$

$\therefore\ x\dfrac{dy}{dx} = y \qquad \therefore\ \dfrac{dy}{dx} = \dfrac{y}{x}.$

## EXAMPLES FOR PRACTICE 3.4

**1. Find $\frac{dy}{dx}$, if :**

(1) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ (2) $x^2+y^2=3axy$

(3) $x^3+y^3=3ax^2y$ (4) $\sqrt{x^3+y^3}=2axy$

(5) $x^2 \cdot y^2=x^2-y^2$

(6) $y=x^3+3xy^2+3x^2y$

(7) $ax^2+2hxy+by^2+2gx+2fy+c=0$

(8) $x^3+y^2+xy=10$

(9) $y^3-3y^2x=x^3+3x^2y$

(10) $x+y-x^3y=15$.

**2. Find $\frac{dy}{dx}$, if :**

(1) $y=x^2e^{xy}$ (2) $x^y=2^{x-y}$

(3) $x^y=y^x$ (4) $e^x+e^y=e^{x-y}$

(5) $x^y=xy+25$.

**3.** (1) If $\log\left(\frac{x^3-y^3}{x^3+y^3}\right)=a$, show that $\frac{dy}{dx}=\frac{y}{x}$.

(2) If $x^3y^k=(x+y)^{3+k}$, show that $\frac{dy}{dx}=\frac{y}{x}$.

(3) If $x^{\frac{5}{3}} \cdot y^{\frac{2}{3}}=(x+y)^{\frac{7}{3}}$, show that $\frac{dy}{dx}=\frac{y}{x}$.

(4) If $x^m \cdot y^n=(x+y)^{m+n}$, show that $\frac{dy}{dx}=\frac{y}{x}$.

(5) If $e^x=x^y$, show that $\frac{dy}{dx}=\frac{\log x-1}{(\log x)^2}$.

(6) If $e^y=y^x$, show that $\frac{dy}{dx}=\frac{(\log y)^2}{\log y-1}$.

(7) If $xy=1+\log y$, show that $\frac{dy}{dx}=\frac{-y^2}{\log y}$.

### Answers

**1.** (1) $-\left(\frac{y}{x}\right)^{\frac{1}{3}}$ (2) $\frac{3ay-2x}{2y-3ax}$ (3) $\frac{x(2ay-x)}{y^2-ax^2}$

(4) $\frac{8a^2xy^2-3x^2}{3y^2-8a^2x^2y}$ [**Hint :** Square both sides]

(5) $\frac{x(1-y^2)}{y(1+x^2)}$ (6) $\frac{-3(x^2+y^2+2xy)}{6xy+3x^2-1}$

(7) $-\left(\frac{ax+hy+g}{hx+by+f}\right)$ (8) $\frac{-(3x^2+y)}{2y+x}$

(9) $\frac{x^2+2xy+y^2}{y^2-2xy-x^2}$ (10) $\frac{3x^2y-1}{1-x^3}$.

**2.** (1) $\frac{y(2+xy)}{x(1-xy)}$ (2) $\frac{x\log 2-y}{x\log(2x)}$

(3) $\frac{y(x\log y-y)}{x(y\log x-x)}$ (4) $\frac{e^{x-y}-e^x}{e^y+e^{x-y}}$

(5) $\frac{xy-xy^2-25y}{x(xy+25)\log x-x^2}$.

## 3.5 : DERIVATIVE OF A PARAMETRIC FUNCTION

If $x$ and $y$ are expressed as functions of the same variable $t$, say $x=f(t)$ and $y=g(t)$, then these equations are said to represent **parametric functions.**

The variable $t$ is called a **parameter.**

We state (without proof) the theorem for finding the derivative of parametric functions.

**Theorem 3 :**

If $x=f(t)$ and $y=g(t)$ are differentiable functions of $t$ such that $y$ is a function of $x$, then $y$ is differentiable function of $x$ and

$$\frac{dy}{dx}=\left(\frac{dy}{dt}\right)\Big/\left(\frac{dx}{dt}\right), \text{ if } \frac{dx}{dt}\neq 0.$$

**Note :** If $x=f(t)$ and $y=g(t)$ are differentiable functions of $t$, then

$$\frac{dx}{dt}=f'(t) \quad \text{and} \quad \frac{dy}{dt}=g'(t)$$

$$\therefore \frac{dy}{dx}=\left(\frac{dy}{dt}\right)\Big/\left(\frac{dx}{dt}\right)=\frac{g'(t)}{f'(t)}.$$

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|---|---|

**1. Find $\frac{dy}{dx}$, if :**

**(1)** $x=at^2$, $y=2at$ **(2)** $x=2at^2$, $y=at^4$

**(3)** $x=e^{3t}$, $y=e^{(4t+5)}$.

**Solution :**

**(1)** $x = at^2$, $y = 2at$

Differentiating $x$ and $y$ w.r.t. $t$, we get

$$\frac{dx}{dt} = a\frac{d}{dt}(t^2) = a \times 2t = 2at$$

and $$\frac{dy}{dt} = 2a\frac{d}{dt}(t) = 2a \times 1 = 2a$$

$$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{2a}{2at} = \frac{1}{t}.$$

**(2)** *Refer to the solution of Q. 1 (1).*

**Ans.** $t^2$.

**(3)** $x = e^{3t}$, $y = e^{(4t+5)}$

Differentiating $x$ and $y$ w.r.t. $t$, we get

$$\frac{dx}{dt} = \frac{d}{dt}(e^{3t}) = e^{3t} \cdot \frac{d}{dt}(3t)$$

$$= e^{3t} \times 3 \times 1 = 3e^{3t}$$

and $$\frac{dy}{dt} = \frac{d}{dt}[e^{(4t+5)}] = e^{(4t+5)} \cdot \frac{d}{dt}(4t+5)$$

$$= e^{(4t+5)} \times (4 \times 1 + 0) = 4e^{(4t+5)}$$

$$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{4e^{(4t+5)}}{3e^{3t}}$$

$$= \frac{4}{3}e^{4t+5-3t} = \frac{4}{3}e^{t+5}.$$

**2. Find $\frac{dy}{dx}$, if :**

**(1)** $x = \left(u + \frac{1}{u}\right)^2$, $y = 2^{\left(u+\frac{1}{u}\right)}$

**(2)** $x = \sqrt{1+u^2}$, $y = \log(1+u^2)$.

**Solution :**

**(1)** $x = \left(u + \frac{1}{u}\right)^2$, $y = 2^{\left(u+\frac{1}{u}\right)}$ ... (1)

Differentiating $x$ and $y$ w.r.t. $u$, we get,

$$\frac{dx}{du} = \frac{d}{du}\left(u + \frac{1}{u}\right)^2 = 2\left(u + \frac{1}{u}\right) \cdot \frac{d}{du}\left(u + \frac{1}{u}\right)$$

$$= 2\left(u + \frac{1}{u}\right)\left(1 - \frac{1}{u^2}\right)$$

and $$\frac{dy}{du} = \frac{d}{du}\left[2^{\left(u+\frac{1}{u}\right)}\right]$$

$$= 2^{\left(u+\frac{1}{u}\right)} \cdot \log 2 \cdot \frac{d}{du}\left(u + \frac{1}{u}\right)$$

$$= 2^{\left(u+\frac{1}{u}\right)} \cdot \log 2 \cdot \left(1 - \frac{1}{u^2}\right)$$

$$\therefore \frac{dy}{dx} = \frac{(dy/du)}{(dx/du)} = \frac{2^{\left(u+\frac{1}{u}\right)} \cdot \log 2 \cdot \left(1 - \frac{1}{u^2}\right)}{2\left(u + \frac{1}{u}\right)\left(1 - \frac{1}{u^2}\right)}$$

$$= \frac{2^{\left(u+\frac{1}{u}\right)} \cdot \log 2}{2\left(u + \frac{1}{u}\right)}$$

$$= \frac{y \log 2}{2\sqrt{x}} \quad \text{... [By (1)]}$$

**(2)** $x = \sqrt{1+u^2}$, $y = \log(1+u^2)$

Differentiating $x$ and $y$ w.r.t. $u$, we get

$$\frac{dx}{du} = \frac{d}{du}(\sqrt{1+u^2}) = \frac{1}{2\sqrt{1+u^2}} \cdot \frac{d}{du}(1+u^2)$$

$$= \frac{1}{2\sqrt{1+u^2}} \times (0 + 2u) = \frac{u}{\sqrt{1+u^2}}$$

and $$\frac{dy}{du} = \frac{d}{du}[\log(1+u^2)]$$

$$= \frac{1}{1+u^2} \cdot \frac{d}{du}(1+u^2)$$

$$= \frac{1}{1+u^2} \times (0 + 2u) = \frac{2u}{1+u^2}$$

$$\therefore \frac{dy}{dx} = \frac{(dy/du)}{(dx/du)} = \frac{\left(\frac{2u}{1+u^2}\right)}{\left(\frac{u}{\sqrt{1+u^2}}\right)}$$

$$= \frac{2u}{1+u^2} \times \frac{\sqrt{1+u^2}}{u} = \frac{2}{\sqrt{1+u^2}}.$$

**2. (3) Differentiate $5^x$ with respect to $\log x$.**

**Solution :** Let $u = 5^x$ and $v = \log x$.

Then we want to find $\frac{du}{dv}$.

Differentiating $u$ and $v$ w.r.t. $x$, we get

$$\frac{du}{dx} = \frac{d}{dx}(5^x) = 5^x \cdot \log 5$$

and $\frac{dv}{dx} = \frac{d}{dx}(\log x) = \frac{1}{x}$

$\therefore \frac{du}{dv} = \frac{(du/dx)}{(dv/dx)} = \frac{5^x \cdot \log 5}{\left(\frac{1}{x}\right)}$

$= x \cdot 5^x \cdot \log 5.$

**3. Solve the following :**

**(1) If $x = a\left(1 - \frac{1}{t}\right)$, $y = a\left(1 + \frac{1}{t}\right)$, then show that $\frac{dy}{dx} = -1$.**

**Solution :** $x = a\left(1 - \frac{1}{t}\right)$, $y = a\left(1 + \frac{1}{t}\right)$

Differentiating $x$ and $y$ w.r.t. $t$, we get

$\frac{dx}{dt} = a\frac{d}{dt}\left(1 - \frac{1}{t}\right) = a[0 - (-1)t^{-2}] = \frac{a}{t^2}$

and $\frac{dy}{dt} = a\frac{d}{dt}\left(1 + \frac{1}{t}\right) = a[0 + (-1)t^{-2}] = -\frac{a}{t^2}$

$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{\left(-\frac{a}{t^2}\right)}{\left(\frac{a}{t^2}\right)} = -1.$

**(2) If $x = \frac{4t}{1+t^2}$, $y = 3\left(\frac{1-t^2}{1+t^2}\right)$, then show that $\frac{dy}{dx} = -\frac{9x}{4y}$.**

**Solution :**

$x = \frac{4t}{1+t^2}$, $y = 3\left(\frac{1-t^2}{1+t^2}\right)$

Differentiating $x$ and $y$ w.r.t. $t$, we get

$\frac{dx}{dt} = \frac{d}{dt}\left(\frac{4t}{1+t^2}\right) = \frac{(1+t^2)\cdot\frac{d}{dt}(4t) - 4t\cdot\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$

$= \frac{(1+t^2)(4) - 4t(0+2t)}{(1+t^2)^2}$

$= \frac{4 + 4t^2 - 8t^2}{(1+t^2)^2} = \frac{4 - 4t^2}{(1+t^2)^2}$

$= \frac{4(1-t^2)}{(1+t^2)^2}$

and $\frac{dy}{dt} = 3\frac{d}{dt}\left(\frac{1-t^2}{1+t^2}\right)$

$= 3\left[\frac{(1+t^2)\cdot\frac{d}{dt}(1-t^2) - (1-t^2)\cdot\frac{d}{dt}(1+t^2)}{(1+t^2)^2}\right]$

$= 3\left[\frac{(1+t^2)(0-2t) - (1-t^2)(0+2t)}{(1+t^2)^2}\right]$

$= 3\left[\frac{-2t - 2t^3 - 2t + 2t^3}{(1+t^2)^2}\right]$

$= \frac{-12t}{(1+t^2)^2}$

$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{\left[\frac{-12t}{(1+t^2)^2}\right]}{\left[\frac{4(1-t^2)}{(1+t^2)^2}\right]}$

$\therefore \frac{dy}{dx} = \frac{-3t}{1-t^2}$ ... (1)

$\frac{-9x}{4y} = \frac{-9}{4}\cdot\frac{\left(\frac{4t}{1+t^2}\right)}{3\left(\frac{1-t^2}{1+t^2}\right)} = \frac{-3t}{1-t^2}$ ... (2)

From (1) and (2)

$\frac{dy}{dx} = -\frac{9x}{4y}.$

**(3) If $x = t\cdot\log t$, $y = t^t$, then show that $\frac{dy}{dx} - y = 0$.**

**Solution :** $x = t\cdot\log t$

Differentiating w.r.t. $t$, we get

$\frac{dx}{dt} = \frac{d}{dt}(t\cdot\log t)$

$= t\frac{d}{dt}(\log t) + (\log t)\cdot\frac{d}{dt}(t)$

$= t \times \frac{1}{t} + (\log t) \times 1 = 1 + \log t.$

Also, $y = t^t$

$\therefore \log y = \log t^t = t\log t.$

Differentiating both sides w.r.t. $t$, we get

$\frac{1}{y}\cdot\frac{dy}{dt} = \frac{d}{dt}(t\log t)$

$= t \cdot \frac{d}{dt}(\log t) + (\log t) \cdot \frac{d}{dt}(t)$

$= t \times \frac{1}{t} + (\log t) \times 1$

$\therefore \frac{dy}{dt} = y(1 + \log t)$

$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{y(1+\log t)}{1+\log t} = y$

$\therefore \frac{dy}{dx} - y = 0.$

## ADDITIONAL SOLVED PROBLEMS–3 (C)

**1. Find $\frac{dy}{dx}$, if $x = \left(t + \frac{1}{t}\right)^a$, $y = a^{\left(t+\frac{1}{t}\right)}$, where $a > 0$, $a \neq 0$.**

**Solution :** $x = \left(t + \frac{1}{t}\right)^a$, $y = a^{\left(t+\frac{1}{t}\right)}$

Differentiating $x$ and $y$ w.r.t. $t$, we get

$\frac{dx}{dt} = \frac{d}{dt}\left(t + \frac{1}{t}\right)^a = a\left(t + \frac{1}{t}\right)^{a-1} \cdot \frac{d}{dt}\left(t + \frac{1}{t}\right)$

$= a\left(t + \frac{1}{t}\right)^{a-1} \cdot \left(1 - \frac{1}{t^2}\right)$

and $\frac{dy}{dt} = \frac{d}{dt}\left[a^{\left(t+\frac{1}{t}\right)}\right]$

$= a^{\left(t+\frac{1}{t}\right)} \cdot \log a \cdot \frac{d}{dt}\left(t + \frac{1}{t}\right)$

$= a^{\left(t+\frac{1}{t}\right)} \cdot \log a \cdot \left(1 - \frac{1}{t^2}\right)$

$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{a^{\left(t+\frac{1}{t}\right)} \cdot \log a \cdot \left(1 - \frac{1}{t^2}\right)}{a\left(t + \frac{1}{t}\right)^{a-1} \cdot \left(1 - \frac{1}{t^2}\right)}$

$= \frac{a^{\left(t+\frac{1}{t}-1\right)} \cdot \log a}{\left(t + \frac{1}{t}\right)^{a-1}}.$

**2. Differentiate $7^x$ with respect to $\log_x 7$.**

**Solution :** Let $u = 7^x$, $v = \log_x 7$.

Then we want to find $\frac{du}{dv}$.

Differentiating $u$ and $v$ w.r.t. $x$, we get

$\frac{du}{dx} = \frac{d}{dx}(7^x) = 7^x \cdot \log 7$

and $\frac{dv}{dx} = \frac{d}{dx}(\log_x 7) = \frac{d}{dx}\left(\frac{\log 7}{\log x}\right)$

$= (\log 7) \cdot \frac{d}{dx}(\log x)^{-1}$

$= (\log 7) \cdot (-1)(\log x)^{-2} \cdot \frac{d}{dx}(\log x)$

$= \frac{-\log 7}{(\log x)^2} \times \frac{1}{x} = \frac{-\log 7}{x(\log x)^2}$

$\therefore \frac{du}{dv} = \frac{(du/dx)}{(dv/dx)} = \frac{7^x \cdot \log 7}{\left[\frac{-\log 7}{x(\log x)^2}\right]}$

$= -x \cdot 7^x \cdot (\log x)^2.$

## EXAMPLES FOR PRACTICE 3.5

**1. Find $\frac{dy}{dx}$, if :**

(1) $x = 35t^2$, $y = 70t$

(2) $x = 4 + 25t^2$, $y = 16t^3$

(3) $x = e^{2t}$, $y = e^{\sqrt{t}}$

(4) $x = \log(1 + t^2)$, $y = \log t$.

**2.** Find $\frac{dy}{dx}$ at $t = 3$, if $x = at^2$, $y = 2at$.

**3.** If $x = \frac{7}{1+t^3}$, $y = \frac{7t}{1+t^3}$, then show that $\frac{dy}{dx} = \frac{2t^3 - 1}{3t^2}$.

**4.** If $x = \frac{2t}{1+t^2}$, $y = \frac{1-t^2}{1+t^2}$, show that $\frac{dy}{dx} = -\frac{x}{y}$.

**5.** If $x = a\left(t - \frac{1}{t}\right)$, $y = a\left(t + \frac{1}{t}\right)$, show that $\frac{dy}{dx} = \frac{x}{y}$.

**6.** If $u = \frac{2bt}{1+t^2}$, $v = a\left(\frac{1-t^2}{1+t^2}\right)$, show that $\frac{du}{dv} = \frac{-b^2 v}{a^2 u}$.

**7.** If $x = t \log t$, $y = t^t$, show that $\frac{dy}{dx} = e^x$.

**8. Differentiate :**

(1) $x^2 - 2$ with respect to $x^3$.

(2) $5^x$ with respect to $\log_5 x$.

(3) $\log t$ with respect to $\log(1 + t^2)$.

### Answers

**1.** (1) $\frac{1}{t}$ (2) $\frac{24t}{25}$ (3) $\frac{e^{\sqrt{t}-2t}}{4\sqrt{t}}$ (4) $\frac{1+t^2}{2t^2}$. **2.** $\frac{1}{3}$

**8.** (1) $\frac{2}{3x}$ (2) $x \cdot 5^x \cdot (\log 5)^2$ (3) $\frac{1+t^2}{2t^2}$.

## 3.6 : SECOND ORDER DERIVATIVE

If $y = f(x)$ is a derivable function of $x$, then we have $\frac{dy}{dx} = f'(x)$. This is called the *first order derivative* of $y$ w.r.t. $x$ and is also denoted by $y_1$ or $y'$. It is clear that $f'(x)$ is also a function of $x$. If it is again differentiable, then its derivative

$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(f'(x))$ is called the *second order derivative* of $y$ w.r.t. $x$ and is denoted by $\frac{d^2y}{dx^2}$ or $f''(x)$ or $y_2$ or $y''$.

Thus, the second order derivative is the derivative of the first order derivative.

In general, the *nth order derivative* of $y$ w.r.t. $x$ is denoted by $\frac{d^ny}{dx^n}$ or $f^{(n)}(x)$.

### EXERCISE 3.6 | Textbook page 98

**1. Find $\frac{d^2y}{dx^2}$, if :**

**(1)** $y = \sqrt{x}$ **(2)** $y = x^5$ **(3)** $y = x^{-7}$.

**Solution :**

**(1)** $y = \sqrt{x}$

Differentiating w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

Differentiating again w.r.t. $x$, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{2\sqrt{x}}\right) = \frac{1}{2}\frac{d}{dx}(x^{-\frac{1}{2}})$$

$$= \frac{1}{2}\cdot\left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1} = -\frac{1}{4}x^{-\frac{3}{2}}.$$

**(2)** $y = x^5$

Differentiating w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x^5) = 5x^4$$

Differentiating again w.r.t. $x$, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(5x^4) = 5\frac{d}{dx}(x^4)$$

$$= 5 \times 4x^3 = 20x^3.$$

**(3)** $y = x^{-7}$

Differentiating w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x^{-7}) = -7x^{-8}$$

Differentiating again w.r.t. $x$, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(-7x^{-8}) = -7\frac{d}{dx}(x^{-8})$$

$$= (-7)(-8)x^{-9} = 56x^{-9}.$$

**2. Find $\frac{d^2y}{dx^2}$, if :**

**(1)** $y = e^x$ **(2)** $y = e^{(2x+1)}$ **(3)** $y = e^{\log x}$.

**Solution :**

**(1)** $y = e^x$

Differentiating w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{d}{dx}(e^x) = e^x$$

Differentiating again w.r.t. $x$, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(e^x) = e^x.$$

**(2)** $y = e^{(2x+1)}$

Differentiating w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{d}{dx}[e^{(2x+1)}] = e^{(2x+1)}\cdot\frac{d}{dx}(2x+1)$$

$$= e^{(2x+1)} \times (2 \times 1 + 0) = 2e^{(2x+1)}$$

Differentiating again w.r.t. $x$, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}[2e^{(2x+1)}] = 2\frac{d}{dx}[e^{(2x+1)}]$$

$$= 2e^{(2x+1)}\cdot\frac{d}{dx}(2x+1) = 2e^{(2x+1)} \times (2 \times 1 + 0)$$

$$= 4e^{(2x+1)}.$$

(3) $y = e^{\log x} = x$ ... $[\because a^{\log_a x} = x]$

Differentiating w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x) = 1$$

Differentiating again w.r.t. $x$, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(1) = 0.$$

## EXAMPLES FOR PRACTICE 3.6

**1. Find $\frac{d^2y}{dx^2}$, if :**

(1) $y = x^3$ (2) $y = \frac{1}{x^2}$ (3) $y = \frac{1}{x\sqrt{x}}$

(4) $y = 3x^2 + 1$.

**2. Find $\frac{d^2y}{dx^2}$, if :**

(1) $y = e^{5x}$ (2) $y = e^{(1-3x)}$ (3) $y = \log x^2$

(4) $y = 3^{2x}$ (5) $y = 7^{(5x+1)}$.

### Answers

**1.** (1) $6x$ (2) $\frac{6}{x^4}$ (3) $\frac{15}{4}x^{-\frac{7}{2}}$ (4) 6.

**2.** (1) $25e^{5x}$ (2) $9e^{(1-3x)}$ (3) $-\frac{2}{x^2}$

(4) $4 \cdot 3^{2x} \cdot (\log 3)^2$ (5) $25 \cdot 7^{(5x+1)} \cdot (\log 7)^2$.

## MISCELLANEOUS EXERCISE – 3

**(Textbook pages 99 to 101)**

**(I) Choose the correct alternative :**

**1.** If $y = (5x^3 - 4x^2 - 8x)^9$, then $\frac{dy}{dx} = \ldots\ldots\ldots$

(a) $9(5x^3 - 4x^2 - 8x)^8(15x^2 - 8x - 8)$

(b) $9(5x^3 - 4x^2 - 8x)^9(15x^2 - 8x - 8)$

(c) $9(5x^3 - 4x^2 - 8x)^8(5x^2 - 8x - 8)$

(d) $9(5x^3 - 4x^2 - 8x)^9(5x^2 - 8x - 8)$

**2.** If $y = \sqrt{x + \frac{1}{x}}$, then $\frac{dy}{dx} = \ldots\ldots\ldots$

(a) $\frac{x^2 - 1}{2x^2\sqrt{x^2+1}}$ (b) $\frac{1 - x^2}{2x^2\sqrt{x^2+1}}$

(c) $\frac{x^2 - 1}{2x\sqrt{x}\sqrt{x^2+1}}$ (d) $\frac{1 - x^2}{2x\sqrt{x}\sqrt{x^2+1}}$.

**3.** If $y = e^{\log x}$, then $\frac{dy}{dx} = \ldots\ldots\ldots$

(a) $\frac{e^{\log x}}{x}$ (b) $\frac{1}{x}$ (c) 0 (d) $\frac{1}{2}$

**4.** If $y = 2x^2 + 2^2 + a^2$, then $\frac{dy}{dx} = \ldots\ldots\ldots$

(a) $x$ (b) $4x$ (c) $2x$ (d) $-2x$

**5.** If $y = 5^x x^5$, then $\frac{dy}{dx} = \ldots\ldots\ldots$

(a) $5^x \cdot x^4(5 + \log 5)$ (b) $5^x \cdot x^5(5 + \log 5)$

(c) $5^x \cdot x^4(5 + x\log 5)$ (d) $5^x \cdot x^5(5 + x\log 5)$

**6.** If $y = \log\left(\frac{e^x}{x^2}\right)$, then $\frac{dy}{dx} = \ldots\ldots\ldots$

(a) $\frac{2-x}{x}$ (b) $\frac{x-2}{x}$ (c) $\frac{e-x}{ex}$ (d) $\frac{x-e}{ex}$

**7.** If $ax^2 + 2hxy + by^2 = 0$, then $\frac{dy}{dx} = \ldots\ldots\ldots$

(a) $\frac{(ax+hy)}{(hx+by)}$ (b) $\frac{-(ax+hy)}{(hx+by)}$

(c) $\frac{(ax-hy)}{(hx+by)}$ (d) $\frac{(2ax+by)}{(hx+3by)}$

**8.** If $x^4 \cdot y^5 = (x+y)^{(m+1)}$ and $\frac{dy}{dx} = \frac{y}{x}$, then $m = \ldots\ldots\ldots$

(a) 8 (b) 4 (c) 5 (d) 20

**9.** If $x = \frac{e^t + e^{-t}}{2}$, $y = \frac{e^t - e^{-t}}{2}$, then $\frac{dy}{dx} = \ldots\ldots\ldots$

(a) $\frac{-y}{x}$ (b) $\frac{y}{x}$ (c) $\frac{-x}{y}$ (d) $\frac{x}{y}$

**10.** If $x = 2at^2$, $y = 4at$, then $\frac{dy}{dx} = \ldots\ldots\ldots$

(a) $-\frac{1}{2at^2}$ (b) $\frac{1}{2at^3}$ (c) $\frac{1}{t}$ (d) $\frac{1}{4at^3}$

### Answers

1. (a) $9(5x^3-4x^2-8x)^8(15x^2-8x-8)$

2. (c) $\dfrac{x^2-1}{2x\sqrt{x}\sqrt{x^2+1}}$

[Hint : $\dfrac{dy}{dx}=\dfrac{1}{2\sqrt{x+\dfrac{1}{x}}}\cdot\dfrac{d}{dx}\left(x+\dfrac{1}{x}\right)$

$=\dfrac{\sqrt{x}}{2\sqrt{x^2+1}}\left(1-\dfrac{1}{x^2}\right)=\dfrac{x^2-1}{2x\sqrt{x}\sqrt{x^2+1}}$].

3. (a) $\dfrac{e^{\log x}}{x}$

4. (b) $4x$

5. (c) $5^x\cdot x^4(5+x\log 5)$

6. (b) $\dfrac{x-2}{x}$

[Hint : $y=\log\left(\dfrac{e^x}{x^2}\right)=\log e^x=\log x^2$

$=x-2\log x$ ... [$\because \log e=1$]

$\therefore \dfrac{dy}{dx}=1-\dfrac{2}{x}=\dfrac{x-2}{x}$.]

7. (b) $\dfrac{-(ax+hy)}{(hx+by)}$

8. (a) 8

[Hint : If $x^p\cdot y^q=(x+y)^{p+q}$, then $\dfrac{dy}{dx}=\dfrac{y}{x}$

$\therefore m+1=4+5=9 \quad \therefore m=8$.]

9. (d) $\dfrac{x}{y}$

[Hint : $\dfrac{dx}{dt}=\dfrac{1}{2}(e^t-e^{-t})$, $\dfrac{dy}{dt}=\dfrac{1}{2}(e^t+e^{-t})$

$\therefore \dfrac{dy}{dx}=\dfrac{(dy/dt)}{(dx/dt)}=\left(\dfrac{e^t+e^{-t}}{2}\right)\Big/\left(\dfrac{e^t-e^{-t}}{2}\right)=\dfrac{x}{y}$].

10. (c) $\dfrac{1}{t}$.

### (II) Fill in the blanks :

1. If $3x^2y+3xy^2=0$, then $\dfrac{dy}{dx}=$ .........

2. If $x^m\cdot y^n=(x+y)^{(m+n)}$, then $\dfrac{dy}{dx}=\dfrac{......}{x}$

3. If $0=\log(xy)+a$, then $\dfrac{dy}{dx}=\dfrac{-y}{......}$

4. If $x=t\log t$ and $y=t^t$, then $\dfrac{dy}{dx}=$ .........

5. If $y=x\cdot\log x$, then $\dfrac{d^2y}{dx^2}=$ .........

6. If $y=[\log(x)]^2$, then $\dfrac{d^2y}{dx^2}=$ .........

7. If $x=y+\dfrac{1}{y}$, then $\dfrac{dy}{dx}=$ .........

8. If $y=e^{ax}$, then $x\cdot\dfrac{dy}{dx}=$ .........

9. If $x=t\cdot\log t$, $y=t^t$, then $\dfrac{dy}{dx}=$ .........

10. If $y=(x+\sqrt{x^2-1})^m$, then $\sqrt{(x^2-1)}\dfrac{dy}{dx}=$ .........

### Answers

1. $-1$

[Hint : $3x^2y+3xy^2=0 \quad \therefore 3xy(x+y)=0$

$\therefore x+y=0 \quad \therefore y=-x \quad \therefore \dfrac{dy}{dx}=-1$.]

2. $y$ 3. $x$

4. $y$

[Hint : $x=t\log t=\log t^t=\log y$

$\therefore 1=\dfrac{1}{y}\cdot\dfrac{dy}{dx} \quad \therefore \dfrac{dy}{dx}=y$.]

5. $\dfrac{1}{x}$

6. $\dfrac{2(1-\log x)}{x^2}$

[Hint : $y=(\log x)^2 \quad \therefore \dfrac{dy}{dx}=2\log x\cdot\dfrac{d}{dx}(\log x)$

$=2\log x\times\dfrac{1}{x}=\dfrac{2\log x}{x}$

and $\dfrac{d^2y}{dx^2}=2\dfrac{d}{dx}\left(\dfrac{\log x}{x}\right)$

$=2\left[\dfrac{x\dfrac{d}{dx}(\log x)-(\log x)\cdot\dfrac{d}{dx}(x)}{x^2}\right]$

$=2\left[\dfrac{x\times\dfrac{1}{x}-(\log x)\times 1}{x^2}\right]$

$=\dfrac{2(1-\log x)}{x^2}$.]

[Note : *Answer in the textbook is incorrect.*]

7. $\dfrac{y^2}{y^2-1}$

[Hint : $\dfrac{dx}{dy}=\dfrac{d}{dy}\left(y+\dfrac{1}{y}\right)=1-\dfrac{1}{y^2}=\dfrac{y^2-1}{y^2}$

$\therefore \dfrac{dy}{dx}=\dfrac{1}{\left(\dfrac{dx}{dy}\right)}=\dfrac{y^2}{y^2-1}$.]

**8.** $axy$ **9.** $y$

**10.** $my$

[Hint : $y=(x+\sqrt{x^2-1})^m$

$$\therefore \frac{dy}{dx}=m(x+\sqrt{x^2-1})^{m-1}\cdot\frac{d}{dx}(x+\sqrt{x^2-1})$$

$$=m(x+\sqrt{x^2-1})^{m-1}\cdot\left[1+\frac{1}{2\sqrt{x^2-1}}\cdot\frac{d}{dx}(x^2-1)\right]$$

$$=m(x+\sqrt{x^2-1})^{m-1}\cdot\left[1+\frac{1}{2\sqrt{x^2-1}}\times 2x\right]$$

$$=m(x+\sqrt{x^2-1})^{m-1}\cdot\left[\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}\right]$$

$$\therefore \frac{dy}{dx}=\frac{m(x+\sqrt{x^2-1})^m}{\sqrt{x^2-1}}=\frac{my}{\sqrt{x^2-1}}$$

$$\therefore \sqrt{x^2-1}\cdot\frac{dy}{dx}=my.]$$

**(III) State whether each of the following is *True* or *False* :**

1. If $f'$ is the derivative of $f$, then the derivative of the inverse of $f$ is the inverse of $f'$.
2. The derivative of $\log_a x$, where $a$ is constant is $\frac{1}{x\cdot\log a}$.
3. The derivative of $f(x)=a^x$, where $a$ is constant is $x\cdot a^{x-1}$.
4. The derivative of polynomial is polynomial.
5. $\frac{d}{dx}(10^x)=x\cdot 10^{x-1}$.
6. If $y=\log x$, then $\frac{dy}{dx}=\frac{1}{x}$.
7. If $y=e^2$, then $\frac{dy}{dx}=2e$.
8. The derivative of $a^x$ is $a^x\cdot\log a$.
9. The derivative of $x^m\cdot y^n=(x+y)^{(m+n)}$ is $\frac{x}{y}$.

**Answers**

1. False 2. True 3. False 4. True 5. False
6. True 7. False 8. True 9. False.

**(IV) Solve the following :**

**1. If $y=(6x^3-3x^2-9x)^{10}$, find $\frac{dy}{dx}$.**

**Solution :** Given : $y=(6x^3-3x^2-9x)^{10}$

Let $u=6x^3-3x^2-9x$

Then $y=u^{10}$

$$\therefore \frac{dy}{du}=\frac{d}{du}(u^{10})=10u^9$$

$$=10(6x^3-3x^2-9x)^9$$

and $$\frac{du}{dx}=\frac{d}{dx}(6x^3-3x^2-9x)$$

$$=6\frac{d}{dx}(x^3)-3\frac{d}{dx}(x^2)-9\frac{d}{dx}(x)$$

$$=6\times 3x^2-3\times 2x-9\times 1$$

$$=18x^2-6x-9$$

$$\therefore \frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

$$=10(6x^2-3x^2-9x)^9\cdot(18x^2-6x-9).$$

**2. If $y=\sqrt[5]{(3x^2+8x+5)^4}$, find $\frac{dy}{dx}$.**

**Solution :** Given : $y=\sqrt[5]{(3x^2+8x+5)^4}$

Let $u=3x^2+8x+5$

Then $y=\sqrt[5]{u^4}=u^{\frac{4}{5}}$

$$\therefore \frac{dy}{du}=\frac{d}{du}(u^{\frac{4}{5}})=\frac{4}{5}u^{\frac{4}{5}-1}$$

$$=\frac{4}{5}(3x^2+8x+5)^{-\frac{1}{5}}$$

and $$\frac{du}{dx}=\frac{d}{dx}(3x^2+8x+5)$$

$$=3\frac{d}{dx}(x^2)+8\frac{d}{dx}(x)+\frac{d}{dx}(5)$$

$$=3\times 2x+8\times 1+0=6x+8$$

$$\therefore \frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

$$=\frac{4}{5}(3x^2+8x+5)^{-\frac{1}{5}}\cdot(6x+8).$$

**3. If $y=[\log(\log(\log x))]^2$, find $\frac{dy}{dx}$.**

**Solution :** $y=[\log(\log(\log x))]^2$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}\cdot[\log(\log(\log x))]^2$$

$$=2[\log(\log(\log x))]\cdot\frac{d}{dx}[\log(\log(\log x))]$$

$$=2\log[\log(\log x)]\times\frac{1}{\log(\log x)}\cdot\frac{d}{dx}[\log(\log x)]$$

$$=\frac{2\log[\log(\log x)]}{\log(\log x)}\times\frac{1}{\log x}\cdot\frac{d}{dx}(\log x)$$

$$=\frac{2\log[\log(\log x)]}{\log(\log x)}\times\frac{1}{\log x}\times\frac{1}{x}$$

$$=\frac{2\log[\log(\log x)]}{x\cdot\log x\cdot\log(\log x)}.$$

**4. Find the rate of change of demand ($x$) of a commodity with respect to its price ($y$) if $y=25+30x-x^2$.**

**Solution :** $y=25+30x-x^2$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}(25+30x-x^2)$$

$$=\frac{d}{dx}(25)+30\frac{d}{dx}(x)-\frac{d}{dx}(x^2)$$

$$=0+30\times1-2x$$

$$=30-2x$$

Hence, the rate of change of demand ($x$) w.r.t. price ($y$)

$$=\frac{dx}{dy}=\frac{1}{\left(\frac{dy}{dx}\right)}=\frac{1}{30-2x}.$$

**5. Find the rate of change of demand ($x$) of a commodity with respect to its price ($y$) if $y=\frac{5x+7}{2x-13}$.**

**Solution :** $y=\frac{5x+7}{2x-13}$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}\left(\frac{5x+7}{2x-13}\right)$$

$$=\frac{(2x-13)\cdot\frac{d}{dx}(5x+7)-(5x+7)\cdot\frac{d}{dx}(2x-13)}{(2x-13)^2}$$

$$=\frac{(2x-13)\cdot(5\times1+0)-(5x+7)\cdot(2\times1-0)}{(2x-13)^2}$$

$$=\frac{10x-65-10x-14}{(2x-13)^2}=\frac{-79}{(2x-13)^2}$$

Hence, rate of change of demand ($x$) w.r.t. price ($y$)

$$=\frac{dx}{dy}=\frac{1}{\left(\frac{xy}{dx}\right)}=-\frac{(2x-13)^2}{79}.$$

**6. Find $\frac{dy}{dx}$, if $y=x^x$.**

**Solution :** $y=x^x$

$\therefore$ $\log y=\log x^x=x\log x$

Differentiating both sides w.r.t. $x$, we get

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{d}{dx}(x\log x)$$

$$=x\cdot\frac{d}{dx}(\log x)+(\log x)\cdot\frac{d}{dx}(x)$$

$$=x\times\frac{1}{x}+(\log x)\times1$$

$$\therefore \frac{dy}{dx}=y(1+\log x)$$

$$=x^x(1+\log x).$$

**7. Find $\frac{dy}{dx}$, if $y=2^{x^x}$.**

**Solution :** Given : $y=2^{x^x}$

Let $u=x^x$

Then $y=2^u$

$$\therefore \frac{dy}{du}=\frac{d}{du}(2^u)=2^u\cdot\log 2$$

$$=2^{x^x}\cdot\log 2 \qquad \dots (1)$$

Now, $u=x^x$

$\therefore$ $\log u=\log x^x=x\log x$

Differentiating both sides w.r.t. $x$, we get

$$\frac{1}{u}\cdot\frac{du}{dx}=\frac{d}{dx}(x\log x)$$

$$=x\frac{d}{dx}(\log x)+(\log x)\cdot\frac{d}{dx}(x)$$

$$=x\times\frac{1}{x}+(\log x)\times1$$

$$\therefore \frac{du}{dx}=u(1+\log x)=x^x(1+\log x) \qquad \dots (2)$$

$$\therefore \frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

$$=2^{x^x}\cdot\log 2\cdot x^x(1+\log x) \qquad \dots \text{[By (1) and (2)]}$$

$$=2^{x^x}\cdot x^x(\log 2)(1+\log x).$$

**8. Find $\frac{dy}{dx}$, if $y=\sqrt{\frac{(3x-4)^3}{(x+1)^4(x+2)}}$.**

**Solution :** *Refer to the solution of Q. 2 (3) of Exercise 3.3.*

**Ans.** $\sqrt{\frac{(3x-4)^3}{(x+1)^4(x+2)}}\left[\frac{9}{2(3x-4)}-\frac{2}{x+1}-\frac{1}{2(x+2)}\right].$

**9. Find $\frac{dy}{dx}$, if $y = x^x + (7x-1)^x$.**

**Solution :** $y = x^x + (7x-1)^x$

Let $u = x^x$ and $v = (7x-1)^x$

Then $y = u + v$

$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ... (1)

Take $u = x^x$

$\therefore \log u = \log x^x = x \log x$

Differentiating both sides w.r.t. $x$, we get

$\frac{1}{u} \cdot \frac{du}{dx} = \frac{d}{dx}(x \log x)$

$= x \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(x)$

$= x \times \frac{1}{x} + (\log x) \times 1$

$\therefore \frac{du}{dx} = u(1 + \log x) = x^x(1 + \log x)$ ... (2)

Also, $v = (7x-1)^x$

$\therefore \log v = \log (7x-1)^x = x \log (7x-1)$

Differentiating both sides w.r.t. $x$, we get

$\frac{1}{v} \cdot \frac{dv}{dx} = \frac{d}{dx}[x \log (7x-1)]$

$= x \frac{d}{dx}[\log (7x-1)] + [\log (7x-1)] \cdot \frac{d}{dx}(x)$

$= x \times \frac{1}{7x-1} \cdot \frac{d}{dx}(7x-1) + [\log (7x-1)] \times 1$

$= \frac{x}{7x-1} \times (7 \times 1 - 0) + \log (7x-1)$

$\therefore \frac{dv}{dx} = v\left[\frac{7x}{7x-1} + \log (7x-1)\right]$

$= (7x-1)^x \left[\frac{7x}{7x-1} + \log (7x-1)\right]$ ... (3)

From (1), (2) and (3), we get

$\frac{dy}{dx} = x^x(1 + \log x) + (7x-1)^x \left[\frac{7x}{7x-1} + \log (7x-1)\right]$.

**10. If $y = x^3 + 3xy^2 + 3x^2y$, find $\frac{dy}{dx}$.**

**Solution :** $y = x^3 + 3xy^2 + 3x^2y$

Differentiating both sides w.r.t. $x$, we get

$\frac{dy}{dx} = 3x^2 + 3\left[x \cdot \frac{d}{dx}(y^2) + y^2 \cdot \frac{d}{dx}(x)\right] + 3\left[x^2 \cdot \frac{dy}{dx} + y \cdot \frac{d}{dx}(x^2)\right]$

$\therefore \frac{dy}{dx} = 3x^2 + 3\left[x \times 2y\frac{dy}{dx} + y^2 \times 1\right] + 3\left[x^2 \cdot \frac{dy}{dx} + y \times 2x\right]$

$\therefore \frac{dy}{dx} = 3x^2 + 6xy\frac{dy}{dx} + 3y^2 + 3x^2\frac{dy}{dx} + 6xy$

$\therefore (1 - 6xy - 3x^2)\frac{dy}{dx} = 3x^2 + 3y^2 + 6xy$

$\therefore \frac{dy}{dx} = \frac{3x^2 + 3y^2 + 6xy}{1 - 6xy - 3x^2}$

$= \frac{-3(x^2 + y^2 + 2xy)}{6xy + 3x^2 - 1}$.

**11. If $x^3 + y^2 + xy = 7$, find $\frac{dy}{dx}$.**

**Solution :** $x^3 + y^2 + xy = 7$

Differentiating both sides w.r.t. $x$, we get

$3x^2 + 2y \cdot \frac{dy}{dx} + x\frac{dy}{dx} + y \cdot \frac{d}{dx}(x) = 0$

$\therefore 3x^2 + 2y\frac{dy}{dx} + x\frac{dy}{dx} + y \times 1 = 0$

$\therefore (2y + x)\frac{dy}{dx} = -3x^2 - y$

$\therefore \frac{dy}{dx} = \frac{-(y + 3x^2)}{2y + x}$.

**12. If $x^3y^3 = x^2 - y^2$, find $\frac{dy}{dx}$.**

**Solution :** $x^3y^3 = x^2 - y^2$

Differentiating both sides w.r.t. $x$, we get

$x^3 \cdot \frac{d}{dx}(y^3) + y^3 \cdot \frac{d}{dx}(x^3) = 2x - 2y\frac{dy}{dx}$

$\therefore x^3 \times 3y^2\frac{dy}{dx} + y^3 \times 3x^2 = 2x - 2y\frac{dy}{dx}$

$\therefore (3x^3y^2 + 2y)\frac{dy}{dx} = 2x - 3x^2y^3$

$\therefore y(2 + 3x^3y)\frac{dy}{dx} = x(2 - 3xy^3)$

$\therefore \frac{dy}{dx} = \frac{x(2 - 3xy^3)}{y(2 + 3x^3y)}$.

**13. If $x^7 \cdot y^9 = (x+y)^{16}$, then show that $\frac{dy}{dx} = \frac{y}{x}$.**

**Solution :** *Refer to the solution of Q. 3 (1) of Exercise 3.4.*

**14. If $x^a \cdot y^b = (x+y)^{a+b}$, then show that $\frac{dy}{dx} = \frac{y}{x}$.**

**Solution :** $x^a \cdot y^b = (x+y)^{a+b}$

$\therefore \log(x^a \cdot y^b) = \log(x+y)^{a+b}$

$\therefore \log x^a + \log y^b = \log(x+y)^{a+b}$

$\therefore a \log x + b \log y = (a+b)\log(x+y)$

Differentiating both sides w.r.t. $x$, we get

$$a \times \frac{1}{x} + b \times \frac{1}{y}\frac{dy}{dx} = (a+b) \times \frac{1}{x+y} \cdot \frac{d}{dx}(x+y)$$

$$\therefore \frac{a}{x} + \frac{b}{y}\frac{dy}{dx} = \frac{a+b}{x+y}\left(1 + \frac{dy}{dx}\right)$$

$$\therefore \frac{a}{x} + \frac{b}{y}\frac{dy}{dx} = \frac{a+b}{x+y} + \frac{a+b}{x+y} \cdot \frac{dy}{dx}$$

$$\therefore \left(\frac{b}{y} - \frac{a+b}{x+y}\right)\frac{dy}{dx} = \frac{a+b}{x+y} - \frac{a}{x}$$

$$\therefore \left[\frac{bx+by-ay-by}{y(x+y)}\right]\frac{dy}{dx} = \frac{ax+bx-ax-ay}{a(x+y)}$$

$$\therefore \left[\frac{bx-ay}{y(x+y)}\right]\frac{dy}{dx} = \frac{bx-ay}{x(x+y)}$$

$$\therefore \frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{x}$$

$$\therefore \frac{dy}{dx} = \frac{y}{x}.$$

**15. Find $\frac{dy}{dx}$, if $x = 5t^2$, $y = 10t$.**

**Solution :** $x = 5t^2$, $y = 10t$

Differentiating $x$ and $y$ w.r.t. $t$, we get

$$\frac{dx}{dt} = 5\frac{d}{dt}(t^2) = 5 \times 2t = 10t$$

and $\frac{dy}{dt} = 10\frac{d}{dt}(t) = 10 \times 1 = 10$

$$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{10}{10t} = \frac{1}{t}.$$

**16. Find $\frac{dy}{dx}$, if $x = e^{3t}$, $y = e^{\sqrt{t}}$.**

**Solution :** $x = e^{3t}$, $e = e^{\sqrt{t}}$

Differentiating $x$ and $y$ w.r.t. $t$, we get

$$\frac{dx}{dt} = \frac{d}{dt}(e^{3t}) = e^{3t} \cdot \frac{d}{dt}(3t)$$

$$= e^{3t} \times 3 = 3e^{3t}$$

and $\frac{dy}{dt} = \frac{d}{dt}(e^{\sqrt{t}}) = e^{\sqrt{t}} \cdot \frac{d}{dt}(\sqrt{t})$

$$= e^{\sqrt{t}} \times \frac{1}{2\sqrt{t}} = \frac{e^{\sqrt{t}}}{2\sqrt{t}}$$

$$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{\left(\frac{e^{\sqrt{t}}}{2\sqrt{t}}\right)}{3e^{3t}}$$

$$= \frac{1}{6\sqrt{t}} \cdot e^{(\sqrt{t} - 3t)}.$$

**17. Differentiate $\log(1+x^2)$ with respect to $a^x$.**

**Solution :** Let $u = \log(1+x^2)$ and $v = a^x$

Then we want to find $\frac{du}{dv}$.

Differentiating $u$ and $v$ w.r.t. $x$, we get

$$\frac{du}{dx} = \frac{d}{dx}[\log(1+x^2)] = \frac{1}{1+x^2} \cdot \frac{d}{dx}(1+x^2)$$

$$= \frac{1}{1+x^2} \times (0+2x) = \frac{2x}{1+x^2}$$

and $\frac{dv}{dx} = \frac{d}{dx}(a^x) = a^x \cdot \log a$

$$\therefore \frac{du}{dv} = \frac{(du/dx)}{(dv/dx)} = \frac{\left(\frac{2x}{1+x^2}\right)}{a^x \cdot \log a} = \frac{2x}{(1+x^2) \cdot a^x \log a}.$$

**18. Differentiate $e^{(4x+5)}$ with resepct to $10^{4x}$.**

**Solution :** Let $u = e^{(4x+5)}$ and $v = 10^{4x}$

Then we want to find $\frac{du}{dv}$.

Differentiating $u$ and $v$ w.r.t. $x$, we get

$$\frac{du}{dx} = \frac{d}{dx}[e^{(4x+5)}] = e^{(4x+5)} \cdot \frac{d}{dx}(4x+5)$$

$$= e^{(4x+5)} \times (4 \times 1 + 0) = 4e^{(4x+5)}$$

and $\frac{dv}{dx} = \frac{d}{dx}(10^{4x}) = 10^{4x} \cdot \log 10 \cdot \frac{d}{dx}(4x)$

$$= 10^{4x} \cdot (\log 10) \times 4 = 4 \cdot 10^{4x} \cdot \log 10$$

$$\therefore \frac{du}{dv} = \frac{(du/dx)}{(dv/dx)} = \frac{4e^{(4x+5)}}{4 \cdot 10^{4x} \cdot \log 10} = \frac{e^{(4x+5)}}{10^{4x} \cdot \log 10}.$$

**19. Find $\frac{d^2y}{dx^2}$, if $y = \log x$.**

**Solution :** $y = \log x$

Differentiating w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\log x) = \frac{1}{x}$$

Differentiating again w.r.t. $x$, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}.$$

**20. Find $\frac{d^2y}{dx^2}$, if $y = 2at$, $x = at^2$.**

**Solution :** $x = at^2$, $y = 2at$

Differentiating $x$ and $y$ w.r.t. $t$, we get

$$\frac{dx}{dt} = \frac{d}{dt}(at^2) = a\frac{d}{dt}(t^2)$$

$$= a \times 2t = 2at \qquad \dots (1)$$

and $\frac{dy}{dt} = \frac{d}{dt}(2at) = 2a\frac{d}{dt}(t)$

$$= 2a \times 1 = 2a$$

$$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{2a}{2at} = \frac{1}{t}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{t}\right) = \frac{d}{dt}\left(\frac{1}{t}\right)\cdot\frac{dt}{dx}$$

$$= -\frac{1}{t^2} \times \frac{1}{\left(\frac{dx}{dt}\right)} = -\frac{1}{t^2} \times \frac{1}{2at} \qquad \dots \text{[By (1)]}$$

$$= -\frac{1}{2at^3}.$$

**21. Find $\frac{d^2y}{dx^2}$, if $y = x^2 \cdot e^x$.**

**Solution :** $y = x^2 \cdot e^x$

Differentiating w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x^2e^x) = x^2\frac{d}{dx}(e^x) + e^x\cdot\frac{d}{dx}(x^2)$$

$$= x^2 \cdot e^x + e^x \times 2x = e^x(x^2 + 2x)$$

Differentiating again w.r.t. $x$, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}[e^x(x^2 + 2x)]$$

$$= e^x \cdot \frac{d}{dx}(x^2 + 2x) + (x^2 + 2x)\cdot\frac{d}{dx}(e^x)$$

$$= e^x(2x + 2) + (x^2 + 2x)\cdot e^x$$

$$= e^x(2x + 2 + x^2 + 2x)$$

$$= e^x(x^2 + 4x + 2).$$

**22. If $x^2 + 6xy + y^2 = 10$, then show that $\frac{d^2y}{dx^2} = \frac{80}{(3x+y)^3}$.**

**Solution :** $x^2 + 6xy + y^2 = 10 \qquad \dots (1)$

Differentiating both sides w.r.t. $x$, we get

$$2x + 6\left[x\frac{dy}{dx} + y\cdot\frac{d}{dx}(x)\right] + 2y\frac{dy}{dx} = 0$$

$$\therefore 2x + 6x\frac{dy}{dx} + 6y \times 1 + 2y\frac{dy}{dx} = 0$$

$$\therefore (6x + 2y)\frac{dy}{dx} = -2x - 6y$$

$$\therefore \frac{dy}{dx} = \frac{-2(x + 3y)}{2(3x + y)} = -\left(\frac{x + 3y}{3x + y}\right) \qquad \dots (2)$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{d}{dx}\left(\frac{x + 3y}{3x + y}\right)$$

$$= -\left[\frac{(3x + y)\cdot\frac{d}{dx}(x + 3y) - (x + 3y)\cdot\frac{d}{dx}(3x + y)}{(3x + y)^2}\right]$$

$$= -\left[\frac{(3x + y)\left(1 + 3\frac{dy}{dx}\right) - (x + 3y)\left(3 + \frac{dy}{dx}\right)}{(3x + y)^2}\right]$$

$$= \frac{1}{(3x + y)^2}\left[-(3x + y)\left\{1 - \frac{3(x + 3y)}{3x + y}\right\} + (x + 3y)\left(3 - \frac{x + 3y}{3x + y}\right)\right] \qquad \dots \text{[By (2)]}$$

$$= \frac{1}{(3x + y)^2}\left[-(3x + y)\left(\frac{3x + y - 3x - 9y}{3x + y}\right) + (x + 3y)\left(\frac{9x + 3y - x - 3y}{3x + y}\right)\right]$$

$$= \frac{1}{(3x + y)^2}\left[8y + \frac{(x + 3y)(8x)}{3x + y}\right]$$

$$= \frac{1}{(3x + y)^2}\left[\frac{8y(3x + y) + (x + 3y)8x}{(3x + y)}\right]$$

$$= \frac{24xy + 8y^2 + 8x^2 + 24xy}{(3x + y)^3}$$

$$= \frac{8x^2 + 48xy + 8y^2}{(3x + y)^3} = \frac{8(x^2 + 6xy + y^2)}{(3x + y)^3}$$

$$= \frac{8(10)}{(3x + y)^3} \qquad \dots \text{[By (1)]}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{80}{(3x + y)^3}.$$

**23. If $ax^2 + 2hxy + by^2 = 0$, then show that $\frac{d^2y}{dx^2} = 0$.**

**Solution :** $ax^2 + 2hxy + by^2 = 0$ ... (1)

$\therefore\ ax^2 + hxy + hxy + by^2 = 0$

$\therefore\ x(ax + hy) + y(hx + by) = 0$

$\therefore\ x(ax + hy) = -y(hx + by)$

$\therefore\ \frac{ax + hy}{hx + by} = -\frac{y}{x}$ ... (2)

Differentiating (1) w.r.t. $x$, we get

$$a \times 2x + 2h\left[x\frac{dy}{dx} + y \cdot \frac{d}{dx}(x)\right] + b \times 2y\frac{dy}{dx} = 0$$

$\therefore\ 2ax + 2hx\frac{dy}{dx} + 2hy \times 1 + 2by\frac{dy}{dx} = 0$

$\therefore\ (2hx + 2by)\frac{dy}{dx} = -2ax - 2hy$

$\therefore\ \frac{dy}{dx} = \frac{-2(ax + hy)}{2(hx + by)} = -\left(\frac{ax + hy}{hx + by}\right)$

$\therefore\ \frac{dy}{dx} = \frac{y}{x}$ ... [By (1)]

$\therefore\ \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{y}{x}\right)$

$$= \frac{x\frac{dy}{dx} - y \cdot \frac{d}{dx}(x)}{x^2}$$

$$= \frac{x\left(\frac{y}{x}\right) - y \times 1}{x^2} \quad \ldots \left[\because \frac{dy}{dx} = \frac{y}{x}\right]$$

$$= \frac{y - y}{x^2} = \frac{0}{x^2}$$

$\therefore\ \frac{d^2y}{dx^2} = 0.$

| **ACTIVITIES** | **Textbook pages 101 and 102** |
|---|---|

**1. $y = (6x^4 - 5x^3 + 2x + 3)^5$, find $\frac{dy}{dx}$.**

**Solution :** Given : $y = (6x^4 - 5x^3 + 2x + 3)^5$

Let $u = [6x^4 - 5x^3 + \boxed{2x} + 3]$

$\therefore\ y = u^{\boxed{5}}$

$\therefore\ \frac{dy}{du} = 5u^4$

and $\frac{du}{dx} = 24x^3 - 15\boxed{x^2} + 2$

By chain rule,

$$\frac{dy}{dx} = \frac{dy}{\boxed{du}} \times \frac{\boxed{du}}{dx}$$

$\therefore\ \frac{dy}{dx} = 5(6x^4 - 5x^3 + 2x + 3)^{\boxed{4}} \times (24x^3 - 15x^2 + \boxed{2})$

**2. Find the rate of change of demand ($x$) of a commodity with respect to its price ($y$), if $y = 30 + 25x + x^2$.**

**Solution :** Let $y = 30 + 25x + x^2$

Differentiating w.r.t. $x$, we get

$\therefore\ \frac{dy}{dx} = \boxed{0} + \boxed{25} + \boxed{2x}$

$\therefore\ \frac{dy}{dx} = 25 + 2x$

By derivation of the inverse function

$$\frac{dx}{dy} = \frac{1}{\boxed{\frac{dy}{dx}}},\ \frac{dy}{dx} \neq 0$$

Rate of change of demand with respect to price

$$= \frac{1}{\boxed{25} + \boxed{2x}}.$$

**3. Find $\frac{dy}{dx}$, if $y = x^{(\log x)} + 10^x$.**

**Solution :** $y = x^{(\log x)} + 10^x$

Let $u = x^{\log x}$, $v = 10^x$

Then $y = u + v$

$\therefore\ \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ... (1)

Now, $u = x^{\log x}$

Taking log on both sides, we get

$\log u = \log x^{\log x}$

$\log u = (\log x)(\log x)$

$\log u = (\log x)^2$

Differentiating w.r.t. $x$, we get

$\therefore\ \frac{1}{u}\frac{du}{dx} = 2(\log x) \times \frac{d}{dx}\boxed{\log x}$

$\therefore \dfrac{du}{dx} = u\left[2 \log x \times \dfrac{1}{x}\right]$

$\therefore \dfrac{du}{dx} = x^{\log x}\left[2\boxed{\log x} \times \dfrac{1}{\boxed{x}}\right]$ ... (2)

Now, $v = 10^x$

Differentiating w.r.t. $x$, we get

$\therefore \dfrac{dv}{dx} = 10^x \boxed{\log 10}$ ... (3)

Substituting equation (2) and (3) in equation (1), we get

$\dfrac{dy}{dx} = x^{\log x}\left[2 \log x \times \dfrac{1}{x}\right] + 10^x \cdot \log 10.$

**4. Find $\dfrac{dy}{dx}$, if $y^x = e^{x+y}$.**

**Solution :** Given : $y^x = e^{x+y}$

Taking log on both side, we get

$\therefore \log (y)^x = \log (e)^{x+y}$

$\therefore x \cdot \boxed{\log y} = \boxed{x+y} \cdot \log e$

$\therefore x \cdot \log y = (x + \boxed{y}) \cdot 1$

$\therefore x \cdot \log y = x + \boxed{y}$

Differentiating w.r.t. $x$, we get

$\therefore x \cdot \dfrac{1}{y} \dfrac{d\boxed{y}}{dx} + \log y \cdot 1 = \boxed{1} + \dfrac{dy}{dx}$

$\therefore x \cdot \dfrac{1}{y} \dfrac{dy}{dx} + \log y = 1 + \dfrac{dy}{dx}$

$\therefore \dfrac{x}{y} \dfrac{dy}{dx} - \dfrac{dy}{dx} = 1 - \boxed{\log y}$

$\therefore \dfrac{dy}{dx}\left(\dfrac{x}{y} - 1\right) = \boxed{1} - \log y$

$\therefore \dfrac{dy}{dx} = \dfrac{(\boxed{1} - \log y)(y)}{x - y}.$

**5. Find $\dfrac{dy}{dx}$, if $x = e^t$, $y = e^{\sqrt{t}}$.**

**Solution :** Given : $x = e^t$, $y = e^{\sqrt{t}}$

Now, $y = e^{\sqrt{t}}$

Differentiating w.r.t. $t$, we get

$\therefore \dfrac{dy}{dt} = e^{\sqrt{t}} \dfrac{d}{dt} \boxed{\sqrt{t}}$

$\therefore \dfrac{dy}{dt} = e^{\sqrt{t}} \cdot \dfrac{1}{2\sqrt{t}}$ ... (1)

Now, $x = e^t$

Differentiating w.r.t. $t$, we get

$\therefore \dfrac{dx}{dt} = \boxed{e^t}$ ... (2)

Now, $\dfrac{dy}{dx} = \dfrac{(dy/dt)}{\boxed{(dx/dt)}}$

$= \dfrac{\dfrac{e^{\sqrt{t}}}{\boxed{2\sqrt{t}}}}{e^t}$

$\therefore \dfrac{dy}{dx} = \dfrac{e^{\sqrt{t}}}{2\sqrt{t}\, e^t}.$

## ACTIVITIES FOR PRACTICE

**1. Find $\dfrac{dy}{dx}$, if $y = \left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)^5$.**

**Solution :** Given : $y = \left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)^5$

Let $u = \sqrt{x} + \dfrac{1}{\square}$

Then $y = u^{\square}$

$\therefore \dfrac{dy}{du} = 5u^{\square}$

and $\dfrac{du}{dx} = \dfrac{1}{2\sqrt{x}} - \dfrac{1}{2\square} = \dfrac{x-1}{2\square}$

$\therefore \dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

$= 5\left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)^{\square} \cdot \left(\dfrac{x-1}{2\square}\right)$

**2. Find $\dfrac{dy}{dx}$, if $y = a^{\log(1+\log x)}$**

**Solution :** Given : $y = a^{\log(1+\log x)}$

$\therefore \dfrac{dy}{dx} = \dfrac{d}{dx}[a^{\log(1+\log x)}]$

$= a^{\log(1+\log x)} \cdot \square \cdot \dfrac{d}{dx}[\log(1+\log x)]$

$= a^{\log(1+\log x)} \cdot \square \times \dfrac{1}{1+\log x} \cdot \dfrac{d}{dx}\square$

$= a^{\log(1+\log x)} \cdot \square \times \dfrac{1}{1+\log x} \times \dfrac{1}{\square}$

$= \dfrac{a^{\log(1+\log x)} \cdot \square}{\square(1+\log x)}.$

**3. Find $\frac{dy}{dx}$, if $y = 7^{x+\frac{1}{x}}$.**

**Solution :** Given : $y = 7^{x+\frac{1}{x}}$

Let $u = x + \frac{1}{x}$

Then $y = 7^{\square}$

$\therefore \frac{dy}{du} = \frac{d}{dx}(7^{\square}) = 7^{\square} \cdot \log 7$

$= 7^{x+\frac{1}{x}} \cdot \log 7$

and $\frac{du}{dx} = \frac{d}{dx}\left(x + \frac{1}{x}\right)$

$= 1 - \frac{1}{\square}$

$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= 7^{x+\frac{1}{x}} \cdot \log 7 \cdot \left(1 - \frac{1}{\square}\right).$

**4. Find $\frac{dy}{dx}$, if $y = e^{2x} + (\log x)^x$.**

**Solution :** $y = e^{2x} + (\log x)^x$

Let $u = e^{2x}$ and $v = (\log x)^x$

Then $y = u + v$

$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ... (1)

Now, $u = e^{2x}$

$\therefore \frac{du}{dx} = e^{2x} \cdot \frac{d}{dx}\square$

$= \square e^{2x}$ ... (2)

Also, $v = (\log x)^x$

$\therefore \log v = \log (\log x)^x = x \log (\log x)$

Differentiating both sides w.r.t. $x$, we get

$\frac{1}{v} \cdot \frac{dv}{du} = \frac{d}{dx}[x \cdot \log (\log x)]$

$= x \cdot \frac{d}{dx}\square + \log (\log x) \cdot \frac{d}{dx}\square$

$= x \times \frac{1}{\square} \cdot \frac{d}{dx}\square + \log (\log x) \times \square$

$= \frac{x}{\square} \times \frac{1}{\square} + \log (\log x)$

$\therefore \frac{dv}{dx} = v\left[\frac{1}{\square} + \log (\log x)\right]$

$= (\log x)^x \left[\frac{1}{\square} + \log (\log x)\right]$ ... (3)

From (1), (2) and (3),

$\frac{dy}{dx} = \square e^{2x} + (\log x)^x \left[\frac{1}{\square} + \log (\log x)\right].$

**5. If $x\sqrt{x} + y\sqrt{y} = a\sqrt{a}$, then find $\frac{dy}{dx}$.**

**Solution :** $x\sqrt{x} + y\sqrt{y} = a\sqrt{a}$

$\therefore x^{\square} + y^{\square} = a^{\square}$

Differentiating both sides w.r.t. $x$, we get

$\square x^{\square} + \square y^{\square} \cdot \frac{dy}{dx} = \square$

$\therefore \square y^{\square} \frac{dy}{dx} = -\square x^{\square}$

$\therefore \frac{dy}{dx} = -\sqrt{\frac{\square}{\square}}.$

**6. If $x = \sqrt{1+u^2}$, $y = \log (1+u^2)$, find $\frac{dy}{dx}$ at $u = \sqrt{3}$.**

**Solution :** $x = \sqrt{1+u^2}$, $y = \log (1+u^2)$

Differentiating $x$ and $y$ w.r.t. $u$, we get

$\frac{dx}{du} = \frac{d}{du}(\sqrt{1+u^2}) = \frac{1}{2\square} \times \frac{d}{du}\square$

$= \frac{1}{2\square} \times \square = \frac{\square}{\sqrt{1+u^2}}$

and $\frac{dy}{du} = \frac{d}{du}[\log (1+u^2)]$

$= \frac{1}{1+u^2} \cdot \frac{d}{du}\square$

$= \frac{1}{1+u^2} \cdot \square = \frac{\square}{1+u^2}$

$\therefore \frac{dy}{dx} = \frac{(dy/du)}{(dx/du)} = \frac{\left(\frac{\square}{1+u^2}\right)}{\left(\frac{\square}{\sqrt{1+u^2}}\right)} = \frac{2}{\square}$

$\therefore \left(\frac{dy}{dx}\right)_{\text{at } u=\sqrt{3}} = \frac{2}{\square} = \square.$

**7. Find $\dfrac{d^2y}{dx^2}$, if $y = x^3 \log x$.**

**Solution :** $y = x^3 \log x$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(x^3 \log x)$$

$$= x^3 \frac{d}{dx}(\log x) + (\log x)\cdot\frac{d}{dx}\square$$

$$= x^3\cdot\square + (\log x)\,\square$$

$$= x^2\,(\square + \square \log x)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}[x^2\,(\square + \square \log x)]$$

$$= x^2\cdot\frac{d}{dx}[\square + \square \log x)] + (\square + \square \log x) \times \frac{d}{dx}(x^2)$$

$$= x^2 \times \frac{\square}{x} + (1 + 3 \log x)\cdot\square$$

$$= \square x + (1 + 3\log x)\,\square$$

$$= x\,(\square + \square \log x).$$

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**8. Find $\dfrac{d^2y}{dx^2}$, if $x = 2at^2$, $y = at^4$.**

**Solution :** $x = 2at^2$, $y = at^4$.

Differentiating $x$ and $y$ w.r.t. $t$, we get

$$\frac{dx}{dt} = \frac{d}{dt}(2at^2) = 2a\,\square$$

$$= \square \quad \dots (1)$$

and $\dfrac{dy}{dt} = \dfrac{d}{dt}(at^4) = a\,\square$

$$= \square$$

$$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{\square}{\square} = \square$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}(\square) = \frac{d}{dt}(\square) \times \frac{dt}{dx}$$

$$= \square \times \frac{1}{\left(\frac{dx}{dt}\right)}$$

$$= \square \times \frac{1}{\square} \quad \dots \text{[By (1)]}$$

$$= \frac{1}{\square}.$$

---

## OBJECTIVE SECTION

### MULTIPLE CHOICE QUESTIONS

**Select and write the correct answer from the given alternatives in each of the following questions :**

**1.** If $y = x + \sqrt{x^2+1}$, then $\dfrac{dy}{dx} = \dots\dots\dots$

(a) $\dfrac{1}{\sqrt{x^2+1}}$ (b) $\dfrac{x}{\sqrt{x^2+1}}$

(c) $\dfrac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}}$ (d) $1 + \dfrac{1}{\sqrt{x^2+1}}$

**2.** If $y = e^{(\log x + 8)}$, then $\dfrac{dy}{dx} = \dots\dots\dots$

(a) $e^{(\log x+8)}$ (b) $\dfrac{e^{(\log x+8)}}{x}$

(c) $\dfrac{e^{(\log x+8)}}{8x}$ (d) $x\cdot e^{(\log x+8)}$

**3.** If $u = e^{\log_e 5x} + e^{\log_e 7x}$, $\dfrac{dy}{dx} = \dots\dots\dots$

(a) 12 (b) 5 (c) 7 (d) $12e^{12x}$

**4.** If $f(x) = \left(\dfrac{x^a}{x^b}\right)^{a+b}\left(\dfrac{x^b}{x^c}\right)^{b+c}\left(\dfrac{x^c}{x^a}\right)^{c+a}$, then $f'(x)$ is equal to

(a) 0 (b) 1 (c) $x^{a+b+c}$ (d) $x^{abc}$

**5.** If $y = (1 + x^{\frac{1}{4}})(1 + x^{\frac{1}{2}})(1 - x^{\frac{1}{4}})$, then $\dfrac{dy}{dx} = \dots\dots\dots$

(a) 1 (b) $-1$ (c) $x$ (d) $\sqrt{x}$

**6.** If $y = \dfrac{x+\sqrt{x}}{\sqrt{x}+1}$, $\dfrac{dy}{dx} = \dots\dots\dots$

(a) $\dfrac{1}{\sqrt{x}}$ (b) $\dfrac{1}{2\sqrt{x}}$

(c) $\dfrac{\dfrac{\sqrt{x}+1}{\sqrt{x}} + (x+\sqrt{x})}{(\sqrt{x}+1)^2}$ (d) $-\dfrac{1}{2\sqrt{x}}$

**7.** If $y = \log\left(\dfrac{e^x + e^{-x}}{e^x - e^{-x}}\right)$, $\dfrac{dy}{dx} = \dots\dots\dots$

(a) $\dfrac{4}{e^{2x}+e^{-2x}}$ (b) $\dfrac{2}{e^x - e^{-x}}$

(c) $\dfrac{-4}{e^{2x}-e^{-2x}}$ (d) $\dfrac{2(e^x - e^{-x})}{e^x + e^{-x}}$

**8.** If $y=\frac{8^x}{x^8}$, $\frac{dy}{dx}=$ .........

(a) $\frac{8^{x-1}}{x^8}+8x^{-7}$ (b) $\frac{8^x}{x^8}\left(\log 8-\frac{8}{x}\right)$

(c) $\frac{x8^{x-1}}{9x^7}$ (d) $\frac{8^x}{x^8}\cdot\log 8$

**9.** If $y=\sqrt{x}\log x^2$, $\frac{dy}{dx}$ is

(a) $2x^{3/2}\log x^2$ (b) $\frac{1}{\sqrt{x}}(2+\log x)$

(c) $\frac{2x\log x}{\sqrt{x}}$ (d) $2\sqrt{x}\log x$

**10.** If $y=x^2+1$ and $u=\sqrt{1+x^2}$, then $\frac{dy}{du}=$ .........

(a) $\frac{x}{u}$ (b) $\frac{u}{x}$

(c) $2\sqrt{1+x^2}$ (d) $\frac{\sqrt{1+x^2}}{2}$

**11.** If $y=\log e^{1+\log x}$, then $\frac{dy}{dx}=$ .........

(a) 0 (b) 1 (c) $e$ (d) $\frac{1}{x}$

**12.** If $y=x^{\sqrt{x}}$, $\frac{dy}{dx}=$ .........

(a) $\frac{2+\log x}{2\sqrt{x}}$ (b) $x^{\sqrt{x}}\left[\frac{2+\log x}{\sqrt{x}}\right]$

(c) $x^{\sqrt{x}}\left[\frac{2+\log x}{2\sqrt{x}}\right]$ (d) $x^{\sqrt{x}}\left[\frac{1+\log x}{2\sqrt{x}}\right]$

**13.** *If* $x^y=3^{x-y}$, then $\frac{dy}{dx}$ is

(a) $\frac{x^{y-1}}{3^{x-y}}$ (b) $\frac{x\log 3-y}{x\log 3x}$

(c) $\frac{\log 3}{x^x}$ (d) $\frac{y\log 3x}{x^y}$

**14.** If $\sqrt{x}+\sqrt{y}=20$, $\frac{dy}{dx}$ is

(a) $\sqrt{\frac{x}{y}}$ (b) $-\sqrt{\frac{y}{x}}$ (c) $\sqrt{\frac{y}{x}}$ (d) $\frac{x}{y}$

**15.** If $x^a\cdot y^b=(x+y)^{a+b}$, then $\frac{dy}{dx}$ is

(a) $\sqrt{\frac{y}{x}}$ (b) $\frac{y}{x}$ (c) $\frac{x}{y}$ (d) $\sqrt{\frac{x}{y}}$

**16.** If $2^x+2^y=2^{x+y}$, then $\frac{dy}{dx}$ at $x=y=1$ is

(a) 0 (b) $-1$ (c) 1 (d) 2

**17.** If $xy=1+\log y$, then $\frac{dy}{dx}=$ .........

(a) $\frac{y}{\log y}$ (b) $\frac{-y^2}{\log y}$

(c) $\frac{y^2}{\log y}$ (d) $\frac{-y}{\log y}$

**18.** If $e^x=x^y$, then $\frac{dy}{dx}$ is

(a) $\frac{x-y}{x\log x}$ (b) $\frac{x+y}{x\log x}$

(c) $\frac{xy}{x\log x}$ (d) $\frac{x}{\log y}$

**19.** If $x=4t$, $y=\frac{4}{t}$, $\frac{dy}{dx}$ is

(a) $\frac{1}{t}$ (b) $\frac{1}{t^2}$ (c) $-\frac{1}{t^2}$ (d) 4

**20.** If $x^y=e^{x-y}$, then $\frac{dy}{dx}=$ .........

(a) $\frac{1+x}{1+\log x}$ (b) $\frac{\log x}{(1+\log x)^2}$

(c) $\frac{1-\log x}{1+\log x}$ (d) $\frac{1-x}{1+\log x}$

### Answers

**1.** (c) $\frac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}}$ **2.** (b) $\frac{e^{(\log x+8)}}{x}$

**3.** (a) 12 **4.** (a) 0

**5.** (b) $-1$ **6.** (b) $\frac{1}{2\sqrt{x}}$

**7.** (c) $\frac{-4}{e^{2x}-e^{-2x}}$ **8.** (b) $\frac{8^x}{x^8}\left(\log 8-\frac{8}{x}\right)$

**9.** (b) $\frac{1}{\sqrt{x}}(2+\log x)$ **10.** (c) $2\sqrt{1+x^2}$

**11.** (d) $\frac{1}{x}$ **12.** (c) $x^{\sqrt{x}}\left[\frac{2+\log x}{2\sqrt{x}}\right]$

**13.** (b) $\frac{x\log 3-y}{x\log 3x}$ **14.** (b) $-\sqrt{\frac{y}{x}}$

**15.** (b) $\frac{y}{x}$ **16.** (b) $-1$

**17.** (b) $\frac{-y^2}{\log y}$ **18.** (a) $\frac{x-y}{x\log x}$

**19.** (c) $-\frac{1}{t^2}$ **20.** (b) $\frac{\log x}{(1+\log x)^2}$.

## TRUE OR FALSE

**State whether the following statements are *True* or *False* :**

1. If $y=\frac{1}{\sqrt{3x+7}}$, then $\frac{dy}{dx}=\frac{3}{2(3x+7)^{3/2}}$.
2. If $y=3^{2x}$, then $\frac{dy}{dx}=3^{2x}\log 3$.
3. If $y=(\log x)^2$, then $\frac{dy}{dx}=\frac{2\log x}{x}$.
4. If $y=x^e$, then $\frac{dy}{dx}=ex^{e+1}$.
5. If $x=2at,\ y=2at^2$, then $\frac{dy}{dx}=\frac{1}{2t}$.
6. If $y=e^{\log x}$, then $\frac{dy}{dx}=1$.
7. If $y=e^{(\log 5+\log x)}$, then $\frac{dy}{dx}=5$.
8. If $x^3y^4=(x+y)^{n+1}$ and $\frac{dy}{dx}=\frac{y}{x}$, then $n=7$.
9. If $y=48x+\log(x+3)$, then rate of change of demand $(x)$ w.r.t. price $(y)$ is $\frac{48x+145}{x+3}$.
10. If $y=10^{x^2}$, then $\frac{dy}{dx}=10^{x^2}\cdot\log 10$.
11. If $x^2+y^2=3axy$, then $\frac{dy}{dx}=\frac{3ay-2x}{2y-3ax}$.
12. If $\log(xy)=xy$, then $\frac{dy}{dx}=\frac{y}{x}$.
13. If $y=e^{5x}$, then $\frac{d^2y}{dx^2}=5e^{5x}$.
14. If $ye^x+xe^y=7$, then $\frac{dy}{dx}=\frac{e^y+ye^x}{e^x+xe^y}$.
15. The derivative of $x^3$ with respect to $x^4+2$ is $\frac{3}{4x}$.

### Answers

| | | | | |
|---|---|---|---|---|
| 1. False | 2. False | 3. True | 4. False | 5. False |
| 6. True | 7. True | 8. False | 9. False | 10. False |
| 11. True | 12. False | 13. False | 14. False | 15. True. |

## FILL IN THE BLANKS

**Fill in the following blanks with an appropriate words/numbers :**

1. The rate of change of demand $(x)$ with respect to price $(y)$, if $y=20+15x+x^2$, is ......... .
2. If $y=e^{\sqrt{x}}$, then $\frac{dy}{dx}=$ ......... .
3. If $y=\log 3x$, then $\frac{dy}{dx}=$ .........
4. If $y=\log_x a$, then $\frac{dy}{dx}=$ .........
5. If $y=x+\sqrt{a^2+x^2}$, then $\frac{dy}{dx}=$ .........
6. If $y=\sqrt{(1-x)(1+x)}$, then $(1-x^2)\frac{dy}{dx}=$ .........
7. If $xy=(x+y)^n$ and $\frac{dy}{dx}=\frac{y}{x}$, then $n=$ .........
8. If $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$, then $\frac{dy}{dx}=$ .........
9. If $\log(xy)=1$, then $\frac{dy}{dx}=$ .........
10. The derivative of $\log(\log x)$ w.r.t. $\log x$ is .........
11. If $y=\frac{1+e^x}{e^x}$, then $\frac{d^2y}{dx^2}=$ .........
12. If $y=29x+\log(1+x^2)$, then rate of change of demand $(x)$ with respect to price $(y)$ is ......... .
13. If $y=x^4+4^x$, then $\frac{dy}{dx}=$ .........
14. If $x^2y^k=(x+y)^{2+k}$, then $\frac{dy}{dx}=$ .........
15. If $xy=1+\log y$, then $\frac{dy}{dx}=$ ......... .

### Answers

| | | |
|---|---|---|
| 1. $\frac{1}{15+2x}$ | 2. $\frac{e^{\sqrt{x}}}{2\sqrt{x}}$ | 3. $\frac{1}{x}$ |
| 4. $\frac{-\log a}{x(\log x)^2}$ | 5. $\frac{x+\sqrt{a^2+x^2}}{\sqrt{a^2+x^2}}$ | 6. $-xy$ |
| 7. 2 | 8. $-\left(\frac{y}{x}\right)^{\frac{1}{3}}$ | 9. $-\frac{y}{x}$ |
| 10. $\frac{1}{\log x}$ | 11. $e^{-x}$ | 12. $\frac{1+x^2}{29+2x+29x^2}$ |
| 13. $4x^3+4^x\cdot\log 4$ | 14. $\frac{y}{x}$ | 15. $-\frac{y^2}{\log y}$. |

# 4 APPLICATIONS OF DERIVATIVES

CHAPTER OUTLINE



## IMPORTANT FORMULAE

1. Let $y = f(x)$ by any curve and $P(a, f(a))$ be any point on it, then the slope of the tangent to the curve at the point P is $f'(a)$. It is also called **gradient** of the curve at the point P.
   Hence, equation of tangent at P is
   $y - f(a) = f'(a)(x - a)$.
2. Slope of normal at $P(a, f(a))$ is $\frac{-1}{f'(a)}$, if $f'(a) \neq 0$ and equation of normal at P is $y - f(a) = \frac{-1}{f'(a)}(x - a)$.
3. A function $f$ is said to be
   (i) increasing in $(a, b)$ if $f'(x) > 0$ for all $x \in (a, b)$
   (ii) decreasing in $(a, b)$ if $f'(x) < 0$ for all $x \in (a, b)$.
4. Elasticity of demand $\eta = \frac{-p}{D} \cdot \frac{dD}{dp}$.
5. $R_m = P\left(1 - \frac{1}{\eta}\right) = R_A\left(1 - \frac{1}{\eta}\right)$, where $R_m$ is the marginal revenue, $R_A$ is the average revenue and $\eta$ is the elasticity of demand.
6. For a person with income $x$, consumption expenditure $E_c$ and saving $S$,
   (i) $x = E_c + S$
   (ii) MPC + MPS = 1
   (iii) APC + APS = 1.
7. A function $f$ is said to have
   (i) maximum at $x = c$, if $f'(c) = 0$ and $f''(c) < 0$
   (ii) minimum at $x = c$, if $f'(c) = 0$ and $f''(c) > 0$
   [**Note :** If $f''(c) = 0$, then second derivative test fails. In such a case, we have to apply the first derivative test.]

   **First Derivative Test :**
   A function $f$ is said to have
   (i) maximum at $x = c$ if
   (a) $f'(c) = 0$ (b) $f'(c - h) > 0$ and
   (c) $f'(c + h) < 0$
   (ii) minimum at $x = c$ if
   (a) $f'(c) = 0$ (b) $f'(c - h) < 0$ and
   (c) $f'(c + h) > 0$
   where $h$ is a very small positive number.

## INTRODUCTION

We have used the concept of differentiation to study the marginal demand, i.e. rate of change of demand with respect to price and marginal cost, i.e. rate of change of cost with respect to the number of articles in Std. XI. Now we will study geometrical and physical significance of derivative and some more applications of derivatives to the concepts in Economics, viz. elasticity of demand, marginal propensity to consume, marginal propensity to save, etc. Let us learn a few more mathematical concepts.

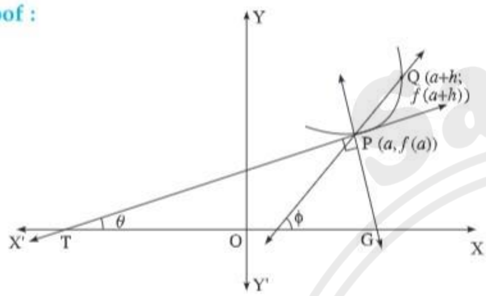
## 4.1 : MEANING OF DERIVATIVE

**Geometrical meaning of derivative (Tangent and Normal at a point to a curve) :**

**Theorem :**

**Prove that the slope of the tangent to a curve $y = f(x)$, at the point $x = a$ on it is $f'(a)$.**

**Proof :**




Let P be the point $x = a$ on the curve $y = f(x)$.
[This means the $x$-coordinate of P is $a$.]
Then P is $(a, f(a))$.

Take a point $Q(a+h, f(a+h))$ on the curve.
Then slope of secant PQ

$$= \frac{f(a+h)-f(a)}{a+h-a} = \frac{f(a+h)-f(a)}{h}$$

Now, as $Q \to P$ along the curve, the limiting position of the secant PQ is called the tangent of the curve at the point P. i.e. at the point $x = a$ and the slope of the tangent at P is called the **gradient** of the curve at the point P.

Now, as $Q \to P$ along the curve, $h \to 0$

$\therefore$ gradient of the curve at P = slope of the tangent PT

$$= \lim_{Q \to P} (\text{slope of secant PQ})$$

$$= \lim_{h \to 0} \frac{f(a+h)-f(a)}{h} = f'(a)$$

Thus, $f'(a)$ represents the slope of the tangent to the curve $y = f(x)$ at the point $(a, f(a))$. [i.e. the point $x = a$ on the curve.]

**Remarks :**

1. If this tangent has inclination $\theta$, then its slope is $\tan\theta$.

   Hence, $\tan\theta = f'(a) = \left(\frac{dy}{dx}\right)_{x=a} = \left(\frac{dy}{dx}\right)_{(a,\, f(a))}$.

2. The above tangent passes through the point $(a, f(a))$ and its slope is $f'(a)$. Hence, by the slope-point form, the equation of the tangent to the curve $y = f(x)$ at the point $x = a$ on it, is

   $y - f(a) = f'(a)(x - a)$.

3. Let line PG be perpendicular to the tangent PT through its point of contact P. Then this line PG is called the **normal** to the curve $y = f(x)$ at the point P. We also say that P is the foot of the normal PG.

   Obviously, the slope of the normal is $\frac{-1}{f'(a)}$ if $f'(a) \neq 0$ and it passes through $P(a, f(a))$. Hence, its equation is

   $y - f(a) = \frac{-1}{f'(a)}(x - a)$.

   If $f'(a) = 0$, then the tangent is parallel to X-axis and hence the normal is parallel to Y-axis. In that case the equation of the normal is $x = a$.

| EXERCISE 4.1 | Textbook page 105 |
|---|---|

**1. Find the equations of tangent and normal to the following curves at the given point on it :**

**(i) $y = 3x^2 - x + 1$ at (1, 3)**

**(ii) $2x^2 + 3y^2 = 5$ at (1, 1)**

**(iii) $x^2 + y^2 + xy = 3$ at (1, 1).**

**Solution :**

(i) $y = 3x^2 - x + 1$

$\therefore \frac{dy}{dx} = \frac{d}{dx}(3x^2 - x + 1)$

$= 3 \times 2x - 1 + 0 = 6x - 1$

$\therefore \left(\frac{dy}{dx}\right)_{\text{at }(1,\,3)} = 6(1) - 1 = 5$

= slope of the tangent at (1, 3)

$\therefore$ the equation of the tangent at (1, 3) is

$y - 3 = 5(x - 1)$

$\therefore y - 3 = 5x - 5$

$\therefore 5x - y - 2 = 0$.

The slope of the normal at (1, 3)

$= \frac{-1}{\left(\frac{dy}{dx}\right)_{\text{at }(1,\,3)}} = -\frac{1}{5}$

$\therefore$ the equation of the normal at (1, 3) is

$y - 3 = -\frac{1}{5}(x - 1)$

$\therefore 5y - 15 = -x + 1$

$\therefore x + 5y - 16 = 0$

Hence, the equations of the tangent and normal are $5x - y - 2 = 0$ and $x + 5y - 16 = 0$ respectively.

**(ii)** $2x^2 + 3y^2 = 5$

Differentiating both sides w.r.t. $x$, we get

$$2 \times 2x + 3 \times 2y\frac{dy}{dx} = 0$$

$\therefore \ 6y\frac{dy}{dx} = -4x \qquad \therefore \ \frac{dy}{dx} = -\frac{2x}{3y}$

$$\therefore \ \left(\frac{dy}{dx}\right)_{\text{at }(1,\ 1)} = \frac{-2(1)}{3(1)} = -\frac{2}{3}$$

$= \text{slope of the tangent at } (1, 1)$

$\therefore$ the equation of the tangent at (1, 1) is

$$y - 1 = \frac{-2}{3}(x - 1)$$

$\therefore \ 3y - 3 = -2x + 2$

$\therefore \ 2x + 3y - 5 = 0.$

The slope of normal at (1, 1)

$$= \frac{-1}{\left(\frac{dy}{dx}\right)_{\text{at }(1,\ 1)}} = \frac{-1}{\left(\frac{-2}{3}\right)} = \frac{3}{2}$$

$\therefore$ the equation of the normal at (1, 1) is

$$y - 1 = \frac{3}{2}(x - 1)$$

$\therefore \ 2y - 2 = 3x - 3$

$\therefore \ 3x - 2y - 1 = 0$

Hence, the equations of the tangent and normal are $2x + 3y - 5 = 0$ and $3x - 2y - 1 = 0$ respectively.

**(iii)** $x^2 + y^2 + xy = 3$

Differentiating both sides w.r.t. $x$, we get

$$2x + 2y\frac{dy}{dx} + x\frac{dy}{dx} + y\frac{d}{dx}(x) = 0$$

$$\therefore \ 2x + 2y\frac{dy}{dx} + x\frac{dy}{dx} + y \times 1 = 0$$

$$\therefore \ (x + 2y)\frac{dy}{dx} = -2x - y$$

$$\therefore \ \frac{dy}{dx} = \frac{-2x - y}{x + 2y} = -\left(\frac{2x + y}{x + 2y}\right)$$

$$\therefore \ \left(\frac{dy}{dx}\right)_{\text{at }(1,\ 1)} = -\left[\frac{2(1) + 1}{1 + 2(1)}\right] = -\frac{3}{3} = -1$$

$= \text{slope of the tangent at } (1, 1)$

$\therefore$ the equation of the tangent at (1, 1) is

$y - 1 = -1(x - 1)$

$\therefore \ y - 1 = -x + 1 \qquad \therefore \ x + y = 2$

The slope of the normal at (1, 1) $= \dfrac{-1}{\left(\frac{dy}{dx}\right)_{\text{at}(1,\ 1)}}$

$$= \frac{-1}{-1} = 1$$

$\therefore$ the equation of the normal at (1, 1) is

$y - 1 = 1(x - 1)$

$\therefore \ y - 1 = x - 1 \qquad \therefore \ x - y = 0$

Hence, the equations of tangent and normal are $x + y = 2$ and $x - y = 0$ respectively.

**2. Find the equations of the tangent and normal to the curve $y = x^2 + 5$ where the tangent is parallel to the line $4x - y + 1 = 0$.**

**Solution :** Let $P(x_1, y_1)$ be the point on the curve $y = x^2 + 5$ where the tangent is parallel to the line $4x - y + 1 = 0$.

Differentiating $y = x^2 + 5$ w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 + 5) = 2x + 0 = 2x$$

$$\therefore \ \left(\frac{dy}{dx}\right)_{\text{at }(x_1,\ y_1)} = 2x_1$$

$= \text{slope of the tangent at } (x_1, y_1)$

Let $m_1 = 2x_1$

The slope of the line $4x - y + 1 = 0$ is

$$m_2 = \frac{-4}{-1} = 4$$

Since, the tangent at $P(x_1, y_1)$ is parallel to the line $4x - y + 1 = 0$, $m_1 = m_2$

$\therefore \ 2x_1 = 4 \qquad \therefore \ x_1 = 2$

Since, $(x_1, y_1)$ lies on the curve $y = x^2 + 5$, $y_1 = {x_1}^2 + 5$

$\therefore \ y_1 = (2)^2 + 5 = 9 \qquad \ldots [\because x_1 = 2]$

$\therefore$ the coordinates of the point are (2, 9) and the slope of the tangent $= m_1 = m_2 = 4$.

$\therefore$ the equation of the tangent at (2, 9) is

$y - 9 = 4(x - 2)$

$\therefore \ y - 9 = 4x - 8$

$\therefore \ 4x - y + 1 = 0$

Slope of the normal $= \dfrac{-1}{m_1} = -\dfrac{1}{4}$

$\therefore$ the equation of the normal at (2, 9) is

$$y - 9 = -\frac{1}{4}(x - 2)$$

$\therefore \ 4y - 36 = -x + 2$

$\therefore \ x + 4y - 38 = 0$

Hence, the equations of tangent and normal are $4x - y + 1 = 0$ and $x + 4y - 38 = 0$ respectively.

[**Note :** *Answer to the equation of tangent in the textbook is incorrect.*]

**3. Find the equations of the tangent and normal to the curve $y = 3x^2 - 3x - 5$ where the tangent is parallel to the line $3x - y + 1 = 0$.**

**Solution :** Let $P(x_1, y_1)$ be the point on the curve $y = 3x^2 - 3x - 5$ where the tangent is parallel to the line $3x - y + 1 = 0$.

Differentiating $y = 3x^2 - 3x - 5$ w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{d}{dx}(3x^2 - 3x - 5)$$

$$= 3 \times 2x - 3 \times 1 - 0 = 6x - 3$$

$$\therefore \left(\frac{dy}{dx}\right)_{\text{at }(x_1, y_1)} = 6x_1 - 3$$

$$= \text{slope of the tangent at } (x_1, y_1)$$

Let $m_1 = 6x_1 - 3$

The slope of the line $3x - y + 1 = 0$

$$= m_2 = \frac{-3}{-1} = 3$$

Since, the tangent at $P(x_1, y_1)$ is parallel to the line $3x - y + 1 = 0$, $m_1 = m_2$

$\therefore\ 6x_1 - 3 = 3 \quad \therefore\ 6x_1 = 6 \quad \therefore\ x_1 = 1$

Since, $(x_1, y_1)$ lies on the curve $y = 3x^2 - 3x - 5$,

$y_1 = 3x_1^2 - 3x_1 - 5$, where $x_1 = 1$

$= 3(1)^2 - 3(1) - 5 = 3 - 3 - 5 = -5$

$\therefore$ the coordinates of the point are $(1, -5)$ and the slope of the tangent $= m_1 = m_2 = 3$.

$\therefore$ the equation of the tangent at $(1, -5)$ is

$y - (-5) = 3(x - 1)$

$\therefore\ y + 5 = 3x - 3 \quad \therefore\ 3x - y - 8 = 0$.

Slope of the normal $= -\frac{1}{m_1} = -\frac{1}{3}$

$\therefore$ the equation of the normal at $(1, -5)$ is

$$y - (-5) = -\frac{1}{3}(x - 1)$$

$\therefore\ 3y + 15 = -x + 1 \quad \therefore\ x + 3y + 14 = 0$

Hence, the equations of tangent and normal are $3x - y - 8 = 0$ and $x + 3y + 14 = 0$ respectively.

## ADDITIONAL SOLVED PROBLEMS – 4 (A)

**1. Find the points on the curve $y = x^3 - 2x^2 - x$, where the tangents are parallel to $3x - y + 1 = 0$.**

**Solution :** Let the required point on the curve $y = x^3 - 2x^2 - x$ be $P(x_1, y_1)$.

Differentiating $y = x^3 - 2x^2 - x$ w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 - 2x^2 - x)$$

$$= 3x^2 - 2 \times 2x - 1 = 3x^2 - 4x - 1$$

$\therefore$ slope of the tangent at $(x_1, y_1)$

$$= \left(\frac{dy}{dx}\right)_{\text{at }(x_1, y_1)} = 3x_1^2 - 4x_1 - 1$$

Since, this tangent is parallel to $3x - y + 1 = 0$ whose slope is $\frac{-3}{-1} = 3$,

slope of the tangent $= 3$.

$\therefore\ 3x_1^2 - 4x_1 - 1 = 3$

$\therefore\ 3x_1^2 - 4x_1 - 4 = 0$

$\therefore\ 3x_1^2 - 6x_1 + 2x_1 - 4 = 0$

$\therefore\ 3x_1(x_1 - 2) + 2(x_1 - 2) = 0$

$\therefore\ (x_1 - 2)(3x_1 + 2) = 0$

$\therefore\ x_1 - 2 = 0 \quad \text{or} \quad 3x_1 + 2 = 0$

$\therefore\ x_1 = 2 \quad \text{or} \quad x_1 = -\frac{2}{3}$

Since, $(x_1, y_1)$ lies on $y = x^3 - 2x^2 - x$,

$y_1 = x_1^3 - 2x_1^2 - x_1$

When $x_1 = 2$, $y_1 = (2)^3 - 2(2)^2 - 2 = 8 - 8 - 2 = -2$

When $x_1 = -\frac{2}{3}$, $y_1 = \left(-\frac{2}{3}\right)^3 - 2\left(-\frac{2}{3}\right)^2 - \left(-\frac{2}{3}\right)$

$$= \frac{-8}{27} - \frac{8}{9} + \frac{2}{3} = -\frac{14}{27}$$

Hence, the required points are

$(2, -2)$ and $\left(-\frac{2}{3}, -\frac{14}{27}\right)$.

**2. If the line $y = 4x - 5$ touches the curve $y^2 = ax^3 + b$ at the point $(2, 3)$, show that $7a + 2b = 0$.**

**Solution :** $y^2 = ax^3 + b$

Differentiating both sides w.r.t. $x$, we get

$$2y\frac{dy}{dx} = a \times 3x^2 + 0$$

$$\therefore\ \frac{dy}{dx} = \frac{3ax^2}{2y}$$

$$\therefore\ \left(\frac{dy}{dx}\right)_{\text{at }(2, 3)} = \frac{3a(2)^2}{2(3)} = 2a$$

$= \text{slope of the tangent at } (2, 3)$

Since the line $y = 4x - 5$ touches the curve at the point $(2, 3)$, slope of the tangent at $(2, 3)$ is 4.

$\therefore\ 2a = 4 \quad \therefore\ a = 2$

Since, $(2, 3)$ lies on the curve $y^2 = ax^3 + b$,

$(3)^2 = a(2)^3 + b \quad \therefore\ 9 = 8a + b$

$\therefore\ 9 = 8(2) + b \qquad \ldots [\because a = 2]$

$\therefore\ b = -7$

$\therefore\ 7a + 2b = 7(2) + 2(-7) = 0$.

## EXAMPLES FOR PRACTICE 4.1

**1. Find the equations of tangent and normal to the following curves at the given points on it :**

(i) $y = x^2 + 4x + 1$ at $(-1, -2)$

(ii) $y = x^2 - 4x + 3$ at $(4, 3)$

(iii) $y = x^3 - 2x^2 + 4$ at the point $x = 2$

(iv) $y = \frac{x}{x^2+1}$ at the origin

(v) $y^2 = 4ax$ at $(a, -2a)$.

**2.** Prove that the tangents to the curve $y = x^2 - 5x + 6$ at the points $(2, 0)$ and $(3, 0)$ are at right angles.

**3.** Find the points on the curve $y = 2x^3 - 3x^2 + 3x - 2$ at which the tangent lines are parallel to the line $3x - y + 7 = 0$.

**4.** Find the equations of the normals to the curve $3x^2 - y^2 = 8$, which are parallel to the line $x + 3y = 4$.

**5.** Find the equations of tangent and normal to the curve $y = 6 - x^2$ where the normal is parallel to the line $x - 4y + 3 = 0$.

**6.** Find the coordinates of the point on the curve $y = 6x - x^2$, the tangent at which is parallel to the line $y = -4x$.

**7.** Find the coordinates of the points on the curve $y = x + \frac{1}{x}$, the tangent at which is parallel to X-axis.

**8.** Find the coordinates of the point on the curve $y = x - \frac{4}{x}$, where the tangent is parallel to the line $y = 2x$.

**9.** If the line $x + y = 0$ touches the curve $y^2 = ax^3 + b$ at $(1, -1)$, find $a$ and $b$.

**10.** The tangent to the curve $y = x + \frac{1}{x}$ at $(a, b)$ is parallel to X-axis. Find $a$ and $b$.

### Answers

**1.** (i) $2x - y = 0$, $x + 2y + 5 = 0$

(ii) $4x - y = 13$, $x + 4y = 16$

(iii) $4x - y = 4$, $x + 4y = 18$

(iv) $x - y = 0$, $x + y = 0$

(v) $x + y + a = 0$, $x - y = 3a$.

**3.** $(0, -2)$ and $(1, 0)$

**4.** $x + 3y - 8 = 0$, $x + 3y + 8 = 0$

**5.** $4x + y - 10 = 0$, $x - 4y + 6 = 0$

**6.** $(5, 5)$

**7.** $(1, 2)$ and $(-1, -2)$

**8.** $(2, 0)$ and $(-2, 0)$

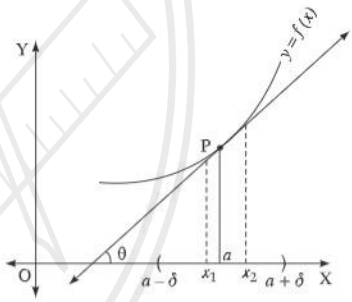
**9.** $a = \frac{2}{3}$, $b = \frac{1}{3}$

**10.** $a = \pm 1$, $b = \pm 2$.

## 4.2 : INCREASING AND DECREASING FUNCTIONS

The increasing and decreasing functions are *always* defined in terms of increasing $x$.

**(1) Increasing Function :** Consider, the function $f$ whose graph is shown below. Let P be the point $x = a$ on this graph. Then P is $(a, f(a))$. We observe that while passing through this point from left to right, the




graph of the function is rising, i.e. the values of the function are increasing. In this case we say that $f$ is an increasing function at P, i.e. at $x = a$.

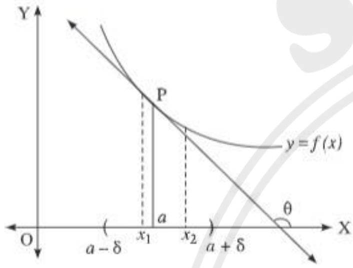
Now, consider a small neighbourhood $(a - \delta, a + \delta)$ about $a$. If we take any two points $x_1$, $x_2$ in this neighbourhood such that $x_1 < x_2$, then $f(x_1) < f(x_2)$. Hence, we have the following definition :

**Definition :** *A function f is said to be* increasing *at $x = a$, if there exists some neighbourhood $(a - \delta, a + \delta)$ of 'a' such that for any two points $x_1$, $x_2$ in this neighbourhood with $x_1 < x_2$, we have $f(x_1) \leqslant f(x_2)$.*

$f$ is said to be *strictly increasing* at $x = a$, if for any two points $x_1$, $x_2$ in this neighbourhood with $x_1 < x_2$, we have $f(x_1) < f(x_2)$.

**Remark :** The tangent to the curve at P makes an acute angle $\theta$ with the positive X-axis. We know that tangent of an acute angle is positive. Hence, $\tan\theta > 0$. But $\tan\theta = f'(a)$. Hence we can say that if $f'(a) > 0$, then the function $f$ is increasing at $x = a$.

**(2) Decreasing Function :** Consider, the function $f$ whose graph is shown below. As before, we can say that the function $f$ is decreasing at $x = a$ and have the following definition :




**Definition :** *A function f is said to be* decreasing *at $x = a$, if there exists some neighbourhood $(a - \delta, a + \delta)$ of 'a' such that for any two points $x_1$, $x_2$ in this neighbourhood with $x_1 < x_2$, we have, $f(x_1) \geqslant f(x_2)$.*

$f$ is said to be *strictly decreasing* at $x = a$, if for any two points $x_1$, $x_2$ in this neighbourhood with $x_1 < x_2$, we have, $f(x_1) > f(x_2)$.

**Remark :** The tangent to the curve at P makes an obtuse angle $\theta$ with the positive X-axis. Since the tangent of an obtuse angle is negative, $\tan\theta < 0$. But $\tan\theta = f'(a)$. Hence, we can say that, if $f'(a) < 0$, then the function $f$ is decreasing at $x = a$.

From the above observations, we have the following result :

Let a function $f$ be differentiable at the point P$(x, y)$. Then

(i) $f$ is increasing at P if $f'(x) > 0$.

(ii) $f$ is decreasing at P if $f'(x) < 0$.

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|---|---|

**1. Test whether the following functions are increasing and decreasing :**

**(i) $f(x) = x^3 - 6x^2 + 12x - 16,\ x \in R$**

**(ii) $f(x) = x - \frac{1}{x},\ x \in R,\ x \neq 0$**

**(iii) $f(x) = \frac{7}{x} - 3,\ x \in R,\ x \neq 0.$**

**Solution :**

(i) $f(x) = x^3 - 6x^2 + 12x - 16$

$\therefore\ f'(x) = \frac{d}{dx}(x^3 - 6x^2 + 12x - 16)$

$= 3x^2 - 6 \times 2x + 12 \times 1 - 0$

$= 3x^2 - 12x + 12$

$= 3(x^2 - 4x + 4)$

$= 3(x - 2)^2 > 0$ for all $x \in R,\ x \neq 2$

$\therefore\ f'(x) > 0$ for all $x \in R - \{2\}$

$\therefore\ f$ is increasing for all $x \in R - \{2\}$.

(ii) $f(x) = x - \frac{1}{x}$

$\therefore\ f'(x) = \frac{d}{dx}\left(x - \frac{1}{x}\right) = 1 - \left(-\frac{1}{x^2}\right)$

$= 1 + \frac{1}{x^2} > 0$ for all $x \in R,\ x \neq 0$

$\therefore\ f'(x) > 0$ for all $x \in R$, where $x \neq 0$

$\therefore\ f$ is increasing for all $x > R$, where $x \neq 0$.

(iii) $f(x) = \frac{7}{x} - 3$

$\therefore\ f'(x) = \frac{d}{dx}\left(\frac{7}{x} - 3\right) = 7\left(-\frac{1}{x^2}\right) - 0$

$= -\frac{7}{x^2} < 0$ for all $x \in R,\ x \neq 0$

$\therefore\ f'(x) < 0$ for all $x \in$ R, where $x \neq 0$.

$\therefore\ f$ is decreasing for all $x \in R$, where $x \neq 0$.

**2. Find the values of $x$, such that $f(x)$ is increasing function :**

**(i) $f(x) = 2x^3 - 15x^2 + 36x + 1$**

**(ii) $f(x) = x^2 + 2x - 5$**

**(iii) $f(x) = 2x^3 - 15x^2 - 144x - 7.$**

**Solution :**

(i) $f(x) = 2x^3 - 15x^2 + 36x + 1$

$\therefore\ f'(x) = \frac{d}{dx}(2x^3 - 15x^2 + 36x + 1)$

$= 2 \times 3x^2 - 15 \times 2x + 36 \times 1 + 0$

$= 6x^2 - 30x + 36 = 6(x^2 - 5x + 6)$

$f$ is increasing, if $f'(x) > 0$

i.e. if $6(x^2 - 5x + 6) > 0$

i.e. if $x^2 - 5x + 6 > 0$

i.e. if $x^2 - 5x > -6$

i.e. if $x^2 - 5x + \frac{25}{4} > -6 + \frac{25}{4}$

i.e. if $\left(x - \frac{5}{2}\right)^2 > \frac{1}{4}$

i.e. if $x - \frac{5}{2} > \frac{1}{2}$ or $x - \frac{5}{2} < -\frac{1}{2}$

i.e. if $x > 3$ or $x < 2$

i.e. if $x \in (-\infty, 2) \cup (3, \infty)$

$\therefore$ $f$ is increasing, if $x \in (-\infty, 2) \cup (3, \infty)$.

**(ii)** $f(x) = x^2 + 2x - 5$

$\therefore$ $f(x) = \frac{d}{dx}(x^2 + 2x - 5)$

$= 2x + 2 \times 1 - 0 = 2x + 2$

$f$ is increasing, if $f'(x) > 0$

i.e. if $2x + 2 > 0$

i.e. if $2x > -2$

i.e. if $x > -1$, i.e. $x \in (-1, \infty)$

$\therefore$ $f$ is increasing, if $x > -1$, i.e. $x \in (-1, \infty)$.

**(iii)** $f(x) = 2x^3 - 15x^2 - 144x - 7$

$\therefore$ $f'(x) = \frac{d}{dx}(2x^3 - 15x^2 - 144x - 7)$

$= 2 \times 3x^2 - 15 \times 2x - 144 \times 1 - 0$

$= 6x^2 - 30x - 144 = 6(x^2 - 5x - 24)$

$f$ is increasing if, $f'(x) > 0$

i.e. if $6(x^2 - 5x - 24) > 0$

i.e. if $x^2 - 5x - 24 > 0$

i.e. if $x^2 - 5x > 24$

i.e. if $x^2 - 5x + \frac{25}{4} > 24 + \frac{25}{4}$

i.e. if $\left(x - \frac{5}{2}\right)^2 > \frac{121}{4}$

i.e. if $x - \frac{5}{2} > \frac{11}{2}$ or $x - \frac{5}{2} < -\frac{11}{2}$

i.e. if $x > 8$ or $x < -3$

i.e. if $x \in (-\infty, -3) \cup (8, \infty)$

$\therefore$ $f$ is increasing, if $x \in (-\infty, -3) \cup (8, \infty)$.

**3. Find the values of $x$ such that $f(x)$ is decreasing function :**

**(i) $f(x) = 2x^3 - 15x^2 - 144x - 7$**

**(ii) $f(x) = x^4 - 2x^3 + 1$**

**(iii) $f(x) = 2x^3 - 15x^2 - 84x - 7$.**

**Solution :**

**(i)** $f(x) = 2x^3 - 15x^2 - 144x - 7$

$\therefore$ $f'(x) = \frac{d}{dx}(2x^3 - 15x^2 - 144x - 7)$

$= 2 \times 3x^2 - 15 \times 2x - 144 \times 1 - 0$

$= 6x^2 - 30x - 144 = 6(x^2 - 5x - 24)$

$f$ is decreasing, if $f'(x) < 0$

i.e. if $6(x^2 - 5x - 24) < 0$

i.e. if $x^2 - 5x - 24 < 0$

i.e. if $x^2 - 5x < 24$

i.e. if $x^2 - 5x + \frac{25}{4} < \frac{121}{4}$

i.e. if $\left(x - \frac{5}{2}\right)^2 < \frac{121}{4}$

i.e. if $-\frac{11}{2} < x - \frac{5}{2} < \frac{11}{2}$

i.e. if $-\frac{11}{2} + \frac{5}{2} < x - \frac{5}{2} + \frac{5}{2} < \frac{11}{2} + \frac{5}{2}$

i.e. if $-3 < x < 8$

$\therefore$ $f$ is decreasing, if $-3 < x < 8$.

**(ii)** $f(x) = x^4 - 2x^3 + 1$

$\therefore$ $f'(x) = \frac{d}{dx}(x^4 - 2x^3 + 1)$

$= 4x^3 - 2 \times 3x^2 + 0 = 4x^3 - 6x^2$

$f$ is decreasing, if $f'(x) < 0$

i.e. if $4x^3 - 6x^2 < 0$

i.e. if $x^2(4x - 6) < 0$

i.e. if $4x - 6 < 0$ ... $[\because x^2 > 0]$

i.e. if $x < \frac{3}{2}$, i.e. $-\infty < x < \frac{3}{2}$

$\therefore$ $f$ is decreasing, if $-\infty < x < \frac{3}{2}$.

**(iii)** $f(x) = 2x^3 - 15x^2 - 84x - 7$

$\therefore$ $f'(x) = \frac{d}{dx}(2x^3 - 15x^2 - 84x - 7)$

$= 2 \times 3x^2 - 15 \times 2x - 84 \times 1 - 0$

$= 6x^2 - 30x - 84 = 6(x^2 - 5x - 14)$

$f$ is decreasing, if $f'(x) < 0$

i.e. if $6(x^2 - 5x - 14) < 0$

i.e. if $x^2 - 5x - 14 < 0$

i.e. if $x^2 - 5x < 14$

i.e. if $x^2 - 5x + \frac{25}{4} < 14 + \frac{25}{4}$

i.e. if $\left(x-\frac{5}{2}\right)^2<\frac{81}{4}$

i.e. if $-\frac{9}{2}<x-\frac{5}{2}<\frac{9}{2}$

i.e. if $-\frac{9}{2}+\frac{5}{2}<x-\frac{5}{2}+\frac{5}{2}<\frac{9}{2}+\frac{5}{2}$

i.e. if $-2<x<7$

$\therefore$ $f$ is decreasing, if $-2<x<7$.

## ADDITIONAL SOLVED PROBLEMS–4 (B)

**1. Show that $f(x)=3x+\left(\frac{1}{3x}\right)$ is**

**(i) an increasing function in the interval $\left(\frac{1}{3}, 1\right)$ and**

**(ii) a decreasing function in the interval $\left(\frac{1}{9}, \frac{1}{3}\right)$.**

**Solution :** $f(x)=3x+\left(\frac{1}{3x}\right)$

$\therefore f'(x)=3\frac{d}{dx}(x)+\frac{1}{3}\frac{d}{dx}(x^{-1})$

$=3\times 1+\frac{1}{3}(-1)x^{-2}=3-\frac{1}{3x^2}$

Now, $f$ is increasing, if $f'(x)>0$ and is decreasing, if $f'(x)<0$.

**(i)** Let $x\in\left(\frac{1}{3}, 1\right)$. Then $\frac{1}{3}<x<1$

$\therefore \frac{1}{9}<x^2<1$

$\therefore \frac{1}{3}<3x^2<3$

$\therefore 3>\frac{1}{3x^2}>\frac{1}{3}$

$\therefore -3<-\frac{1}{3x^2}<-\frac{1}{3}$

$\therefore 3-3<3-\frac{1}{3x^2}<3-\frac{1}{3}$ $\quad\therefore 0<f'(x)<\frac{8}{3}$

$\therefore f'(x)>0$ for all $x\in\left(\frac{1}{3}, 1\right)$

$\therefore$ $f$ is increasing in the interval $\left(\frac{1}{3}, 1\right)$.

**(ii)** Let $x\in\left(\frac{1}{9}, \frac{1}{3}\right)$. Then $\frac{1}{9}<x<\frac{1}{3}$

$\therefore \frac{1}{81}<x^2<\frac{1}{9}$

$\therefore \frac{1}{27}<3x^2<\frac{1}{3}$ $\quad\therefore 27>\frac{1}{3x^2}>3$

$\therefore -27<-\frac{1}{3x^2}<-3$

$\therefore 3-27<3-\frac{1}{3x^2}<3-3$

$\therefore -24<f'(x)<0$

$\therefore f'(x)<0$ for all $x\in\left(\frac{1}{9}, \frac{1}{3}\right)$

$\therefore$ $f$ is decreasing in the interval $\left(\frac{1}{9}, \frac{1}{3}\right)$.

**2. Show that the function $f(x)=\frac{4x^2+1}{x}$ is a decreasing function in the interval $\left(\frac{1}{4}, \frac{1}{2}\right)$.**

**Solution :** $f(x)=\frac{4x^2+1}{x}=4x+\frac{1}{x}$

$\therefore f'(x)=\frac{d}{dx}\left(4x+\frac{1}{x}\right)=4\times 1+(-1)x^{-2}$

$=4-\frac{1}{x^2}=\frac{4x^2-1}{x^2}$

$=\frac{4\left(x^2-\frac{1}{4}\right)}{x^2}=\frac{4\left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)}{x^2}$

Let $x\in\left(\frac{1}{4}, \frac{1}{2}\right)$. Then $\frac{1}{4}<x<\frac{1}{2}$

$\therefore x+\frac{1}{2}>0$ and $x-\frac{1}{2}<0$

$\therefore \left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)<0$ and $x^2>0$

$\therefore \frac{4\left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)}{x^2}<0$

$\therefore f'(x)<0$ for all $x\in\left(\frac{1}{4}, \frac{1}{2}\right)$

$\therefore$ $f$ is decreasing in the interval $\left(\frac{1}{4}, \frac{1}{2}\right)$.

## EXAMPLES FOR PRACTICE 4.2

**1. Test whether the following functions are increasing or decreasing :**

(i) $f(x)=x^3-3x^2+3x-100$

(ii) $f(x)=2-3x+3x^2-x^3$.

**2. Find the values of $x$ such that $f(x)$ is increasing function :**

(i) $f(x)=x^3-3x^2-9x+25$

(ii) $f(x)=x^3-6x^2+12x+10$
(iii) $f(x)=x^3-9x^2+27x+7$
(iv) $f(x)=x^3-6x^2-15x+1$
(v) $f(x)=2x^3-15x^2+36x+5$
(vi) $f(x)=2x+\frac{1}{2x}$.

**3. Find the values of $x$ such that $f(x)$ is decreasing function :**

(i) $f(x)=3x^2-15x+9$
(ii) $f(x)=x^3-6x^2-15x+12$
(iii) $f(x)=2x^3-9x^2+12x+2$
(iv) $f(x)=8x+\frac{2}{x}$.

**4.** Show that the function $f(x)=x^3+6x^2-36x+7$ is a decreasing function in the interval $(-6, 2)$.

**5.** Show that the function $f(x)=\frac{2x-3}{2x+1}$, for $x \neq -\frac{1}{2}$ is increasing.

**6.** Show that $f(x)=2x+\frac{1}{2x}$ is

(i) an increasing function in the interval $\left(\frac{1}{2}, 1\right)$, and

(ii) a decreasing function in the interval $\left(\frac{1}{4}, \frac{1}{2}\right)$.

**7.** Show that the function $f(x)=x^3-6x^2+15x+7$ is always increasing.

**8.** Show that $f(x)=3x^2-4x-x^3$ is decreasing for all real values of $x$.

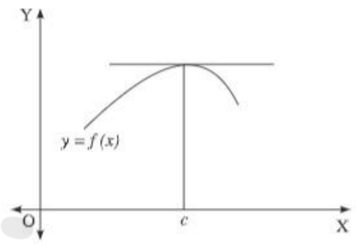
### Answers

**1.** (i) increasing for all $x \in R-\{1\}$
(ii) decreasing for all $x \in R-\{1\}$

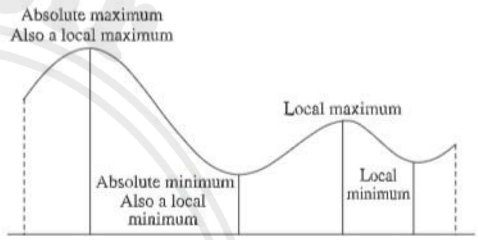
**2.** (i) $(-\infty, -1) \cup (3, \infty)$ (ii) $R-\{2\}$
(iii) $R-\{3\}$ (iv) $(-\infty, -1) \cup (5, \infty)$
(v) $(-\infty, 2) \cup (3, \infty)$
(vi) $\left(-\infty, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$.

**3.** (i) $x<\frac{5}{2}$ (ii) $-1<x<5$
(iii) $1<x<2$ (iv) $-\frac{1}{2}<x<\frac{1}{2}$.

## 4.3 : MAXIMA AND MINIMA



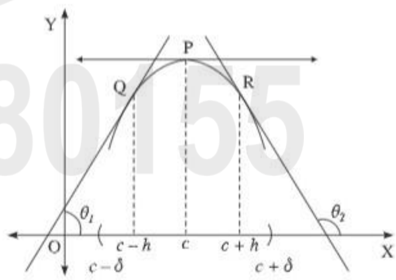

$f$ has a local maximum at $x=c$




Consider, the graph of the function $f$ shown above. At $x=c$ the function $y=f(x)$ has a maximum value. If we move away from $c$ to either side, the curve falls and the function value gets smaller. However, if we sketch more of the curve we might find that the function takes larger values elsewhere also. We would then know that $f(c)$ was not the absolute maximum of the function. It would still be relative or local maximum.

Now, we define local and absolute Maxima and Minima of a function $f$.

**(A) Maxima :** Consider, the graph of the function $f$ shown below. The function $f$ takes the maximum value at $x=c$, i.e. $f(c)$ is the maximum value of $f$. Here we observe that for any $x$, in a small neighbourhood $(c-\delta, c+\delta)$ of $c$, $f(x) \leqslant f(c)$.




We also see that in this neighbourhood, $f$ increases from the left up to $c$ and decreases to the right of $c$. Hence, we have the following definition :

**Definition :** *A function f is said to have a* local maximum *at $x = c$, if in some neighbourhood $(c-\delta, c+\delta)$ of c, the function f increases from the left upto c and decreases to the right of c.*

**Remarks :**

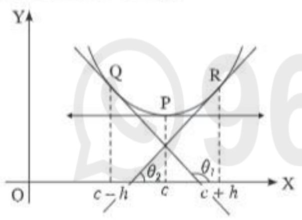
1. The tangent to this graph at $x=c$ is parallel to the X-axis. Hence, the slope of this tangent which is $f'(c)$ is equal to 0.
2. If $c-h$ is a point in this neighbourhood and to the left of $c$, then $f$ is increasing at $x=c-h$. Hence, $f'(c-h)>0$.
3. If $c+h$ is a point in this neighbourhood and to the right of $c$, then $f$ is decreasing at $x=c+h$. Hence, $f'(c+h)<0$.

   From these remarks, we can say that a function $f$ has a *local maximum* at $x=c$, if $f'(c)=0$; $f'(c-h)>0$ and $f'(c+h)<0$ ... (1)

   This is the **first derivative test** for maximum.
4. Also we observe that at every point $x\in(c-\delta, c)$, i.e. to the left of $c$, $f'(x)>0$, $f'(c)=0$ and at every point $x\in(c, c+\delta)$, i.e. to the right of $c$, $f'(x)<0$. Thus the first derivative function $f'$ is decreasing at $x=c$. Therefore, the derivative of $f'$ at $c$, which is $f''(c)$ is negative, i.e. $f''(c)<0$. Hence, we have the following **second derivative test** for maximum : A function $f$ has a *local maximum* at $x=c$, if $f'(c)=0$, and $f''(c)<0$.

**(B) Minima :** Consider, the graph of the function $f$ shown below. The function $f$ takes minimum value at $x=c$, i.e. $f(c)$ is the minimum value of $f$. Here we observe that for any $x$, in a small neighbourhood $(c-\delta, c+\delta)$ of $c$, $f(x)\geqslant f(c)$.

We also see that in this neighbourhood, $f$ decreases from the left up to $c$ and increases to the right of $c$. Hence, we have the following :




**Definition :** *A function f is said to have a* local minimum *at $x=c$, if in some neighbourhood $(c-\delta, c+\delta)$ of c, the function decreases from the left up to c and increases to the right of c.*

**Remarks :**

1. The tangent to this graph at $x=c$ is parallel to the X-axis. Hence the slope of this tangent which is $f'(c)$ is equal to 0.
2. If $c-h$ is a point in this neighbourhood and to the left of $c$, then $f$ is decreasing at $x=c-h$. Hence $f'(c-h)<0$.
3. If $c+h$ is a point in this neighbourhood and to the right of $c$, then $f$ is increasing at $x=c+h$. Hence $f'(c+h)>0$.

   From these remarks, we can say that a function $f$ has a *local minimum* at $x=c$, if $f'(c)=0$; $f'(c-h)<0$ and $f'(c+h)>0$. ... (2)

   This is the **first derivative test** for minimum.
4. Also we observe that at every point $x\in(c-\delta, c)$, i.e. to the left of $c$, $f'(x)<0$, $f'(c)=0$ and at every point $x\in(c, c+\delta)$, i.e. to the right of $c$, $f'(x)>0$. Thus the first derivative function $f'$ is increasing at $x=c$. Therefore, the derivative of $f'$ at $c$, which is $f''(c)$ is positive, i.e. $f''(c)>0$.

Hence, we have the following **second derivative test** for minimum : A function $f$ has a *local minimum* at $x=c$, if $f'(c)=0$ and $f''(c)>0$.

Note that *maxima* is the plural of maximum and *minima* is the plural of minimum. Maximum and minimum values are termed as *extremum values.*

**Procedure for Finding Maxima or Minima :**

1. **Second Derivative Test :**
   (1) Find $f'(x)$ and $f''(x)$.
   (2) Find the roots of the equation $f'(x)=0$.
   (3) If $c$ is a root of this equation, find $f''(c)$.
      (i) If $f''(c)<0$, then $f$ has a maximum at $x=c$.
      (ii) If $f''(c)>0$, then $f$ has a minimum at $x=c$.
   (4) Use the same procedure for the other roots of $f'(x)=0$, as in (3).
2. **First Derivative Test :**
   (1) Find $f'(x)$.
   (2) Find the roots of the equation $f'(x)=0$.
   (3) If $c$ is a root of this equation, find $f'(c-h)$ and $f'(c+h)$, where $h$ is a very small positive real number.

(i) If $f'(c-h)>0$ and $f'(c+h)<0$, then $f$ has a maximum at $x=c$.

(ii) If $f'(c-h)<0$ and $f'(c+h)>0$, then $f$ has a minimum at $x=c$.

(4) Use the same procedure for the other roots of $f'(x)=0$, as in (3).

**Remark :** We observe that out of the above two procedures, the second derivative test is easier to apply than the first derivative test. However, the second derivative test fails, if $f''(c)=0$. In such a case, we have to apply the first derivative test.

If the first derivative test also fails, then $x=c$ is neither point of local maximum nor point of local minimum. It is called as *point of inflexion.*

| EXERCISE 4.3 | Textbook page 109 |
|---|---|

**1. Determine the maximum and minimum values of the following functions :**

**(i)** $f(x)=2x^3-21x^2+36x-20$

**(ii)** $f(x)=x\cdot\log x$

**(iii)** $f(x)=x^2+\frac{16}{x}$.

**Solution :**

**(i)** $f(x)=2x^3-21x^2+36x-20$

$\therefore\ f'(x)=\frac{d}{dx}(2x^3-21x^2+36x-20)$

$=2\times 3x^2-21\times 2x+36\times 1-0$

$=6x^2-42x+36$

and $f''(x)=\frac{d}{dx}(6x^2-42x+36)$

$=6\times 2x-42\times 1+0=12x-42$

$f'(x)=0$ gives $6x^2-42x+36=0$.

$\therefore\ x^2-7x+6=0 \quad \therefore\ (x-1)(x-6)=0$

$\therefore$ the roots of $f'(x)=0$ are $x_1=1$ and $x_2=6$.

For $x=1$, $f''(1)=12(1)-42=-30<0$

$\therefore$ by the second derivative test, $f$ has maximum at $x=1$ and maximum value of $f$ at $x=1$

$=f(1)=2(1)^3-21(1)^2+36(1)-20$

$=2-21+36-20=-3$

For $x=6$, $f''(6)=12(6)-42=30>0$

$\therefore$ by the second derivative test, $f$ has minimum at $x=6$ and minimum value of $f$ at $x=6$

$=f(6)=2(6)^3-21(6)^2+36(6)-20$

$=432-756+216-20=-128$

Hence, the function $f$ has maximum value $-3$ at $x=1$ and minimum value $-128$ at $x=6$.

**(ii)** $f(x)=x\cdot\log x$

$\therefore\ f'(x)=\frac{d}{dx}(x\cdot\log x)$

$=x\cdot\frac{d}{dx}(\log x)+\log x\cdot\frac{d}{dx}(x)$

$=x\times\frac{1}{x}+(\log x)\times 1=1+\log x$

and $f''(x)=\frac{d}{dx}(1+\log x)=0+\frac{1}{x}=\frac{1}{x}$

Now, $f'(x)=0$, if $1+\log x=0$

i.e. if $\log x=-1=-\log e$

i.e. if $\log x=\log(e^{-1})=\log\frac{1}{e}$ i.e. if $x=\frac{1}{e}$

When $x=\frac{1}{e}$, $f''(x)=\frac{1}{(1/e)}=e>0$

$\therefore$ by the second derivative test,

$f$ is minimum at $x=\frac{1}{e}$.

Minimum value of $f$ at $x=\frac{1}{e}$

$=\frac{1}{e}\log\left(\frac{1}{e}\right)=\frac{1}{e}\cdot\log(e^{-1})$

$=\frac{1}{e}\cdot(-1)\log e$

$=-\frac{1}{e}$ ... [$\because\ \log e=1$]

Hence, the function $f$ has minimum at $x=\frac{1}{e}$ and minimum value is $-\frac{1}{e}$.

**(iii)** $f(x)=x^2+\frac{16}{x}$

$\therefore\ f'(x)=\frac{d}{dx}\left(x^2+\frac{16}{x}\right)$

$=2x+16(-1)x^{-2}=2x-\frac{16}{x^2}$

and $f''(x)=\frac{d}{dx}\left(2x-\frac{16}{x^2}\right)$

$=2\times 1-16(-2)x^{-3}=2+\frac{32}{x^3}$

$\therefore\ x^3 = 8 \qquad \therefore\ x = 2$

For $x = 2$, $f''(2) = 2 + \frac{32}{(2)^3} = 6 > 0$

$\therefore$ by the second derivative test, $f$ has minimum at $x = 2$ and minimum value of $f$ at $x = 2$

$= f(2) = (2)^2 + \frac{16}{2} = 4 + 8 = 12$

Hence, the function $f$ has minimum at $x' = 2$ and minimum value is 12.

**2. Divide the number 20 into two parts such that their product is maximum.**

**Solution :** Let the first part of 20 be $x$.

Then the second part is $20 - x$.

$\therefore$ their product $= x(20 - x)$

$= 20x - x^2 = f(x)$ ... (Say)

$\therefore\ f'(x) = \frac{d}{dx}(20x - x^2)$

$= 20 \times 1 - 2x = 20 - 2x$

and $f''(x) = \frac{d}{dx}(20 - 2x)$

$= 0 - 2 \times 1 = -2$

The root of the equation $f'(x) = 0$

i.e. $20 - 2x = 0$ is $x = 10$

and $f''(10) = -2 < 0$

$\therefore$ by the second derivative test, $f$ is maximum at $x = 10$.

Hence, the required parts of 20 are 10 and 10.

**3. A metal wire of 36 cm long is bent to form a rectangle. Find its dimensions where its area is maximum.**

**Solution :** Let $x$ cm and $y$ cm be the length and breadth of the rectangle.

Then its perimeter is $2(x + y) = 36$

$\therefore\ x + y = 18 \qquad \therefore\ y = 18 - x$

Area of the rectangle $= xy = x(18 - x)$

Let $f(x) = x(18 - x) = 18x - x^2$

Then $f'(x) = \frac{d}{dx}(18x - x^2)$

$= 18 \times 1 - 2x = 18 - 2x$

Now, $f'(x) = 0$, if $18 - 2x = 0$

i.e. if $x = 9$

and $f''(9) = -2 < 0$

$\therefore$ by the second derivative test, $f$ has maximum value at $x = 9$

When $x = 9$, $y = 18 - 9 = 9$

Hence, the rectangle is a square of side 9 cm.

**4. The total cost of producing $x$ units is ₹ $(x^2 + 60x + 50)$ and the price is ₹$(180 - x)$ per unit. For what units is the profit maximum?**

**Solution :** Let the number of units sold be $x$.

Then profit = S.P. – C.P.

$\therefore\ P(x) = (180 - x)x - (x^2 + 60x + 50)$

$= 180x - x^2 - x^2 - 60x - 50$

$\therefore\ P(x) = 120x - 2x^2 - 50$

$\therefore\ P'(x) = \frac{d}{dx}(120x - 2x^2 - 50)$

$= 120 \times 1 - 2 \times 2x - 0 = 120 - 4x$

and $P''(x) = \frac{d}{dx}(120 - 4x)$

$= 0 - 4 \times 1 = -4$

$P'(x) = 0$ if $120 - 4x = 0$

i.e. if $x = 30$

and $P''(30) = -4 < 0$

$\therefore$ by the second derivative test, $P(x)$ is maximum when $x = 30$.

Hence, the number of units sold for maximum profit is 30.

## ADDITIONAL SOLVED PROBLEMS – 4 (C)

**1. If sum of two numbers is 6, then find the maximum value of the product of square of first number and the other number.**

**Solution :** Let the two numbers be $x$ and $y$.

Then $x + y = 6 \qquad \therefore\ y = 6 - x$

$\therefore$ the product of square of first number and the other number $= x^2y$

$= x^2(6 - x) = 6x^2 - x^3$

Let $f(x) = 6x^2 - x^3$

Then $f'(x) = \frac{d}{dx}(6x^2 - x^3)$

$= 6 \times 2x - 3x^2 = 12x - 3x^2$

and $f''(x) = \frac{d}{dx}(12x - 3x^2)$

$= 12 \times 1 - 3 \times 2x = 12 - 6x$

$f'(x) = 0$ gives $12x - 3x^2 = 0$

$\therefore\ 3x(4 - x) = 0$

$\therefore\ x = 0$ or $x = 4$

$f''(0) = 12 - 6 \times 0 = 12 > 0$

$\therefore$ $f$ has a minimum value at $x = 0$.

Also, $f''(4) = 12 - 6(4) = -12 < 0$

$\therefore$ $f$ has a maximum value at $x = 4$.

When $x = 4$, $y = 6 - 4 = 2$

$\therefore$ maximum value of $x^2y = (4)^2(2) = 32$.

Hence, the required numbers are 4, 2 and maximum value of the product $= 32$.

**2. Divide 16 into two parts such that sum of square of first part and ten times of the second part is minimum.**

**Solution :** Let one part of 16 be $x$.

Then the other part is $16 - x$.

$\therefore$ the sum of square of first part and ten times of the second part $= x^2 + 10(16 - x) = x^2 - 10x + 160$

Let $f(x) = x^2 - 10x + 160$

Then $f'(x) = \frac{d}{dx}(x^2 - 10x + 160)$

$= 2x - 10 \times 1 + 0 = 2x - 10$

and $f''(x) = \frac{d}{dx}(2x - 10)$

$= 2 \times 1 - 0 = 2$

$f'(x) = 0$ gives $2x - 10 = 0$

$\therefore\ x = 5$

and $f''(5) = 2 > 0$

$\therefore$ by the second derivative test, $f$ is minimum at $x = 5$.

Hence, the parts of 16 are 5 and 11.

**3. If $x$ curtains are ordered, the price $p$ per curtain is given as $p = \left(3x + \frac{1296}{x^2} - \frac{8}{x}\right)$. Find the number of curtains Mrs. Joshi should order for the most economical deal.**

**Solution :** Let Mrs. Joshi order $x$ curtains.

Then the total price for $x$ curtains

$= p \cdot x = \left(3x + \frac{1296}{x^2} - \frac{8}{x}\right)x$

$= 3x^2 + \frac{1296}{x} - 8$

Let $f(x) = 3x^2 + \frac{1296}{x} - 8$

$\therefore\ f'(x) = \frac{d}{dx}\left(3x^2 + \frac{1296}{x} - 8\right)$

$= 3 \times 2x + (1296)(-1)x^{-2} - 0 = 6x - \frac{1296}{x^2}$

and $f''(x) = \frac{d}{dx}\left(6x - \frac{1296}{x^2}\right)$

$= 6 \times 1 - (1296)(-2)x^{-3} = 6 + \frac{2592}{x^3}$

$f'(x) = 0$ gives $6x - \frac{1296}{x^2} = 0$

$\therefore\ 6x^3 - 1296 = 0$

$\therefore\ x^3 = 216 \qquad \therefore\ x = 6$

and $f''(6) = 6 + \frac{2592}{(6)^3} = 6 + 12 = 18 > 0$

$\therefore$ by the second derevative test, $f$ is minimum at $x = 6$.

Hence, Mrs. Joshi should order 6 curtains for the most economical deal.

**4. $x$ tons of output is produced by a firm at total cost $C = ₹\left(\frac{2x^3}{3} - 4x^2 + 43x + 71\right)$. For what level of output is the marginal cost minimum? Also, find marginal cost at that output.**

**Solution :** The cost function is given as

$C = \frac{2x^3}{3} - 4x^2 + 43x + 71$

$\therefore$ marginal cost $= C_M = \frac{dC}{dx}$

$= \frac{d}{dx}\left(\frac{2x^3}{3} - 4x^2 + 43x + 71\right)$

$= \frac{2}{3} \times 3x^2 - 4 \times 2x + 43 \times 1 + 0$

$\therefore\ C_M = 2x^2 - 8x + 43$

$\therefore\ \frac{dC_M}{dx} = \frac{d}{dx}(2x^2 - 8x + 43)$

$= 2 \times 2x - 8 \times 1 + 0 = 4x - 8$

and $\frac{d^2C_M}{dx^2} = \frac{d}{dx}(4x-8)$

$= 4 \times 1 + 0 = 4$

$\frac{dC_M}{dx} = 0$ gives $4x - 8 = 0$

$\therefore\ 4x = 8 \qquad \therefore\ x = 2$

and $\left(\frac{d^2C_M}{dx^2}\right)_{\text{at } x=2} = 4 > 0$

$\therefore$ by the second derivative test, $C_M$ is minimum when $x = 2$.

The marginal cost at $x = 2$ is

$2(2)^2 - 8(2) + 43 = 8 - 16 + 43 = 35$

Hence, the marginal cost is minimum for $x = 2$ and $C_M = 35$.

## EXAMPLES FOR PRACTICE 4.3

1. **Examine the following functions for maxima and minima :**
   (i) $f(x) = x^3 - 5x^2 + 8x - 4$
   (ii) $f(x) = x^3 - 9x^2 + 15x + 3$
   (iii) $f(x) = 4x^3 - 18x^2 + 24x + 35$
   (iv) $f(x) = x^3 - 6x^2 + 9x - 2$
   (v) $f(x) = 2x^3 - 9x^2 + 12x + 5$
   (vi) $f(x) = 3x^3 - 9x^2 - 27x + 15$
   (vii) $f(x) = x^4 - 32x$
   (viii) $f(x) = x + \frac{25}{x}$.
2. Discuss the extreme values of the function
   (i) $f(x) = xe^x$ (ii) $f(x) = x^2e^x$.
3. Show that $f(x) = \frac{\log x}{x}$, $x \neq 0$ is maximum at $x = e$.
4. Divide 70 into two parts, so that the sum of their squares is minimum.
5. Divide 12 into two parts, so that the product of the square of one part and the fourth power of the other is maximum.
   [Hint : $f(x) = x^4(12-x)^2$.]
6. Find two positive numbers $x$ and $y$, such that $x + y = 60$ and $xy^3$ is maximum.
7. A rod 72 metres long is bent to form a rectangle. Find its dimensions, if its area is maximum.
8. A manufacturer can sell $x$ items at a price of ₹ $(280 - x)$ each. The cost of producing $x$ items is ₹ $(x^2 + 40x + 35)$. Find the number of items to be sold so that the manufacturer can make maximum profit.
9. The total cost of producing $x$ items per day is ₹ $\left(\frac{x^2}{4} + 35x + 25\right)$ and price per item at which they may be sold is ₹ $\left(50 - \frac{x}{2}\right)$. Find the daily output to maximize the profit.
10. The revenue $R$ on selling $x$ items is given by $R = 15x^3 - 135x^2 + 360x + 102$. Find $x$ for which the revenue is maximum.
11. For producing $x$ units of a product, revenue function is $R(x) = 40x - x^2$ and cost function is $C(x) = 30 + 4x^2$. Find $x$ so that
    (i) revenue is maximum (ii) cost is minimum.
12. In a firm, the cost function for output $x$ is given as $C = \frac{x^3}{3} - 20x^2 + 70x$. Find the output for which
    (i) marginal cost $(C_m)$ is minimum.
    (ii) average cost $(C_A)$ is minimum.

### Answers

**1.**

| | Max. at | Max. value | Min. at | Min. value |
|---|---|---|---|---|
| (i) | $\frac{4}{3}$ | $\frac{4}{27}$ | 2 | 0 |
| (ii) | 1 | 10 | 5 | −22 |
| (iii) | 1 | 45 | 2 | 43 |
| (iv) | 1 | 2 | 3 | −2 |
| (v) | 1 | 10 | 2 | 9 |
| (vi) | −1 | 30 | 3 | −66 |
| (vii) | – | – | 2 | −48 |
| (viii) | −5 | −10 | 5 | 10 |

**2.** (i) min. at $x = -1$ and minimum value $= -\frac{1}{e}$.
(ii) min. at $x = 0$, minimum value $= 0$
max. at $x = -2$, maximum value $= \frac{4}{e^2}$.

**4.** 35 and 35 **5.** 8 and 4

6. $x = 45$ and $y = 15$
7. square of side 18 metres
8. 60 9. 10 10. $x = 2$
11. (i) $x = 20$ (ii) $x = 0$.
12. (i) $C_m$ is minimum for $x = 20$
(ii) $C_A$ is minimum for $x = 30$.

## 4.4 : APPLICATIONS OF DERIVATIVE IN ECONOMICS

We have studied the following functions in Std. XI.

1. Demand function $D = f(P)$

Marginal Demand (MD) $= D_m = \frac{dD}{dP}$.

2. Supply function $S = g(P)$

Marginal Supply (MS) $= \frac{dS}{dP}$

3. Total cost function $C = f(x)$, where $x$ is the number of articles produced.

Marginal Cost (MC) $= C_m = \frac{dC}{dx}$

Average Cost (AC) $= C_A = \frac{C}{x}$

4. Total revenue $R = P \cdot D$, where $P$ is the price and $D$ is the demand.

Average Revenue $= R_A = \frac{R}{D} = \frac{P \cdot D}{D} = P$

Total profit $= \pi = R - C$.

**Note :** $\pi$ is just a symbol for total profit. It should not be mistaken for the constant value $\frac{22}{7} \approx 3.141$.

**(1) Elasticity of Demand :**

If Demand ($D$) of a commodity is a function of its Price ($P$), then Elasticity of Demand with respect to price is given by

$$\eta = -\frac{P}{D} \cdot \frac{dD}{dP}$$

**Remarks :**

(i) Demand is decreasing function of price

$$\therefore \frac{dD}{dP} < 0$$

Also, price $P$ and the demand $D$ are always positive.

$$\therefore \eta = -\frac{P}{D} \cdot \frac{dD}{dP} > 0$$

(ii) If $\eta = 0$, the demand $D$ is constant function of price $P$.

$$\therefore \frac{dD}{dP} = 0$$

In this case demand is perfectly inelastic.

(iii) If $0 < \eta < 1$, the demand is relatively inelastic.

(iv) If $\eta = 1$, the demand is exactly proportional to the price and demand is said to unitary elastic.

(v) If $\eta > 1$, the demand is relatively elastic.

**Relation between marginal revenue ($R_m$), average revenue ($R_A$) and elasticity of demand ($\eta$) :**

$R_m = \frac{dR}{dD}$, where $R = P \cdot D$

$$\therefore R_m = \frac{d}{dD}(P \cdot D)$$

$$= P + D \cdot \frac{dP}{dD}$$

$$= P\left(1 + \frac{D}{P} \cdot \frac{dP}{dD}\right) \quad \ldots (1)$$

But $\eta = -\frac{P}{D} \cdot \frac{dD}{dP}$

$$\therefore -\frac{1}{\eta} = \frac{D}{P} \cdot \frac{dP}{dD} \quad \ldots (2)$$

From (1) and (2), we get

$$R_m = P\left(1 - \frac{1}{\eta}\right)$$

$$\therefore \boxed{R_m = R_A\left(1 - \frac{1}{\eta}\right)} \quad \ldots [\because R_A = P]$$

**(2) Marginal Propensity :**

**Marginal propensity to consume :**

The consumption expenditure ($E_c$) of any person depends on $x$, where $x$ is his income.

i.e. $E_c = f(x)$

Marginal propensity to consume (MPC) $= \frac{dE_c}{dx}$

Average propensity to consume (APC) $= \frac{E_c}{x}$

**Marginal propensity to save :**

If $S$ is the saving of a person with income $x$, then

Marginal propensity to save (MPS) $= \frac{dS}{dx}$.

Average propensity to save (APS) $= \frac{S}{x}$.

**Note :** Income = Expenditure + Saving

i.e. $x = E_c + S$

$$\therefore\ 1 = \frac{dE_c}{dx} + \frac{dS}{dx}$$

$$\therefore\ \boxed{\text{MPC} + \text{MPS} = 1}$$

Also, $x = E_c + S$

$$\therefore\ 1 = \frac{E_c}{x} + \frac{S}{x}$$

$$\therefore\ \boxed{\text{APC} + \text{APS} = 1}$$

| EXERCISE 4.4 | Textbook pages 112 and 113 |
|---|---|

**1. The demand function of a commodity at price $P$ is given as $D = 40 - \frac{5P}{8}$. Check whether it is increasing or decreasing function.**

**Solution :** $D = 40 - \frac{5P}{8}$

$$\therefore\ \frac{dD}{dP} = \frac{d}{dP}\left(40 - \frac{5P}{8}\right)$$

$$= 0 - \frac{5}{8} \times 1 = \frac{-5}{8} < 0$$

Hence, the given function is decreasing function.

**2. The price $P$ for demand $D$ is given as $P = 183 + 120D - 3D^2$, find $D$ for which price is increasing.**

**Solution :** $P = 183 + 120D - 3D^2$

$$\therefore\ \frac{dP}{dD} = \frac{d}{dD}(183 + 120D - 3D^2)$$

$$= 0 + 120 \times 1 - 3 \times 2D$$

$$= 120 - 6D$$

If price $P$ is increasing, then $\frac{dP}{dD} > 0$

$\therefore\ 120 - 6D > 0$

$\therefore\ 120 > 6D$

$\therefore\ D < 20$

Hence, the price is increasing when $D < 20$.

**3. The total cost function for production of $x$ articles is given as $C = 100 + 600x - 3x^2$. Find the values of $x$ for which total cost is decreasing.**

**Solution :** The cost function is given as

$C = 100 + 600x - 3x^2$

$$\therefore\ \frac{dC}{dx} = \frac{d}{dx}(100 + 600x - 3x^2)$$

$$= 0 + 600 \times 1 - 3 \times 2x$$

$$= 600 - 6x$$

If the total cost is decreasing, then $\frac{dC}{dx} < 0$

$\therefore\ 600 - 6x < 0$

$\therefore\ 600 < 6x$

$\therefore\ x > 100$

Hence, the total cost is decreasing for $x > 100$.

**4. The manufacturing company produces $x$ items at the total cost of ₹ $(180 + 4x)$. The demand function for this product is $P = (240 - x)$. Find $x$ for which**

**(i) revenue is increasing**

**(ii) profit is increasing.**

**Solution :**

**(i)** Let $R$ be the total revenue.

Then $R = P \cdot x = (240 - x)x$

$\therefore\ R = 240x - x^2$

$$\therefore\ \frac{dR}{dx} = \frac{d}{dx}(240x - x^2)$$

$$= 240 \times 1 - 2x = 240 - 2x$$

$R$ is increasing, if $\frac{dR}{dx} > 0$

i.e. if $240 - 2x > 0$

i.e. if $240 > 2x$

i.e. if $x < 120$

Hence, the revenue is increasing, if $x < 120$.

**(ii)** Profit $\pi = R - C$

$\therefore\ \pi = (240x - x^2) - (180 + 4x)$

$= 240x - x^2 - 180 - 4x$

$= 236x - x^2 - 180$

$$\therefore\ \frac{d\pi}{dx} = \frac{d}{dx}(236x - x^2 - 180)$$

$$= 236 \times 1 - 2x - 0 = 236 - 2x$$

Profit is increasing, if $\frac{d\pi}{dx} > 0$

i.e. if $236 - 2x > 0$

i.e. if $236 > 2x$

i.e. if $x < 118$

Hence, the profit is increasing, if $x < 118$.

**5. For manufacturing $x$ units, labour cost is $150 - 54x$ and processing cost is $x^2$. Price of each unit is $p = 10800 - 4x^2$. Find the values of $x$ for which**

**(i) total cost is decreasing**

**(ii) revenue is increasing.**

**Solution :**

**(i)** Total cost $C$ = labour cost + processing cost

$\therefore\ C = 150 - 54x + x^2$

$\therefore\ \frac{dC}{dx} = \frac{d}{dx}(150 - 54x + x^2)$

$= 0 - 54 \times 1 + 2x = -54 + 2x$

The total cost is decreasing, if $\frac{dC}{dx} < 0$

i.e. if $-54 + 2x < 0$

i.e. if $2x < 54$

i.e. if $x < 27$

Hence, the total cost is decreasing, if $x < 27$.

**(ii)** The total revenue $R$ is given as

$R = p \cdot x = (10800 - 4x^2)x$

$\therefore\ R = 10800x - 4x^3$

$\therefore\ \frac{dR}{dx} = \frac{d}{dx}(10800x - 4x^3)$

$= 10800 \times 1 - 4 \times 3x^2 = 10800 - 12x^2$

The revenue is increasing, if $\frac{dR}{dx} > 0$

i.e. if $10800 - 12x^2 > 0$

i.e. if $10800 > 12x^2$

i.e. if $x^2 < 900$

i.e. if $x < 30$ ... $[\because\ x > 0]$

Hence, the revenue is increasing, if $x < 30$.

**6. The total cost of manufacturing $x$ articles is $C = 47x + 300x^2 - x^4$. Find $x$, for which average cost is (i) increasing (ii) decreasing.**

**Solution :** The total cost is given as

$C = 47x + 300x^2 - x^4$

$\therefore$ average cost is given by

$C_A = \frac{C}{x} = \frac{47x + 300x^2 - x^4}{x}$

$\therefore\ C_A = 47 + 300x - x^3$

$\therefore\ \frac{dC_A}{dx} = \frac{d}{dx}(47 + 300x - x^3)$

$= 0 + 300 \times 1 - 3x^2 = 300 - 3x^2$

**(i)** $C_A$ is increasing, if $\frac{dC_A}{dx} > 0$

i.e. if $300 - 3x^2 > 0$

i.e. if $300 > 3x^2$

i.e. if $x^2 < 100$

i.e. if $x < 10$ ... $[\because\ x > 0]$

Hence, the average cost is increasing, if $x < 10$.

**(ii)** $C_A$ is decreasing, if $\frac{dC_A}{dx} < 0$

i.e. if $300 - 3x^2 < 0$

i.e. if $300 < 3x^2$

i.e. if $x^2 > 100$

i.e. if $x > 10$ ... $[\because\ x > 0]$

Hence, the average cost is decreasing, if $x > 10$.

**7. (i) Find the marginal revenue, if the average revenue is 45 and elasticity of demand is 5.**

**Solution :** Given $R_A = 45$ and $\eta = 5$

Now, $R_m = R_A\left(1 - \frac{1}{\eta}\right)$

$= 45\left(1 - \frac{1}{5}\right) = 45\left(\frac{4}{5}\right) = 36$

Hence, the marginal revenue = 36.

**(ii) Find the price, if the marginal revenue is 28 and elasticity of demand is 3.**

**Solution :** Given $R_m = 28$ and $\eta = 3$

Now, $R_m = R_A\left(1 - \frac{1}{\eta}\right)$

$\therefore\ 28 = R_A\left(1 - \frac{1}{3}\right) = \frac{2}{3}R_A$

$\therefore\ R_A = \frac{28 \times 3}{2} = 42$

Hence, the price = 42.

**(iii) Find the elasticity of demand, if the marginal revenue is 50 and price is ₹ 75.**

**Solution :** Given $R_m = 50$ and $R_A = 75$

Now, $R_m = R_A\left(1 - \frac{1}{\eta}\right)$

$\therefore\ 50 = 75\left(1 - \frac{1}{\eta}\right)$ $\quad\therefore\ 1 - \frac{1}{\eta} = \frac{50}{75} = \frac{2}{3}$

$\therefore\ \frac{1}{\eta} = 1 - \frac{2}{3} = \frac{1}{3}$ $\quad\therefore\ \eta = 3$

Hence, the elasticity of demand = 3.

**8. If the demand function is $D=\frac{p+6}{p-3}$, find the elasticity of demand at $p=4$.**

**Solution :** The demand function is

$$D=\frac{p+6}{p-3}$$

$$\therefore \frac{dD}{dp}=\frac{d}{dp}\left(\frac{p+6}{p-3}\right)$$

$$=\frac{(p-3)\frac{d}{dp}(p+6)-(p+6)\frac{d}{dp}(p-3)}{(p-3)^2}$$

$$=\frac{(p-3)(1+0)-(p+6)(1-0)}{(p-3)^2}$$

$$=\frac{p-3-p-6}{(p-3)^2}=\frac{-9}{(p-3)^2}$$

Elasticity of demand is given by

$$\eta=\frac{-p}{D}\cdot\frac{dD}{dp}$$

$$=\frac{-p}{\left(\frac{p+6}{p-3}\right)}\times\frac{-9}{(p-3)^2}$$

$$=\frac{9p}{(p+6)(p-3)}$$

When $p=4$, then

$$\eta=\frac{9(4)}{(4+6)(4-3)}=\frac{36}{10\times 1}=3.6.$$

Hence, the elasticity of demand at $p=4$ is 3.6.

**9. Find the price for the demand function $D=\frac{2p+3}{3p-1}$, when elasticity of demand is $\frac{11}{14}$.**

**Solution :** The demand function is

$$D=\frac{2p+3}{3p-1}$$

$$\therefore \frac{dD}{dp}=\frac{d}{dp}\left(\frac{2p+3}{3p-1}\right)$$

$$=\frac{(3p-1)\frac{d}{dp}(2p+3)-(2p+3)\frac{d}{dp}(3p-1)}{(3p-1)^2}$$

$$=\frac{(3p-1)(2\times 1+0)-(2p+3)(3\times 1-0)}{(3p-1)^2}$$

$$=\frac{6p-2-6p-9}{(3p-1)^2}=\frac{-11}{(3p-1)^2}$$

Elasticity of demand is given by

$$\eta=-\frac{p}{D}\cdot\frac{dD}{dp}$$

$$=\frac{-p}{\left(\frac{2p+3}{3p-1}\right)}\times\frac{-11}{(3p-1)^2}$$

$$=\frac{11p}{(2p+3)(3p-1)}=\frac{11p}{6p^2+7p-3}$$

If $\eta=\frac{11}{14}$, then

$$\frac{11}{14}=\frac{11p}{6p^2+7p-3}$$

$\therefore\ 66p^2+77p-33=154p$

$\therefore\ 66p^2-77p-33=0$

$\therefore\ 6p^2-7p-3=0$

$\therefore\ (2p-3)(3p+1)=0$

$\therefore\ 2p-3=0$ ... $[\because p\geqslant 0]$

$\therefore\ p=\frac{3}{2}.$

**10. If the demand function is $D=50-3p-p^2$. Find the elasticity of demand at (i) $p=5$ (ii) $p=2$. Comment on the result.**

**Solution :** The demand function is $D=50-3p-p^2$

$$\therefore \frac{dD}{dp}=\frac{d}{dp}(50-3p-p^2)$$

$$=0-3\times 1-2p=-3-2p$$

Elasticity of demand is given by

$$\eta=-\frac{p}{D}\cdot\frac{dD}{dp}$$

$$=\frac{-p}{(50-3p-p^2)}\times(-3-2p)$$

$$=\frac{p(3+2p)}{50-3p-p^2}$$

(i) When $p=5$, then

$$\eta=\frac{5(3+2\times 5)}{50-3(5)-(5)^2}=\frac{5\times 13}{50-15-25}$$

$$=\frac{65}{10}=6.5.$$

Since, $\eta>1$, the demand is elastic.

**(ii)** When $p = 2$, then

$$\eta = \frac{2(3+2\times 2)}{50-3(2)-(2)^2} = \frac{2\times 7}{50-6-4}$$

$$= \frac{14}{40} = \frac{7}{20}.$$

Since, $0 < \eta < 1$, the demand is inelastic.

**11. For the demand function $D = 100 - \frac{p^2}{2}$, find the elasticity of demand at (i) $p = 10$ (ii) $p = 6$ and comment on the results.**

**Solution :** The demand function is

$$D = 100 - \frac{p^2}{2}$$

$$\therefore \frac{dD}{dp} = \frac{d}{dp}\left(100 - \frac{p^2}{2}\right)$$

$$= 0 - \frac{1}{2} \times 2p = -p$$

Elasticity of demand is given by

$$\eta = \frac{-p}{D} \cdot \frac{dD}{dp}$$

$$= \frac{-p}{\left(100 - \frac{p^2}{2}\right)} \times (-p) = \frac{p^2}{\left(100 - \frac{p^2}{2}\right)}$$

**(i)** When $p = 10$, then

$$\eta = \frac{(10)^2}{100 - \frac{(10)^2}{2}} = \frac{100}{100-50}$$

$$= \frac{100}{50} = 2$$

Since, $\eta > 1$, the demand is elastic.

**(ii)** When $p = 6$, then

$$\eta = \frac{(6)^2}{100 - \frac{(6)^2}{2}} = \frac{36}{100-18}$$

$$= \frac{36}{82} = \frac{18}{41}$$

Since, $0 < \eta < 1$, the demand is inelastic.

**12. A manufacturing company produces $x$ items at a total cost of ₹ $(40 + 2x)$. Their price is given as $p = 120 - x$. Find the value of $x$ for which**

**(i) revenue is increasing**

**(ii) profit is increasing.**

**(iii) Also find elasticity of demand for price 80.**

**Solution :**

**(i)** The total revenue $R$ is given by

$$R = p \cdot x = (120 - x)x$$

$$\therefore R = 120x - x^2$$

$$\therefore \frac{dR}{dx} = \frac{d}{dx}(120x - x^2)$$

$$= 120 \times 1 - 2x = 120 - 2x$$

If the revenue is increasing, then $\frac{dR}{dx} > 0$

$\therefore 120 - 2x > 0$

$\therefore 120 > 2x \quad \therefore x < 60$

Hence, the revenue is increasing when $x < 60$.

**(ii)** Profit $\pi = R - C$

$$= (120x - x^2) - (40 + 2x)$$

$$= 120x - x^2 - 40 - 2x$$

$$\therefore \pi = 118x - x^2 - 40$$

$$\therefore \frac{d\pi}{dx} = \frac{d}{dx}(118x - x^2 - 40)$$

$$= 118 \times 1 - 2x - 0 = 118 - 2x$$

If the profit is increasing, then $\frac{d\pi}{dx} > 0$

$\therefore 118 - 2x > 0$

$\therefore 118 > 2x \quad \therefore x < 59$

Hence, the profit is increasing when $x < 59$.

**(iii)** $p = 120 - x \quad \therefore x = 120 - p$

$$\therefore \frac{dx}{dp} = \frac{d}{dp}(120 - p)$$

$$= 0 - 1 = -1$$

Elasticity of demand is given by

$$\eta = \frac{-p}{x} \cdot \frac{dx}{dp}$$

$$= \frac{-p}{120-p} \times (-1) = \frac{p}{120-p}$$

When $p = 80$, then

$$\eta = \frac{80}{120-80} = \frac{80}{40} = 2.$$

**13. Find MPC, MPS, APC and APS, if the expenditure $E_c$ of a person with income $I$ is given as $E_c = (0.0003)I^2 + (0.075)I$, when $I = 1000$.**

**Solution :** $E_c = (0.0003)I^2 + (0.075)I$

$$MPC = \frac{dE_c}{dI} = \frac{d}{dI}[(0.0003)I^2 + (0.075)I]$$

$$= (0.0003)(2I) + (0.075)(1)$$

$$= (0.0006)I + 0.075$$

When $I = 1000$, then

$$MPC = (0.0006)(1000) + 0.075$$

$$= 0.6 + 0.075 = 0.675.$$

$\because$ MPC + MPS = 1

$\therefore$ 0.675 + MPS = 1

$\therefore$ MPS = 1 − 0.675 = 0.325

Now, $APC = \frac{E_c}{I} = \frac{(0.0003)I^2 + (0.075)I}{I}$

$$= (0.0003)I + (0.075)$$

When $I = 1000$, then

$$APC = (0.0003)(1000) + 0.075$$

$$= 0.3 + 0.075 = 0.375$$

$\because$ APC + APS = 1

$\therefore$ 0.375 + APS = 1

$\therefore$ APS = 1 − 0.375 = 0.625

Hence, MPC = 0.675, MPS = 0.325

APC = 0.375, APS = 0.625.

## EXAMPLES FOR PRACTICE 4.4

1. The total cost function $C$ is given by $C = x^2 - 14x + 8$, where $x$ is the output. Find the output for which the total cost is increasing.
2. Total cost ($C$) to manufacture $x$ items is given by $C = 150x - 2x^3 + 100$. Find $x$ for which the total cost is decreasing.
3. The daily production cost '$C$' to manufacture '$x$' items of a commodity is given by $C = 200 + 12x + \frac{x^2}{2}$. Find $x$ for which the average cost is decreasing.

   [**Hint :** Average cost = AC = $\frac{C}{x}$.]
4. The cost $C$ of producing $x$ articles is given as $C = x^3 - 16x^2 + 47x$. For what values of $x$ will the average cost be decreasing?
5. Find the value of $x$ at which the cost function given by $C(x) = 2x^3 + 3x^2 - 12x + 7$ is neither increasing nor decreasing.

   [**Hint :** $C$ is neither increasing nor decreasing, if $\frac{dC}{dx} = 0$.]
6. The daily production cost to manufacture '$x$' items is given by $C = 50 + 14x + \frac{x^2}{2}$. Find $x$ for which the average cost decreases.
7. The total revenue $R = 720x - 3x^2$, where $x$ is the number of items sold. Find $x$ for which total revenue $R$ is increasing.
8. The manufacturing cost of '$x$' items is $C = 110 + 2x$. If $p$ is the price and $x$ is the demand, $p$ is given by $p = 200 - x$. For what values of $x$, the profit will increase?

   [**Hint :** Profit $(P) = R - C$, where $R = p \cdot x$].
9. If for a commodity, demand function is given by $D = 2000 - 10p - p^2$, determine elasticity of demand when price $(p)$ is 10 units.
10. The demand function for a commodity is $D = \frac{2}{3 + 5p}$. Find elasticity of demand when price is 10 units.
11. At what price, price elasticity of demand for a commodity would be $(1/2)$, if the demand function is given by $D = \frac{10}{p + 4}$?
12. The demand function is $D = \sqrt{100 - p^2}$. Find the elasticity of demand when the price is 5.
13. The demand function is $D = \frac{2p + 5}{p - 2}$. Find the elasticity of demand when price is 8.
14. Demand function $x$, for a certain commodity is given as $x = 200 - 4p$, where $p$ is unit price. Find
    (i) elasticity of demand as a function of $p$.
    (ii) elasticity of demand when $p = 10$. Interpret your result.
    (iii) the price $p$ for which elasticity of demand is equal to one.
15. Find the elasticity of demand if the marginal revenue is 40 and the price is 50.
16. Find the marginal revenue if the average revenue is 30 and the elasticity of demand is 2.

17. Find MPC, MPS, APC and APS if the expenditure $E_c$ of a person with income $I$ is given as $E_c = (0.0001)I^2 + (0.016)I$, when $I = 4000$.

18. The expenditure $E_c$ of a person with income $I$ is given by $E_c = (0.000035)I^2 + (0.045)I$. Find marginal propensity to consume (MPC) and average propensity to consume (APC), when $I = 5000$.

19. The consumption expenditure $E_c$ of a person with the income $x$, is given by $E_c = 0.0006x^2 + 0.003x$. Find MPC, MPS, APC and APS when the income $x = 200$.

20. Find MPC, MPS, APC and APS, if the expenditure $E_c$ of a person with income $I$ is given as : $E_c = (0.00002)I^2 + (0.008)I$, when $I = 8000$.

### Answers

**1.** $x > 7$ **2.** $x > 5$ **3.** $0 < x < 20$ **4.** $x < 8$

**5.** $x = 1$ **6.** $0 < x < 10$ **7.** $x < 120$ **8.** $x < 99$ **9.** $\eta = \frac{1}{6}$

**10.** $\eta = 0.94$ **11.** $p = 4$ **12.** $\eta = \frac{1}{3}$ **13.** $\eta = \frac{4}{7}$

**14.** (i) $\eta = \frac{p}{50 - p}$, $p < 50$
(ii) 0.25, inelastic
(iii) 25

**15.** $\eta = 5$ **16.** $R_m = 15$

**17.** MPC = 0.816, MPS = 0.184, APC = 0.416, APS = 0.584

**18.** MPC = 0.395, APC = 0.22

**19.** MPC = 0.243, MPS = 0.757, APC = 0.123, APS = 0.877

**20.** MPC = 0.328, MPS = 0.672, APC = 0.168, APS = 0.832.

## MISCELLANEOUS EXERCISE – 4

**(Textbook pages 113 and 114)**

**(I) Choose the correct alternative :**

1. The equation of tangent to the curve $y = x^2 + 4x + 1$ at $(-1, -2)$ is
(a) $2x - y = 0$ (b) $2x + y - 5 = 0$
(c) $2x - y - 1 = 0$ (d) $x + y - 1 = 0$

2. The equation of tangent to the curve $x^2 + y^2 = 5$, where the tangent is parallel to the line $2x - y + 1 = 0$ are
(a) $2x - y + 5 = 0$; $2x - y - 5 = 0$
(b) $2x + y + 5 = 0$; $2x + y - 5 = 0$
(c) $x - 2y + 5 = 0$; $x - 2y - 5 = 0$
(d) $x + 2y + 5$; $x + 2y - 5 = 0$

3. If elasticity of demand $\eta = 1$, then demand is
(a) constant (b) inelastic
(c) unitary elastic (d) elastic

4. If $0 < \eta < 1$, then the demand is
(a) constant (b) inelastic
(c) unitary elastic (d) elastic

5. The function $f(x) = x^3 - 3x^2 + 3x - 100$, $x \in R$ is
(a) increasing for all $x \in R$, $x \neq 1$
(b) decreasing
(c) neither increasing nor decreasing
(d) decreasing for all $x \in R$, $x \neq 1$

6. If $f(x) = 3x^3 - 9x^2 - 27x + 15$, then
(a) $f$ has maximum value 66
(b) $f$ has minimum value 30
(c) $f$ has maxima at $x = -1$
(d) $f$ has minima at $x = -1$

### Answers

**1.** (a) $2x - y = 0$
**2.** (a) $2x - y + 5 = 0$; $2x - y - 5 = 0$
**3.** (c) unitary elastic
**4.** (b) inelastic
**5.** (a) increasing for all $x \in R$, $x \neq 1$
**6.** (c) $f$ has maxima at $x = -1$.

**(II) Fill in the blanks :**

1. The slope of tangent at any point $(a, b)$ is called as .........

2. If $f(x) = x^3 - 3x^2 + 3x - 100$, $x \in R$, then $f''(x)$ is .........

3. If $f(x) = \frac{7}{x} - 3$, $x \in R$, $x \neq 0$, then $f''(x)$ is .........

4. A rod of 108 m length is bent to form a rectngle. If area at the rectangle is maximum, then its dimensions are .........

5. If $f(x) = x \cdot \log x$, then its maximum value is .........

### Answers

**1.** gradient **2.** $6x - 6 = 6(x - 1)$ **3.** $14x^{-3}$

**4.** 27 and 27 **5.** $-\frac{1}{e}$.

**(III) State whether each of the following is *True* or *False* :**

1. The equation of tangent to the curve $y = 4xe^x$ at $\left(-1, \frac{-4}{e}\right)$ is $y \cdot e + 4 = 0$.

2. $x + 10y + 21 = 0$ is the equation of normal to the curve $y = 3x^2 + 4x - 5$ at (1, 2).

3. An absolute maximum must occur at a critical point or at an end point.

4. The function $f(x) = x \cdot e^{x(1-x)}$ is increasing on $\left(\frac{-1}{2}, 1\right)$.

### Answers

1. True 2. False 3. True
4. True.

[**Hint :** $f(x) = xe^{x(1-x)} = xe^{x-x^2}$

$$\therefore f'(x) = x \cdot \frac{d}{dx}(e^{x-x^2}) + e^{x-x^2} \cdot \frac{d}{dx}(x)$$
$$= x \cdot e^{x-x^2} \cdot \frac{d}{dx}(x - x^2) + e^{x-x^2} \times 1$$
$$= x \cdot e^{x-x^2} \times (1 - 2x) + e^{x-x^2}$$
$$= e^{x-x^2}(x - 2x^2 + 1)$$
$$= -2e^{x-x^2}\left(x^2 - \frac{1}{2}x - \frac{1}{2}\right)$$
$$= -2e^{x-x^2}\left[\left(x^2 - \frac{1}{2}x + \frac{1}{16}\right) - \frac{1}{2} - \frac{1}{16}\right]$$
$$\therefore f'(x) = -2e^{x-x^2}\left[\left(x - \frac{1}{4}\right)^2 - \frac{9}{16}\right] \quad \dots (1)$$

Now, $x \in \left(-\frac{1}{2}, 1\right)$ $\quad \therefore -\frac{1}{2} < x < 1$

$$\therefore -\frac{1}{2} - \frac{1}{4} < x - \frac{1}{4} < 1 - \frac{1}{4} = \frac{3}{4}$$
$$\therefore 0 < \left(x - \frac{1}{4}\right)^2 < \frac{9}{16}$$
$$\therefore \left(x - \frac{1}{4}\right)^2 - \frac{9}{16} < 0 \quad \dots (2)$$

Also, $2e^{x-x^2} > 0$ for $x \in \left(-\frac{1}{2}, 1\right)$

$$\therefore -2e^{x-x^2} < 0 \quad \dots (3)$$

From (2) and (3),

$$-2e^{x-x^2}\left[\left(x - \frac{1}{4}\right)^2 - \frac{9}{16}\right] > 0$$
$$\therefore f'(x) > 0 \quad \dots \text{[By (1)]}$$

Hence, function $f(x)$ is increasing on $\left(-\frac{1}{2}, 1\right)$]

**(IV) Solve the following :**

**1. Find the equations of tangent and normal to the following curves :**

**(i)** $xy = c^2$ **at** $\left(ct, \frac{c}{t}\right)$, **where** $t$ **is a parameter.**

**(ii)** $y = x^2 + 4x$ **at the point whose ordinate is** $-3$.

**(iii)** $x = \frac{1}{t}$, $y = t - \frac{1}{t}$, **at** $t = 2$.

**(iv)** $y = x^3 - x^2 - 1$ **at the point whose abscissa is** $-2$.

**Solution :**

(i) $xy = c^2$

Differentiating both sides w.r.t. $x$, we get

$$x\frac{dy}{dx} + y \cdot \frac{d}{dx}(x) = 0$$
$$\therefore x\frac{dy}{dx} + y \times 1 = 0$$
$$\therefore x\frac{dy}{dx} = -y \qquad \therefore \frac{dy}{dx} = -\frac{y}{x}$$
$$\therefore \left(\frac{dy}{dx}\right)_{\text{at }\left(ct, \frac{c}{t}\right)} = -\frac{\left(\frac{c}{t}\right)}{ct} = -\frac{1}{t^2}$$
$$= \text{slope of the tangent at } \left(ct, \frac{c}{t}\right)$$

$\therefore$ the equation of the tangent at $\left(ct, \frac{c}{t}\right)$ is

$$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$$
$$\therefore t^2y - ct = -x + ct$$
$$\therefore x + t^2y - 2ct = 0$$

The slope of the normal at $\left(ct, \frac{c}{t}\right)$

$$= \frac{-1}{\left(\frac{dy}{dx}\right)_{\text{at }\left(ct, \frac{c}{t}\right)}} = \frac{-1}{\left(-\frac{1}{t^2}\right)} = t^2$$

$\therefore$ the equation of the normat at $\left(ct, \frac{c}{t}\right)$ is

$$y - \frac{c}{t} = t^2(x - ct)$$
$$\therefore ty - c = t^3x - ct^4$$
$$\therefore t^3x - ty - c(t^4 - 1) = 0$$

Hence, equations of tangent and normal are $x + t^2y - 2ct = 0$ and $t^3x - ty - c(t^4 + 1) = 0$ respectively.

**(ii)** Let $P(x_1, y_1)$ be the point on the curve $y = x^2 + 4x$, where $y_1 = -3$

$\therefore\ y_1 = x_1^2 + 4x_1$

$\therefore\ x_1^2 + 4x_1 = -3$

$\therefore\ x_1^2 + 4x_1 + 3 = 0$

$\therefore\ (x_1 + 3)(x_1 + 1) = 0$

$\therefore\ x_1 = -3$ or $x_1 = -1$

$\therefore$ coordinates of the points are $(-3, -3)$ and $(-1, -3)$.

Differentiating $y = x^2 + 4x$ w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 + 4x) = 2x + 4$$

$$\therefore\ \left(\frac{dy}{dx}\right)_{\text{at }(-3,\ -3)} = 2(-3) + 4 = -2$$

$= $ slope of the tangent at $(-3, -3)$

$\therefore$ equation of the tangent at $(-3, -3)$ is

$y - (-3) = -2[x - (-3)]$

$\therefore\ y + 3 = -2x - 6$

$\therefore\ 2x + y + 9 = 0$

The slope of the normal at $(-3, -3)$

$$= \frac{-1}{\left(\frac{dy}{dx}\right)_{\text{at }(-3,\ -3)}} = \frac{-1}{-2} = \frac{1}{2}$$

$\therefore$ the equation of the normal at $(-3, -3)$ is

$$y - (-3) = \frac{1}{2}[x - (-3)]$$

$\therefore\ 2y + 6 = x + 3$

$\therefore\ x - 2y - 3 = 0$

$$\left(\frac{dy}{dx}\right)_{\text{at }(-1,\ -3)} = 2(-1) + 4 = 2$$

$= $ slope of the tangent at $(-1, -3)$

$\therefore$ the equation of the tangent at $(-1, -3)$ is

$y - (-3) = 2[x - (-1)]$

$\therefore\ y + 3 = 2x + 2$

$\therefore\ 2x - y - 1 = 0$

The slope of the normal at $(-1, -3)$

$$= \frac{-1}{\left(\frac{dy}{dx}\right)_{\text{at }(-1,\ -3)}} = -\frac{1}{2}$$

The equation of the normal at $(-1, -3)$ is

$$y - (-3) = -\frac{1}{2}[x - (-1)]$$

$\therefore\ 2y + 6 = -x - 1$

$\therefore\ x + 2y + 7 = 0$

Hence, the equations of tangent and normal at

(i) $(-3, -3)$ are $2x + y + 9 = 0$ and $x - 2y - 3 = 0$

(ii) $(-1, -3)$ are $2x - y - 1 = 0$ and $x + 2y + 7 = 0$

**(iii)** When $t = 2$, $x = \frac{1}{2}$ and $y = 2 - \frac{1}{2} = \frac{3}{2}$

Hence, the point P at which we want to find the equations of tangent and normal is $\left(\frac{1}{2}, \frac{3}{2}\right)$.

Now, $x = \frac{1}{t}$, $y = t - \frac{1}{t}$

Differentiating $x$ and $y$ w.r.t. $t$, we get

$$\frac{dx}{dt} = \frac{d}{dt}\left(\frac{1}{t}\right) = -\frac{1}{t^2}$$

and $$\frac{dy}{dt} = \frac{d}{dt}\left(t - \frac{1}{t}\right) = 1 - \left(-\frac{1}{t^2}\right) = 1 + \frac{1}{t^2} = \frac{t^2 + 1}{t^2}$$

$$\therefore\ \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{\left(\frac{t^2+1}{t^2}\right)}{\left(-\frac{1}{t^2}\right)} = -(t^2 + 1)$$

$$\therefore\ \left(\frac{dy}{dx}\right)_{\text{at } t=2} = -(4 + 1) = -5$$

$= $ slope of the tangent at $t = 2$

$\therefore$ the equation of the tangent at $\left(\frac{1}{2}, \frac{3}{2}\right)$ is

$$y - \frac{3}{2} = -5\left(x - \frac{1}{2}\right)$$

$$\therefore\ y - \frac{3}{2} = -5x + \frac{5}{2}$$

$\therefore\ 5x + y - 4 = 0$

The slope of the normal at $t = 2$

$$= \frac{-1}{\left(\frac{dy}{dx}\right)_{\text{at } t=2}} = \frac{-1}{-5} = \frac{1}{5}$$

$\therefore$ the equation of the normal at $\left(\frac{1}{2}, \frac{3}{2}\right)$ is

$$y-\frac{3}{2}=\frac{1}{5}\left(x-\frac{1}{2}\right)$$

$$\therefore\ 5y-\frac{15}{2}=x-\frac{1}{2}$$

$$\therefore\ x-5y+7=0$$

Hence, the equations of tangent and normal are $5x+y-4=0$ and $x-5y+7=0$ respectively.

**(iv)** $y=x^3-x^2-1$

$$\therefore\ \frac{dy}{dx}=\frac{d}{dx}(x^3-x^2-1)$$

$$=3x^2-2x-0=3x^2-2x$$

$$\therefore\ \left(\frac{dy}{dx}\right)_{\text{at } x=-2}=3(-2)^2-2(-2)=16$$

= slope of the tangent at $x=-2$

When $x=-2$, $y=(-2)^3-(-2)^2-1=-13$

$\therefore$ the point P is $(-2, -13)$

$\therefore$ the equation of the tangent at $(-2, -13)$ is

$$y-(-13)=16[x-(-2)]$$

$$\therefore\ y+13=16x+32$$

$$\therefore\ 16x-y+19=0$$

The slope of the normal at $x=-2$

$$=\frac{-1}{\left(\frac{dy}{dx}\right)_{\text{at } x=-2}}=\frac{-1}{16}$$

$\therefore$ the equation of the normal at $(-2, -13)$ is

$$y-(-13)=-\frac{1}{16}[x-(-2)]$$

$$\therefore\ 16y+208=-x-2$$

$$\therefore\ x+16y+210=0$$

Hence, equations of tangent and normal are $16x-y+19=0$ and $x+16y+210=0$ respectively.

**2. Find the equation of the normal to the curve $y=\sqrt{x-3}$ which is perpendicular to the line $6x+3y-4=0$.**

**Solution :** Let $P(x_1, y_1)$ be the foot of the required normal to the curve $y=\sqrt{x-3}$.

Differentiating $y=\sqrt{x-3}$ w.r.t. $x$, we get

$$\frac{dy}{dx}=\frac{d}{dx}(\sqrt{x-3})=\frac{1}{2\sqrt{x-3}}\cdot\frac{d}{dx}(x-3)$$

$$=\frac{1}{2\sqrt{x-3}}\times(1-0)=\frac{1}{2\sqrt{x-3}}$$

$$\therefore\ \left(\frac{dy}{dx}\right)_{\text{at } (x_1, y_1)}=\frac{1}{2\sqrt{x_1-3}}$$

= slope of the tangent at $P(x_1, y_1)$

$\therefore$ slope of the normal at $P(x_1, y_1)$

$$=m_1=\frac{-1}{\left(\frac{dy}{dx}\right)_{\text{at } (x_1, y_1)}}=-2\sqrt{x_1-3}$$

The slope of the line $6x+3y-4=0$ is

$$m_2=\frac{-6}{3}=-2$$

Since, the normal at $P(x_1, y_1)$ is perpendicular to the line $6x+3y-4=0$, $m_1\cdot m_2=-1$, i.e. $m_1=\frac{-1}{m_2}$

$$\therefore\ -2\sqrt{x_1-3}=\frac{-1}{-2}=\frac{1}{2}$$

$$\therefore\ x_1-3=\frac{1}{16}\qquad \therefore\ x_1=\frac{49}{16}$$

Since, $(x_1, y_1)$ lies on the curve $y=\sqrt{x-3}$

$$\therefore\ y_1=\sqrt{x_1-3}$$

$$\therefore\ y_1=\sqrt{\frac{49}{16}-3}=\pm\frac{1}{4}$$

$\therefore$ the coordinates of the point P are $\left(\frac{49}{16}, \frac{1}{4}\right)$ or $\left(\frac{49}{16}, -\frac{1}{4}\right)$

and the slopes of the normal is $m_1=-\frac{1}{m_2}=\frac{1}{2}$.

$\therefore$ the equation of the normal at $\left(\frac{49}{16}, \frac{1}{4}\right)$ is

$$y-\frac{1}{4}=\frac{1}{2}\left(x-\frac{49}{16}\right)$$

$$\therefore\ 2y-\frac{1}{2}=x-\frac{49}{16}$$

$$\therefore\ x-2y-\frac{41}{16}=0$$

$$\therefore\ 16x-32y-41=0$$

and the equation of the normal at $\left(\frac{49}{16}, -\frac{1}{4}\right)$ is

$$y-\left(-\frac{1}{4}\right)=\frac{1}{2}\left(x-\frac{49}{16}\right)$$

$$\therefore\ 2y+\frac{1}{2}=x-\frac{49}{16}$$

$\therefore\ x - 2y - \frac{57}{16} = 0$, i.e. $16x - 32y - 57 = 0$

Hence, the equation of the normals are $16x - 32y - 41 = 0$ and $16x - 32y - 57 = 0$.

[**Note :** *Answer in the textbook is incorrect.*]

**3. Show that the function $f(x) = \frac{x-2}{x+1}$, $x \neq -1$ is increasing.**

**Solution :** $f(x) = \frac{x-2}{x+1}$

$$\therefore\ f'(x) = \frac{d}{dx}\left(\frac{x-2}{x+1}\right) = \frac{(x+1)\cdot\frac{d}{dx}(x-2) - (x-2)\cdot\frac{d}{dx}(x+1)}{(x+1)^2}$$

$$= \frac{(x+1)\cdot(1-0) - (x-2)\cdot(1+0)}{(x+1)^2}$$

$$= \frac{x+1-x+2}{(x+1)^2} = \frac{3}{(x+1)^2}$$

$\because\ x \neq -1 \qquad \therefore\ x+1 \neq 0$

$\therefore\ (x+1)^2 > 0 \qquad \therefore\ \frac{3}{(x+1)^2} > 0$

$\therefore\ f'(x) > 0$, for all $x \in R$, $x \neq -1$

Hence, the function $f$ is increasing for all $x \in R$, where $x \neq -1$.

**4. Show that the function $f(x) = \frac{3}{x} + 10$, $x \neq 0$ is decreasing.**

**Solution :** $f(x) = \frac{3}{x} + 10$

$$\therefore\ f'(x) = \frac{d}{dx}\left(\frac{3}{x} + 10\right) = 3\left(-\frac{1}{x^2}\right) + 0$$

$$= -\frac{3}{x^2}$$

$\because\ x \neq 0 \qquad \therefore\ x^2 > 0 \qquad \therefore\ \frac{3}{x^2} > 0$

$\therefore\ -\frac{3}{x^2} < 0$

$\therefore\ f'(x) < 0$ for all $x \in R$, $x \neq 0$

Hence, the function $f$ is decreasing for all $x \in R$, where $x \neq 0$.

**5. If $x + y = 3$, show that the maximum value of $x^2y$ is 4.**

**Solution :** $x + y = 3 \qquad \therefore\ y = 3 - x$

$\therefore\ x^2y = x^2(3-x) = 3x^2 - x^3$

Let $f(x) = 3x^2 - x^3$

Then $f'(x) = \frac{d}{dx}(3x^2 - x^3)$

$= 3 \times 2x - 3x^2 = 6x - 3x^2$

and $f''(x) = \frac{d}{dx}(6x - 3x^2)$

$= 6 \times 1 - 3 \times 2x = 6 - 6x$

Now, $f'(x) = 0$ gives $6x - 3x^2 = 0$

$\therefore\ 3x(2-x) = 0$

$\therefore\ x = 0$ or $x = 2$

$f''(0) = 6 - 0 = 6 > 0$

$\therefore\ f$ has minimum value at $x = 0$

Also, $f''(2) = 6 - 12 = -6 < 0$

$\therefore\ f$ has maximum value at $x = 2$

When $x = 2$, $y = 3 - 2 = 1$

$\therefore$ maximum value of $x^2y = (2)^2(1) = 4$.

**6. Examine the function $f$ for maxima and minima, where $f(x) = x^3 - 9x^2 + 24x$.**

**Solution :** $f(x) = x^3 - 9x^2 + 24x$

$\therefore\ f'(x) = \frac{d}{dx}(x^3 - 9x^2 + 24x)$

$= 3x^2 - 9 \times 2x + 24 \times 1 = 3x^2 - 18x + 24$

and $f''(x) = \frac{d}{dx}(3x^2 - 18x + 24)$

$= 3 \times 2x - 18 \times 1 + 0 = 6x - 18$

$f'(x) = 0$ gives $3x^2 - 18x + 24 = 0$

$\therefore\ x^2 - 6x + 8 = 0 \qquad \therefore\ (x-2)(x-4) = 0$

$\therefore$ the roots of $f'(x) = 0$ are $x_1 = 2$ and $x_2 = 4$.

(a) $f''(2) = 6(2) - 18 = -6 < 0$

$\therefore$ by the second derivative test, $f$ has maximum at $x = 2$ and maximum value of $f$ at $x = 2$

$= f(2) = (2)^3 - 9(2)^2 + 24(2)$

$= 8 - 36 + 48 = 20$

(b) $f''(4) = 6(4) - 18 = 6 > 0$

$\therefore$ by the second derivative test, $f$ has minimum at $x = 4$ and minimum value of $f$ at $x = 4$

$= f(4) = (4)^3 - 9(4)^2 + 24(4)$

$= 64 - 144 + 96 = 16.$

**ACTIVITIES** | **Textbook pages 114 and 115**

**1. Find the equation of tangent to the curve $\sqrt{x}-\sqrt{y}=1$ at P(9, 4).**

**Solution :** Given equation of curve is

$\sqrt{x}-\sqrt{y}=1$

Differentiating w.r.t. $x$, we get

$\therefore \frac{1}{2\sqrt{x}}-\frac{1}{2\boxed{\sqrt{y}}}\frac{dy}{dx}=0$

$\therefore \frac{1}{2\sqrt{y}}\frac{dy}{dx}=\frac{1}{2}\boxed{\frac{1}{\sqrt{x}}}$

$\therefore \frac{1}{\sqrt{y}}\frac{dy}{dx}=\frac{1}{\sqrt{x}}$

$\therefore \frac{dy}{dx}=\frac{\sqrt{y}}{\sqrt{x}}$

$\therefore \left(\frac{dy}{dx}\right)_{P(9,\,4)}=\frac{\sqrt{9}}{\boxed{\sqrt{4}}}=\frac{3}{2}$

$\therefore$ slope of tangent is $\frac{3}{2}$

$\therefore$ equation of the tangent at P(9, 4) is

$y-4=\boxed{\frac{3}{2}}(x-9)$

$\therefore 2(y-4)=3(x-9)$

$\therefore 2y-\boxed{8}=\boxed{3x}-27$

$\therefore 3x-2y+8+\boxed{-27}=0$

$\therefore 3x-2y-19=0.$

**2. A rod of 108 metres long is bent to form rectangle. Find its dimensions if the area of rectangle is maximum.**

**Solution :** Let $x$ be the length and $y$ be breadth of the rectangle.

$\therefore 2x+2y=108$

$\therefore x+y=\boxed{54}$

$\therefore y=54-\boxed{x}$

Now, area of the rectangle $=xy$

$=x\boxed{54-x}$

$\therefore f(x)=54x-\boxed{x^2}$

$\therefore f'(x)=\boxed{54}-2x$

$\therefore f''(x)=\boxed{-2}$

For extream values, $f'(x)=0$

$\therefore 54-2x=0$

$\therefore -2x=\boxed{-54}$

$\therefore x=\frac{-54}{-2}$

$\therefore x=\boxed{27}$

$\therefore f''(27)=-2<0$

$\therefore$ area is maximum when $x=27$, $y=27$

$\therefore$ the dimensions of rectangle are 27 m × 27 m.

**3. Find the value of $x$ for which the function $f(x)=2x^3-9x^2+12x+2$ is decreasing.**

**Solution :** Given $f(x)=2x^3-9x^2+12x+2$

$\therefore f'(x)=\boxed{6}x^2-\boxed{18x}+\boxed{12}$

$\therefore f'(x)=6(x-1)(\boxed{x-2})$

Now, $f'(x)<0$

$\therefore 6(x-1)(x-2)<0$

Since, $ab<0 \Leftrightarrow a<0$ and $b>0$ or $a>0$ and $b<0$

**Case I :** $(x-1)<0$ and $x-2>0$

$\therefore x<\boxed{1}$ and $x>\boxed{2}$

which is contradiction.

**Case II :** $x-1>0$ and $x-2<0$

$\therefore x>\boxed{1}$ and $x<\boxed{2}$

$1<\boxed{x}<2$

$f(x)$ is decreasing if and only if $x\in(1, 2)$.

## ACTIVITIES FOR PRACTICE

**1. Find the equation of the tangent to the curve $3x^2+2y^2=5$ at (1, 1).**

**Solution :** $3x^2+2y^2=5$

Differentiating both sides w.r.t. $x$, we get

$6x+\boxed{\phantom{0}}\frac{dy}{dx}=0$

$\therefore \boxed{\phantom{0}}\frac{dy}{dx}=-6x$

$\therefore \dfrac{dy}{dx} = \dfrac{-3x}{\square}$

$\therefore \left(\dfrac{dy}{dx}\right)_{\text{at }(1,\,1)} = \dfrac{-3(1)}{\square} = \dfrac{-3}{\square}$

$= \text{slope of the tangent at } (1, 1)$

$\therefore$ the equation of the tangent at (1, 1) is

$y - 1 = \square(x - 1)$

$\therefore \square(y-1) = \square(x-1)$

$\therefore \square y - \square = \square x - \square$

$\therefore \square x + \square y - 5 = 0.$

**2. Find the equation of the normal to the curve $xy = a$ at the point $(\sqrt{a}, \sqrt{a})$.**

**Solution :** $xy = a$

Differentiating both sides w.r.t. $x$, we get

$x\dfrac{dy}{dx} + y \cdot \dfrac{d}{dx}\square = 0$

$\therefore x\dfrac{dy}{dx} + y\square = 0$

$\therefore x\dfrac{dy}{dx} = \square$

$\therefore \dfrac{dy}{dx} = \dfrac{\square}{x}$

$\therefore \left(\dfrac{dy}{dx}\right)_{\text{at }(\sqrt{a},\,\sqrt{a})} = \dfrac{\square}{\sqrt{a}} = \square$

$\therefore$ slope of normal at $(\sqrt{a}, \sqrt{a})$

$= \dfrac{-1}{\left(\dfrac{dy}{dx}\right)_{\text{at }(\sqrt{a},\,\sqrt{a})}} = \dfrac{-1}{\square} = \square$

$\therefore$ the equation of the normal at $(\sqrt{a}, \sqrt{a})$ is

$y - \sqrt{a} = \square(x - \sqrt{a})$

$\therefore x - y = 0.$

**3. The rectangle has area of 50 $cm^2$. Find its dimensions for least perimeter.**

**Solution :** Let $x$ cm and $y$ cm be the length and breadth of a rectangle.

Then its area is $xy = 50 \therefore y = \dfrac{50}{\square}$

Perimeter of the rectangle $= 2(x + y)$

$= 2\left(x + \dfrac{50}{\square}\right)$

Let $f(x) = 2\left(x + \dfrac{50}{\square}\right)$

Then $f'(x) = 2\left(1 - \dfrac{50}{\square}\right)$

and $f''(x) = 2\left(0 - \square\right) = \dfrac{200}{\square}$

Now, $f'(x) = 0$, if $1 - \dfrac{50}{\square} = 0$

i.e. if $x^2 = \square$

i.e. if $x = \pm\square$

But $x$ is not negative.

$\therefore x = \square$ and $f''\square = \dfrac{200}{\square} > 0$

$\therefore$ by the second derivative test, $f$ is minimum at $x = \square$

When $x = \square$, $y = \dfrac{50}{\square} = \square$

$\therefore x = \square$ cm, $y = \square$ cm

Hence, the rectangle is a square of side $\square$ cm.

**4. Find the values of $x$ for which $f(x) = x + \dfrac{1}{x}$ is decreasing.**

**Solution :** $f(x) = x + \dfrac{1}{x}$

$\therefore f'(x) = \dfrac{d}{dx}\left(x + \dfrac{1}{x}\right)$

$= 1 - \dfrac{1}{\square}$

$f$ is decreasing, if $f'(x) < 0$

i.e. if $1 - \dfrac{1}{\square} < 0$

i.e. if $1 < \dfrac{1}{\square}$

i.e. if $\square < 1$

i.e. if $-1 < \square < 1$

$\therefore f$ is decreasing if $x \in \square$.

**5. If Mr. Rane orders $x$ chairs at the price $p = (2x^2 - 12x - 192)$ per chair. How many chairs should he order so that the cost of deal is minimum?**

**Solution :** Let Mr. Rane order $x$ chairs.

Then the total price of $x$ chairs

$= p \cdot x = (2x^2 - 12x - 192)x$

$= 2x^3 - 12x^2 - 192x$

Let $f(x) = 2x^3 - 12x^2 - 192x$

$\therefore\ f'(x) = \frac{d}{dx}(2x^3 - 12x^2 - 192x)$

$= 2\square - 12\square - 192 \times 1$

$= \square x^2 - \square x - 192$

and $f''(x) = \square \times 2x - \square \times 1 - 0$

$= \square x - \square$

$f'(x) = 0$ gives, $\square x^2 - \square x - 192 = 0$

$\therefore\ x^2 - \square x - \square = 0$

$\therefore\ (x - \square)(x + \square) = 0$

$\therefore\ x = \square$ ... $[\because\ x > 0]$

and $f''(8) = \square(8) - \square = \square > 0$

$\therefore\ f$ is minimum when $x = 8$

Hence, Mr Rane should order 8 chairs for minimum cost of deal.

## OBJECTIVE SECTION

### MULTIPLE CHOICE QUESTIONS

**Select and write the correct answer from the given alternatives in each of the following questions :**

1. If normal to the curve $y = f(x)$ is parallel to X-axis, then

(a) $\frac{dy}{dx} = 0$ (b) $\frac{dy}{dx} = 1$

(c) $\frac{dx}{dy} = 0$ (d) $\frac{dx}{dy} = 1$

2. The equation of tangent to the curve $y = 3x^2 - x + 1$ at (1, 3) is

(a) $5x - y = 2$ (b) $x + 5y = 16$

(c) $5x - y + 2 = 0$ (d) $5x = y$

3. Equation of tangent to the curve $2x^2 + 3y^2 - 5 = 0$ at (1, 1) is

(a) $2x - 3y - 5 = 0$ (b) $2x + 3y - 5 = 0$

(c) $2x + 3y + 5 = 0$ (d) $3x + 2y + 5 = 0$

4. The slope of tangent to the curve $x = \frac{1}{t}$, $y = t - \frac{1}{t}$ at $t = 2$ is

(a) 5 (b) $-5$ (c) $-\frac{1}{5}$ (d) $\frac{1}{5}$

5. The tangents to the curve $y = 2x^3 - 2$ at the points $x = \pm 2$ are

(a) parallel

(b) mutually perpendicular

(c) increasing but not perpendicular

(d) same

6. If the tangent to the curve $x^2 = px + y$ at the origin is perpendicular to $6x + 3y - 11 = 0$, then the value of $p$ is

(a) $\frac{1}{2}$ (b) 2 (c) $-2$ (d) $-\frac{1}{2}$

7. The equation of normal to the curve $y^2(2a - x) = x^3$ at the point $(a,\ -a)$ is

(a) $x - 2y = 3a$ (b) $x + 2y + 3a = 0$

(c) $x - 2y + 3a = 0$ (d) $x + 2y = 3a$

8. If the tangent to the curve $y = x(x - 1)$ at the point $(a, b)$ is parallel to the Y-axis, then

(a) $a = \frac{1}{2}, b = \frac{1}{4}$ (b) $a = \frac{1}{2}, b = -\frac{1}{4}$

(c) $a = -\frac{1}{2}, b = \frac{1}{4}$ (d) $a = -\frac{1}{2}, b = -\frac{1}{4}$

9. The value of $x$ for which $f(x) = 2x^3 - 15x^2 + 36x + 5$ is decreasing are

(a) $x < 2$ or $x > 3$ (b) $2 < x < 3$

(c) $x < 2$ and $x > 3$ (d) 2 and 3

10. $f(x) = x - \sqrt{x}$ is increasing in

(a) $\left(-\infty, \frac{1}{4}\right)$ (b) $\left(\frac{1}{4}, \infty\right)$

(c) $\left(0, \frac{1}{4}\right)$ (d) $R^+$

11. The values of $x$ for which $f(x) = 2x + \frac{1}{2x}$ is increasing are

(a) $x < -\frac{1}{2}$ or $x > \frac{1}{2}$ (b) $x < \frac{1}{2}$ and $x > 1$

(c) $x < \frac{1}{2}$ or $x > 1$ (d) $\frac{1}{2}$ and 1

12. The function $f(x)=x^3-3x^2+3x-100,\ x\in R$ is
(a) increasing
(b) decreasing
(c) increasing and decreasing
(d) neither increasing nor decreasing

13. $f(x)=8x^3-60x^2+144x+27$ is decreasing function in
(a) $[-3, -2]$ (b) $[2, 3]$
(c) $[1, 5]$ (d) $[-3, 2]$

14. At $x=-1$, the function $f(x)=2x^3-3x^2-12x+12$ is
(a) maximum
(b) minimum
(c) neither maximum nor minimum
(d) increasing

15. $f(x)=\dfrac{\log x}{x}$ is increasing in
(a) $(1, 2e)$ (b) $(0, e)$
(c) $(2, 2e)$ (d) $\left(\dfrac{1}{e}, 2e\right)$

16. Function $f(x)=x^2-3x+4$ has minimum value at $x=$ .........
(a) 0 (b) $-\dfrac{3}{2}$ (c) 1 (d) $\dfrac{3}{2}$

17. If the sum of two numbers is 15 and the sum of their squares is minimum, the numbers are
(a) $\dfrac{15}{2}, \dfrac{15}{2}$ (b) 10, 5 (c) 8, 7 (d) 11, 4

18. Two numbers $x$ and $y$ such that $x+y=2$ and $x^3\cdot y$ is maximum are
(a) $\dfrac{1}{3}, \dfrac{5}{3}$ (b) $\dfrac{3}{2}, \dfrac{1}{2}$ (c) 1, 1 (d) $\dfrac{6}{5}, \dfrac{4}{5}$

19. The function $y=a\log x+bx^2+x$ has its extreme values at $x=-1$ and $x=2$, then
(a) $a=2, b=\dfrac{1}{2}$ (b) $a=-2, b=\dfrac{1}{2}$
(c) $a=2, b=-1$ (d) $a=2, b=-\dfrac{1}{2}$

20. A rod 72 metres long is bent to form a rectangle. Its dimensions, when its area is maximum, are
(a) 18 m, 18 m (b) 20 m, 16 m
(c) 24 m, 12 m (d) 30 m, 6 m

### Answers

1. (c) $\dfrac{dx}{dy}=0$
2. (a) $5x-y=2$
3. (b) $2x+3y-5=0$
4. (b) $-5$
5. (a) parallel
6. (d) $-\dfrac{1}{2}$
7. (a) $x-2y=3a$
8. (b) $a=\dfrac{1}{2}, b=-\dfrac{1}{4}$
9. (b) $2<x<3$
10. (b) $\left(\dfrac{1}{4}, \infty\right)$
11. (a) $x<-\dfrac{1}{2}$ or $x>\dfrac{1}{2}$
12. (a) increasing
13. (b) $[2, 3]$
14. (a) maximum
15. (b) $(0, e)$
16. (d) $\dfrac{3}{2}$
17. (a) $\dfrac{15}{2}, \dfrac{15}{2}$
18. (b) $\dfrac{3}{2}, \dfrac{1}{2}$
19. (d) $a=2, b=-\dfrac{1}{2}$
20. (a) 18 m, 18 m.

## TRUE OR FALSE

**State whether the following statements are *True* or *False* :**

1. The gradient of the curve $y=x^3-2x^2+4$ at the point $x=2$ is $-4$.
2. The function $f(x)$ will have maximum at $x=c$ if $f(x)$ is increasing for $x<c$ and $f(x)$ is decreasing for $x>c$.
3. A function $f$ has minimum at $x=a$, if $f'(a)=0$ and $f''(a)<0$.
4. $f(x)=x-\sqrt{x}$ is increasing in $\left(0, \dfrac{1}{4}\right)$.
5. If the demand function is $D=f(P)$, then marginal demand $=\dfrac{dD}{dP}$.
6. The function $f(x)=x^3+5x^2-1$ is decreasing in the interval $\left(-\infty, -\dfrac{10}{3}\right)$.
7. The function $f(x)=x^3-6x^2+9x-2$ has maximum at $x=-1$.
8. At $x=-1$, the function $f(x)=2x^3-3x^2-12x+12$ is maximum.
9. If $f(x)=x^2+ax+b$ has a minimum at $x=3$ and minimum value is 5, then $a=-6, b=-14$.
10. Elasticity of demand $\eta=\dfrac{P}{d}\cdot\dfrac{dD}{dP}$.

**Answers**

| | | | | |
|---|---|---|---|---|
| **1.** False | **2.** True | **3.** False | **4.** False | **5.** True |
| **6.** False | **7.** False | **8.** True | **9.** False | **10.** False |

## FILL IN THE BLANKS

**Fill in the following blanks with an appropriate words/numbers :**

1. The slope of the normal to the curve $y = f(x)$ at $P(a, f(a))$ is .........
2. A function $f$ has maximum at $x = a$ if $f'(a) = 0$ and $f''(a)$ ..........
3. If the slope of the tangent to the curve $y = \frac{x}{b-x}$ at the point (1, 1) is 2, then the value of $b$ is .........
4. The function $f(x) = x^3 + 6x^2 - 36x + 7$ is decreasing if .........
5. The function $f(x) = x^2 - 2x$ is increasing if .........
6. The function $f(x) = 2x^3 - 9x^2 + 12x + 5$ has minimum at $x =$ .........
7. At $x = 3$, then function $f(x) = 2x^3 - 15x^2 + 36x + 10$ is .........
8. If the total cost function $C = f(x)$, where $x$ is the number of items produced, then marginal cost = .........
9. If the total cost function $C = f(x)$, where $x$ is the number of items produced, then average cost = .........
10. If $0 < \eta < 1$, then demand is .........

**Answers**

| | | | |
|---|---|---|---|
| **1.** $\frac{-1}{f'(a)}, f'(a) \neq 0$ | **2.** $< 0$ | **3.** 2 or $\frac{1}{2}$ | **4.** $-6 < x < 2$ |
| **5.** $x > 1$ | **6.** 2 | **7.** minimum | |
| **8.** $\frac{dC}{dx}$ | **9.** $\frac{C}{x}$ | **10.** relatively inelastic. | |

❄❄❄

# 5 INTEGRATION

## CHAPTER OUTLINE



## IMPORTANT FORMULAE

1. $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
2. $\int kf(x)\,dx = k\int f(x)\,dx$, where $k$ is a constant
3. If $\int f(x)\,dx = g(x) + c$, then
   $\int f(ax+b)\,dx = \frac{1}{a}\cdot g(ax+b) + c,\ a \neq 0$
4. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c,\ (n \neq -1)$
5. $\int 1\,dx = x + c$
6. $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c,\ (n \neq -1)$
7. $\int a^x\,dx = \frac{a^x}{\log a} + c,\ (a \neq 1)$
8. $\int e^x\,dx = e^x + c$
9. $\int \frac{1}{x}\,dx = \log|x| + c$
10. $\int \frac{1}{ax+b}\,dx = \frac{\log|ax+b|}{a} + c$
11. $\int a^{px+q}\,dx = \frac{a^{px+q}}{p\cdot\log a} + c$
12. $\int e^{ax+b}\,dx = \frac{e^{ax+b}}{a} + c$
13. $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$
14. $\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$
15. $\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \log\left|x+\sqrt{a^2+x^2}\right| + c$
16. $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \log\left|x+\sqrt{x^2-a^2}\right| + c$
17. $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + c$
18. $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + c$

19. $\int \frac{f'(x)}{f(x)}\, dx = \log |f(x)| + c$

20. $\int [f(x)]^n \cdot f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} + c, \ (n \neq -1)$

21. $\int \frac{f'(x)}{\sqrt{f(x)}}\, dx = 2\sqrt{f(x)} + c$

22. $\int uv dx = u \int v dx - \int \left(\frac{du}{dx} \int v dx\right) dx$

The order in which $u$ and $v$ are to be chosen, is according to the serial order of the letters of the word *LAE*.

*L* : Logarithmic,

*A* : Algebraic and

*E* : Exponential functions.

23. $\int e^x [f(x) + f'(x)] dx = e^x \cdot f(x) + c$

24. **Integration by Partial Fractions :**

Integral of the type $\int \frac{P(x)}{Q(x)}\, dx$, where

(i) $P(x)$ and $Q(x)$ are polynomials in $x$

(ii) degree $P(x) <$ degree $Q(x)$

(iii) no common polynomial factors in $P(x)$ and $Q(x)$.

**Case (i) :** If the denominator $Q(x)$ consists of distinct linear factors

i.e. $\frac{P(x)}{Q(x)} = \frac{P(x)}{(a_1x + b_1)(a_2x + b_2) \ \ldots \ (a_nx + b_n)}$,

then we need to find the constants $A_1, A_2, A_3, \ldots, A_n$ such that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \ldots + \frac{A_n}{a_nx + b_n}$$

**Case (ii) :** If the denominator has repeated linear factor, i.e. $Q(x) = (x - a)^k (x - a_1)(x - a_2) \ \ldots \ (x - a_r)$, then we assume

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x - a} + \frac{A_2}{(x - a)^2} + \ldots + \frac{A_k}{(x - a)^k} + \frac{B_1}{x - a_1} + \frac{B_2}{x - a_2} + \ldots + \frac{B_r}{x - a_r}$$

where $A_1, A_2, \ldots, A_k, B_1, B_2, \ldots, B_r$ are constants.

**Case (iii) :** If the denominator $Q(x)$ has non-repeated quadratic factors then corresponding to each quadratic factor $ax^2 + bx + c$, we assume the partial fraction $\frac{Ax + B}{ax^2 + bx + c}$, where $A$ and $B$ are constants.

**Remark :** If degree $P(x) >$ degree $Q(x)$, then divide $P(x)$ by $Q(x)$ till the degree of remainder $f(x)$ is less than $Q(x)$.

$\therefore \ \frac{P(x)}{Q(x)} = r(x) + \frac{f(x)}{Q(x)}$, where $r(x)$ is the quotient.

## INTRODUCTION

In Chapter 3, we have studied about the derivative of a function and different methods to obtain it. We shall now study antidifferentiation, i.e. finding antiderivatives of a function.

Consider the following examples :

(i) $\frac{d}{dx}(x^2) = 2x$;

(ii) $\frac{d}{dx}(\log x) = \frac{1}{x}$.

Here we say that

(i) antiderivative of $2x$ is $x^2$;

(ii) antiderivative of $\frac{1}{x}$ is $\log x, \ x > 0$.

The process of finding an antiderivative is called **integration.**

## 5.1 : DEFINITION OF INTEGRAL OR PRIMITIVE OR ANTIDERIVATIVE OF A FUNCTION

**Definition :** If $f$ and $g$ are two functions such that $\frac{d}{dx}[g(x)] = f(x)$, then the function $g$ is called an antiderivative or a primitive or an indefinite integral of the function $f$ and we write this as :

$\int f(x)\, dx = g(x)$

and read it as : 'integral of $f(x)$ w.r.t. $x$ is $g(x)$.'

Observe that,

$\frac{d}{dx}(x^2) = 2x \qquad \therefore \ \int 2x\, dx = x^2$

$\frac{d}{dx}(x^2 + 4) = 2x \qquad \therefore \ \int 2x\, dx = x^2 + 4$

$\frac{d}{dx}(x^2 - 7) = 2x \qquad \therefore \ \int 2x\, dx = x^2 - 7$

This shows that antiderivative of $2x$ is not unique. Each of these integrals differs from any of the rest by a constant. Hence, we can write $\int 2x\,dx = x^2 + c$ where $c$ is a constant, which can be any real number.

In general, if $\frac{d}{dx}[g(x)] = f(x)$, then

$\int f(x)dx = g(x) + c$, where $c$ is an arbitrary constant. It is called the *constant of integration*.

**Note :** In $\int f(x)\,dx$, $f(x)$ is an integrand and $x$ is a variable of integration.

### Integrals of some standard functions :

Since, integration is the reverse process of differentiation, an integral of a function can be obtained by inverting the formula of the derivatives. From the formulae of the derivatives, we can obtain directly the corresponding formulae for integrals as shown below :

**1.** $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n,\ n \neq -1$

$\therefore \int x^n\,dx = \frac{x^{n+1}}{n+1} + c,\ n \neq -1$

**2.** $\frac{d}{dx}(\log x) = \frac{1}{x},\ x > 0$

$\therefore \int \frac{1}{x}\,dx = \log x + c,\ x > 0$

**3.** $\frac{d}{dx}(e^x) = e^x \quad \therefore \int e^x\,dx = e^x + c$

**4.** $\frac{d}{dx}(a^x) = a^x \cdot \log a,\ a > 0,\ a \neq 1$

$\therefore \int a^x\,dx = \frac{a^x}{\log a} + c,\ a > 0,\ a \neq 1$

### Rules of Integration :

If $f(x)$ and $g(x)$ are integrable functions of $x$, then

1. $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$
2. $\int [f(x) - g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx$
3. $\int k f(x)\,dx = k\int f(x)\,dx$, where $k$ is a constant.

**Remark :** If $f_1, f_2, \ldots, f_n$ are real integrable function of $x$ and $k_1, k_2, \ldots, k_n$ are constants, then

$\int [k_1 f_1(x) + k_2 f_2(x) + \ldots + k_n f_n(x)]\,dx$

$= k_1 \int f_1(x)\,dx + k_2 \int f_2(x)\,dx + \ldots + k_n \int f_n(x)\,dx.$

**Result :** If $\int f(x)\,dx = \phi(x) + c$, then

$\int f(ax + b)\,dx = \frac{1}{a}\phi(ax + b) + c,\ a \neq 0$

This result tells us that if $x$ is replaced by $ax + b$ in integrals of standard function, then in standard formulae we replace $x$ by $ax + b$ and divide the obtained answer by $a$.

**For example :**

(1) $\int (ax + b)^n dx = \frac{(ax + b)^{n+1}}{(n+1)a} + c$, if $n \neq -1$

(2) $\int \frac{1}{ax + b}\,dx = \frac{\log(ax + b)}{a} + c$, if $(ax + b) > 0$.

(3) $\int e^{ax + b}\,dx = \frac{e^{ax + b}}{a} + c$

(4) $\int a^{kx + b}\,dx = \frac{a^{kx + b}}{k \cdot \log a} + c$

**Note :** In the integrals of standard functions, we have seen that $\frac{1}{x}$ can be integrated only if $x > 0$, because $\log x$ is not defined when $x \leqslant 0$.

Now, we find integral of $\frac{1}{x}$ when $x < 0$.

If $x$ is negative then $-x$ is positive.

$\therefore \int \frac{1}{x}\,dx = \int \frac{-1}{(-x)}\,dx = -\int \frac{1}{-x}\,dx$

$= \frac{-\log(-x)}{-1} + c = \log(-x) + c$

$\therefore \int \frac{1}{x}\,dx = \log x + c$, if $x > 0$

$= \log(-x) + c$, if $x < 0$

We can combine both these results into a single result, namely,

$\int \frac{1}{x}\,dx = \log|x| + c$

Henceforth, we shall use the above formula to integrate $\frac{1}{x}$, even if $x$ takes negative value.

$\therefore \int \frac{1}{ax + b}\,dx = \frac{\log|ax + b|}{a} + c.$

## EXERCISE 5.1 Textbook page 119

**1. Evaluate :** $\int \frac{-2}{\sqrt{5x - 4} - \sqrt{5x - 2}}\,dx.$

**Solution :** $\int \frac{-2}{\sqrt{5x - 4} - \sqrt{5x - 2}}\,dx$

$= \int \frac{-2}{\sqrt{5x - 4} - \sqrt{5x - 2}} \times \frac{\sqrt{5x - 4} + \sqrt{5x - 2}}{\sqrt{5x - 4} + \sqrt{5x - 2}}\,dx$

$$= \int \frac{-2(\sqrt{5x-4}+\sqrt{5x-2})}{(5x-4)-(5x-2)}\,dx$$

$$= \int (\sqrt{5x-4}+\sqrt{5x-2})\,dx$$

$$= \int (\sqrt{5x-4})^{\frac{1}{2}}dx + \int (\sqrt{5x-2})^{\frac{1}{2}}\,dx$$

$$= \frac{(5x-4)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} \times \frac{1}{5} + \frac{(5x-2)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} \times \frac{1}{5} + c$$

$$= \frac{2}{15}\left[(5x-4)^{\frac{3}{2}} + (5x-2)^{\frac{3}{2}}\right] + c.$$

**2. Evaluate : $\int\left(1+x+\frac{x^2}{2!}\right)dx$.**

**Solution :** $\int\left(1+x+\frac{x^2}{2!}\right)dx$

$$= \int 1dx + \int x dx + \frac{1}{2!}\int x^2 dx$$

$$= x + \frac{x^2}{2} + \frac{1}{2!} \times \frac{x^3}{3} + c$$

$$= x + \frac{x^2}{2} + \frac{x^3}{6} + c.$$

**3. Evaluate : $\int \frac{3x^3 - 2\sqrt{x}}{x}\,dx$.**

**Solution :** $\int \frac{3x^3 - 2\sqrt{x}}{x}\,dx$

$$= \int \left(\frac{3x^3}{x} - \frac{2\sqrt{x}}{x}\right)dx$$

$$= \int \left(3x^2 - \frac{2}{\sqrt{x}}\right)dx$$

$$= 3\int x^2 dx - 2\int x^{-\frac{1}{2}}\,dx$$

$$= 3\cdot\left(\frac{x^3}{3}\right) - 2\cdot\left[\frac{x^{\frac{1}{2}}}{\left(\frac{1}{2}\right)}\right] + c$$

$$= x^3 - 4\sqrt{x} + c.$$

**4. Evaluate : $\int (3x^2-5)^2 dx$.**

**Solution :** $\int (3x^2-5)^2 dx = \int (9x^4 - 30x^2 + 25)\,dx$

$$= 9\int x^4 dx - 30\int x^2 dx + 25\int 1\,dx$$

$$= 9\left(\frac{x^5}{5}\right) - 30\left(\frac{x^3}{3}\right) + 25x + c$$

$$= \frac{9x^5}{5} - 10x^3 + 25x + c.$$

**5. Evaluate : $\int \frac{1}{x(x-1)}dx$.**

**Solution :** $\int \frac{1}{x(x-1)}dx = \int \frac{x-(x-1)}{x(x-1)}\,dx$

$$= \int \left(\frac{1}{x-1} - \frac{1}{x}\right)dx$$

$$= \int \frac{1}{x-1}\,dx - \int \frac{1}{x}\,dx$$

$$= \log|x-1| - \log|x| + c$$

$$= \log\left|\frac{x-1}{x}\right| + c.$$

**6. If $f'(x) = x^2 + 5$ and $f(0) = -1$, then find the value of $f(x)$.**

**Solution :** By the definition of integral

$$f(x) = \int f'(x)\,dx = \int (x^2+5)\,dx$$

$$= \int x^2 dx + 5\int 1\,dx$$

$$= \frac{x^3}{3} + 5x + c \qquad \ldots (1)$$

Now, $f(0) = -1$ gives

$f(0) = 0 + 0 + c = -1$

$\therefore\ c = -1$

$\therefore$ from (1), $f(x) = \frac{x^3}{3} + 5x - 1.$

**7. If $f'(x) = 4x^3 - 3x^2 + 2x + k$, $f(0) = -1$ and $f(1) = 4$, find $f(x)$.**

**Solution :** By the definition of integral

$$f(x) = \int f'(x)\,dx = \int (4x^3 - 3x^2 + 2x + k)\,dx$$

$$= 4\int x^3\,dx - 3\int x^2\,dx + 2\int x\,dx + k\int 1\,dx$$

$$= 4\left(\frac{x^4}{4}\right) - 3\left(\frac{x^3}{3}\right) + 2\left(\frac{x^2}{2}\right) + kx + c$$

$$\therefore\ f(x) = x^4 - x^3 + x^2 + kx + c \qquad \ldots (1)$$

Now, $f(0) = 1$ gives

$f(0) = 0 - 0 + 0 + 0 + c = 1 \qquad \therefore\ c = 1$

$\therefore$ from (1), $f(x) = x^4 - x^3 + x^2 + kx + 1 \qquad \ldots (2)$

Further $f(1) = 4$ gives

$f(1) = 1 - 1 + 1 + k + 1 = 4 \qquad \therefore\ k = 2$

$\therefore$ from (2), $f(x) = x^4 - x^3 + x^2 + 2x + 1.$

**8. If $f'(x)=\frac{x^2}{2}-kx+1$, $f(0)=2$ and $f(3)=5$, find $f(x)$.**

**Solution :** By the definition of integral

$f(x)=\int f'(x)dx=\int\left(\frac{x^2}{2}-kx+1\right)dx$

$=\frac{1}{2}\int x^2dx-k\int xdx+\int 1\,dx$

$=\frac{1}{2}\left(\frac{x^3}{3}\right)-k\left(\frac{x^2}{2}\right)+x+c$

$\therefore\ f(x)=\frac{x^3}{6}-\frac{kx^2}{2}+x+c$ … (1)

Now, $f(0)=2$ gives

$f(0)=0-0+0+c=2\qquad\therefore\ c=2$

$\therefore$ from (1), $f(x)=\frac{x^3}{6}-\frac{kx^2}{2}+x+2$ … (2)

Further $f(3)=5$ gives

$f(3)=\frac{27}{6}-\frac{9k}{2}+3+2=5$

$\therefore\ \frac{9k}{2}=\frac{9}{2}\qquad\therefore\ k=1$

$\therefore$ from (2), $f(x)=\frac{x^3}{6}-\frac{x^2}{2}+x+2.$

## EXAMPLES FOR PRACTICE 5.1

**1. Integrate the following w.r.t. $x$ :**

(1) $(2x^2-1)^2$ (2) $x^2\left(1-\frac{2}{x}\right)^2$

(3) $\left(9-\frac{x}{2}\right)^8+(4x+5)^3$ (4) $(7x-2)^2$

(5) $\frac{1}{(6x+5)^4}-\frac{1}{(8-3x)^9}$ (6) $\sqrt{3x+4}-\frac{1}{\sqrt{5-3x}}$

(7) $\frac{5x+4}{2x+1}$ (8) $\frac{1}{\sqrt{3x+a}-\sqrt{3x-a}}$

(9) $\frac{3x+5}{\sqrt{2x+1}}$ (10) $\left(x+\frac{1}{x}\right)^3$

(11) $\frac{5(x^6+1)}{x^2+1}$ (12) $x^3\left(2-\frac{3}{x}\right)^2$

(13) $\frac{x^3+4x^2-6x+5}{x}$ (14) $e^{a\log x}+e^{x\log a}$

(15) $e^{(2-5x)}+\frac{2}{6x+1}$.

**2.** If $f'(x)=8x^3+3x^2-10x+k$, $f(0)=-3$ and $f(-1)=0$, find $f(x)$.

### Answers

**1.** (1) $\frac{4}{5}x^5-\frac{4}{3}x^3+\frac{x^2}{2}+c$

(2) $\frac{(x-2)^3}{3}+c$

(3) $-\frac{2}{9}\left(9-\frac{x}{2}\right)^9+\frac{1}{16}(4x+5)^4+c$

(4) $\frac{(7x-2)^3}{21}+c$

(5) $-\frac{1}{18(6x+5)^3}-\frac{1}{24(8-3x)^8}+c$

(6) $\frac{2}{9}(3x+4)^{\frac{3}{2}}+\frac{2}{3}(5-3x)^{\frac{1}{2}}+c$

(7) $\frac{5x}{2}+\frac{3}{4}\log|2x+1|+c$

(8) $\frac{1}{9a}[(3x+a)^{\frac{3}{2}}+(3x-a)^{\frac{3}{2}}]+c$

(9) $\frac{1}{2}(2x-1)^{\frac{3}{2}}+\frac{13}{2}(2x-1)^{\frac{1}{2}}+c$

(10) $\frac{x^4}{4}+\frac{3x^2}{2}+3\log|x|-\frac{1}{2x^2}+c$

(11) $x^5-\frac{5}{3}x^3+5x+c$

(12) $x^4-4x^3+\frac{9}{2}x^2+c$

(13) $\frac{x^3}{3}+2x^2-6x+5\log|x|+c$

(14) $\frac{x^{a+1}}{a+1}+\frac{a^x}{\log a}+c$

(15) $-\frac{1}{5}e^{(2-5x)}+\frac{1}{3}\log|6x+1|+c.$

**2.** $f(x)=2x^4+x^3-5x^2-7x-3.$

## 5.2 : METHOD OF CHANGE OF VARIABLE OR METHOD OF SUBSTITUTION

Consider, the integral of $\int f(x)dx$. Here we are integrating the function $f$ w.r.t. $x$ and hence $x$ is the variable of integration. Many times, changing this variable of integration by a suitable substitution, the function to be integrated can be reduced to some standard form. Hence, this method is called **integration by substitution.**

**Result :** If $x = \phi(t)$ is the differentiable function of $t$, then $\int f(x)\,dx = \int f[\phi(t)] \cdot \phi'(t)\,dt$

**Remark :**

Comparing $\int f(x)\,dx$ with $\int f[\phi(t)] \cdot \phi'(t)\,dt$, we observe that $dx$ has been replaced by $\phi'(t)\,dt$. Thus if we have the substitution $x = t^2$, then we replace $dx$ by $2t\,dt$.

**Corollary 1 : If $\int f(x)\,dx = g(x) + c$, then prove that**

$$\int f(ax+b)\,dx = \frac{g(ax+b)}{a} + c,$$

**where $a \neq 0$ and $b$ are constants.**

**Proof :** Put $ax + b = t$. Then

$$a \cdot dx = dt \qquad \therefore\ dx = \frac{dt}{a}$$

$$\therefore\ \int f(ax+b)\,dx = \int f(t) \cdot \frac{dt}{a}$$

$$= \frac{1}{a}\int f(t)\,dt = \frac{1}{a} \cdot g(t) + c$$

$$= \frac{g(ax+b)}{a} + c \qquad (a \neq 0).$$

**Corollary 2 : Prove that**

(i) $\int [f(x)]^n \cdot f'(x)\,dx = \dfrac{[f(x)]^{n+1}}{n+1} + c,$ if $n \neq -1$

(ii) $\int \dfrac{f'(x)}{f(x)}\,dx = \log|f(x)| + c$

(iii) $\int \dfrac{f'(x)}{\sqrt[n]{f(x)}}\,dx = \dfrac{[f(x)]^{1-(1/n)}}{1-(1/n)} + c$

(iv) $\int \dfrac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)} + c.$

**Proof :**

(i) Put $f(x) = t \quad \therefore\ f'(x)\,dx = dt.$

If $n \neq -1$, then

$$\int [f(x)]^n \cdot f'(x)\,dx = \int t^n\,dt$$

$$= \frac{t^{n+1}}{n+1} + c = \frac{[f(x)]^{n+1}}{n+1} + c$$

(ii) Put $f(x) = t \quad \therefore\ f'(x)\,dx = dt$

$$\therefore\ \int \frac{f'(x)}{f(x)}\,dx = \int \frac{dt}{t} = \log|t| + c$$

$$= \log|f(x)| + c$$

(iii) Put $f(x) = t \quad \therefore\ f'(x)\,dx = dt$

$$\therefore\ \int \frac{f'(x)}{\sqrt[n]{f(x)}}\,dx = \int [(f(x)]^{-1/n} f'(x)\,dx$$

$$= \int t^{-1/n}\,dt = \frac{t^{1-(1/n)}}{1-(1/n)} + c$$

$$= \frac{[f(x)]^{1-(1/n)}}{1-(1/n)} + c$$

(iv) Put $f(x) = t \quad \therefore\ f'(x)\,dx = dt$

$$\therefore\ \int \frac{f'(x)}{\sqrt{f(x)}}\,dx = \int \frac{1}{\sqrt{t}}\,dt$$

$$= \int t^{-\frac{1}{2}}\,dt = \frac{t^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} + c$$

$$= 2\sqrt{f(x)} + c.$$

## EXERCISE 5.2 | Textbook page 123

**Evaluate the following :**

**1.** $\int x\sqrt{1+x^2}\,dx$

**Solution :** Let $I = \int x\sqrt{1+x^2}\,dx = \int \sqrt{1+x^2} \cdot x\,dx$

Put $1 + x^2 = t$

$$\therefore\ 2x\,dx = dt \qquad \therefore\ x\,dx = \frac{dt}{2}$$

$$\therefore\ I = \int \sqrt{t}\,\frac{dt}{2} = \frac{1}{2}\int t^{\frac{1}{2}}\,dt$$

$$= \frac{1}{2} \cdot \frac{t^{\frac{3}{2}}}{(3/2)} + c$$

$$= \frac{1}{3}(1+x^2)^{\frac{3}{2}} + c.$$

**2.** $\int \dfrac{x^3}{\sqrt{1+x^4}}\,dx$

**Solution :** Let $I = \int \dfrac{x^3}{\sqrt{1+x^4}}\,dx$

Put $1 + x^4 = t \qquad \therefore\ 4x^3\,dx = dt$

$$\therefore\ x^3\,dx = \frac{dt}{4}$$

$$\therefore\ I = \int \frac{1}{\sqrt{t}} \cdot \frac{dt}{4} = \frac{1}{4}\int t^{-\frac{1}{2}}\,dt$$

$$= \frac{1}{4} \cdot \frac{t^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} + c$$

$$= \frac{1}{2}\sqrt{1+x^4} + c.$$

**3.** $\int (e^x + e^{-x})^2 (e^x - e^{-x})\, dx$

**Solution :** Let $I = \int (e^x + e^{-x})^2 (e^x - e^{-x})\, dx$

Put $e^x + e^{-x} = t$

$\therefore\ (e^x - e^{-x})\, dx = dt$

$\therefore\ I = \int t^2 dt = \dfrac{t^3}{3} + c$

$= \dfrac{(e^x + e^{-x})^3}{3} + c.$

**4.** $\displaystyle\int \frac{1+x}{x+e^{-x}}\, dx$

**Solution :** Let $I = \displaystyle\int \frac{1+x}{x+e^{-x}}\, dx$

$= \displaystyle\int \frac{(1+x)e^x}{(x+e^{-x})e^x}\, dx$

$= \displaystyle\int \frac{(1+x)e^x}{xe^x+1}\, dx$

Put $xe^x + 1 = t$

$\therefore\ (xe^x + e^x \times 1)\, dx = dt$

$\therefore\ (1+x)e^x\, dx = dt$

$\therefore\ I = \displaystyle\int \frac{1}{t}\, dt = \log|t| + c$

$= \log|xe^x + 1| + c.$

**5.** $\int (x+1)(x+2)^7(x+3)\, dx$

**Solution :** Let $I = \int (x+1)(x+2)^7(x+3)\, dx$

$= \int (x+2)^7 (x+1)(x+3)\, dx$

$= \int (x+2)^7 [(x+2)-1][(x+2)+1]\, dx$

$= \int (x+2)^7 [(x+2)^2 - 1]\, dx$

$= \int [(x+2)^9 - (x+2)^7]\, dx$

$= \int (x+2)^9 dx - \int (x+2)^7 dx$

$= \dfrac{(x+2)^{10}}{10} - \dfrac{(x+2)^8}{8} + c.$

**6.** $\displaystyle\int \frac{1}{x \log x}\, dx$

**Solution :**

Put $\log x = t \quad \therefore\ \dfrac{1}{x}\, dx = dt$

$\therefore\ \displaystyle\int \frac{dx}{x \cdot \log x} = \int \frac{1}{\log x} \cdot \frac{1}{x}\, dx$

$= \displaystyle\int \frac{1}{t}\, dt = \log|t| + c$

$= \log|\log x| + c.$

**7.** $\displaystyle\int \frac{x^5}{x^2+1}\, dx$

**Solution :** Let $I = \displaystyle\int \frac{x^5}{x^2+1}\, dx = \int \frac{x^4}{x^2+1} \cdot x\, dx$

$= \displaystyle\int \frac{(x^2)^2}{x^2+1} \cdot x\, dx$

Put $x^2 + 1 = t \quad \therefore\ 2x\, dx = dt$

$\therefore\ x\,dx = \dfrac{dt}{2}$ and $x^2 = t - 1$

$\therefore\ I = \displaystyle\int \frac{(t-1)^2}{t} \cdot \frac{dt}{2} = \frac{1}{2} \int \left(\frac{t^2 - 2t + 1}{t}\right) dt$

$= \dfrac{1}{2} \displaystyle\int \left(t - 2 + \frac{1}{t}\right) dt$

$= \dfrac{1}{2} \int t\,dt - \int 1\,dt + \dfrac{1}{2} \displaystyle\int \frac{1}{t}\, dt$

$= \dfrac{1}{2} \cdot \dfrac{t^2}{2} - t + \dfrac{1}{2} \log|t| + c$

$= \dfrac{1}{4}(x^2+1)^2 - (x^2+1) + \dfrac{1}{2} \log|x^2+1| + c.$

**8.** $\displaystyle\int \frac{2x+6}{\sqrt{x^2+6x+3}}\, dx$

**Solution :** Let $I = \displaystyle\int \frac{2x+6}{\sqrt{x^2+6x+3}}\, dx$

Put $x^2 + 6x + 3 = t$

$\therefore\ (2x+6)dx = dt$

$\therefore\ I = \displaystyle\int \frac{1}{\sqrt{t}}\, dt = \int t^{-\frac{1}{2}}\, dt$

$= \dfrac{t^{\frac{1}{2}}}{1/2} + c = 2\sqrt{x^2+6x+3} + c.$

**9.** $\displaystyle\int \frac{1}{\sqrt{x}+x}\, dx$

**Solution :** Let $I = \displaystyle\int \frac{1}{\sqrt{x}+x}\, dx = \int \frac{1}{\sqrt{x}(1+\sqrt{x})}\, dx$

$= \displaystyle\int \frac{1}{1+\sqrt{x}} \cdot \frac{1}{\sqrt{x}}\, dx$

Put $1 + \sqrt{x} = t \quad \therefore\ \dfrac{1}{2\sqrt{x}}\, dx = dt$

$\therefore\ \dfrac{1}{\sqrt{x}}\, dx = 2\, dt$

$\therefore \; I = \int \frac{1}{t} \cdot 2dt = 2\int \frac{1}{t}\, dt$

$= 2 \log |t| + c = 2 \log |1 + \sqrt{x}| + c.$

**10.** $\displaystyle\int \frac{1}{x(x^6+1)}\, dx$

**Solution :** Let $I = \displaystyle\int \frac{1}{x(x^6+1)}\, dx$

$= \displaystyle\int \frac{x^5}{x^6(x^6+1)}\, dx$

Put $x^6 = t \quad \therefore \; 6x^5\, dx = dt$

$\therefore \; x^5\, dx = \frac{1}{6}\, dt$

$\therefore \; I = \displaystyle\int \frac{1}{t(t+1)} \cdot \frac{dt}{6}$

$= \frac{1}{6}\displaystyle\int \frac{(t+1)-t}{t(t+1)}\, dt = \frac{1}{6}\int \left(\frac{1}{t} - \frac{1}{t+1}\right) dt$

$= \frac{1}{6}\left[\displaystyle\int \frac{1}{t}\, dt - \int \frac{1}{t+1}\, dt\right]$

$= \frac{1}{6}\left[\log (t) - \log |t+1|\right] + c$

$= \frac{1}{6} \log \left|\frac{t}{t+1}\right| + c = \frac{1}{6} \log \left|\frac{x^6}{x^6+1}\right| + c.$

## ADDITIONAL SOLVED PROBLEMS – 5 (A)

**1. Evaluate :** $\displaystyle\int \frac{e^{3x}}{e^{3x}+1}\, dx.$

**Solution :**

Let $I = \displaystyle\int \frac{e^{3x}}{e^{3x}+1}\, dx$

Put $e^{3x} + 1 = t$

$\therefore \; 3e^{3x}\, dx = dt$

$\therefore \; e^{3x}\, dx = \frac{dt}{3}$

$\therefore \; I = \displaystyle\int \frac{1}{t} \cdot \frac{dt}{3} = \frac{1}{3}\int \frac{1}{t}\, dt$

$= \frac{1}{3} \log |t| + c = \frac{1}{3} \log |e^{3x} + 1| + c.$

**2. Evaluate :** $\displaystyle\int \frac{e^{x-1} + x^{e-1}}{e^x + x^e}\, dx.$

**Solution :** Put $e^x + x^e = t$

$\therefore \; (e^x + ex^{e-1})\, dx = dt$

$\therefore \; e\,(e^{x-1} + x^{e-1})\, dx = dt$

$\therefore \; (e^{x-1} + x^{e-1})\, dx = \frac{1}{e} \cdot dt$

$\therefore \; \displaystyle\int \frac{e^{x-1} + x^{e-1}}{e^x + x^e}\, dx = \int \frac{1}{t} \cdot \frac{dt}{e}$

$= \frac{1}{e}\displaystyle\int \frac{1}{t}\, dt = \frac{1}{e} \cdot \log |t| + c$

$= \frac{1}{e} \log |e^x + x^e| + c.$

**3. Evaluate :** $\displaystyle\int \frac{1}{x + x^{-n}}\, dx.$

**Solution :**

Let $I = \displaystyle\int \frac{1}{x + x^{-n}}\, dx = \int \frac{dx}{x + \left(\frac{1}{x^n}\right)}$

$= \displaystyle\int \frac{x^n\, dx}{x^{n+1} + 1}$

Put $x^{n+1} + 1 = t \quad \therefore \; (n+1)x^n\, dx = dt$

$\therefore \; x^n\, dx = \frac{dt}{n+1}$

$\therefore \; I = \displaystyle\int \frac{dt}{(n+1)\,t} = \frac{1}{n+1}\int \frac{dt}{t}$

$= \frac{1}{n+1} \log |t| + c$

$= \frac{1}{n+1} \log |x^{n+1} + 1| + c.$

**4. Evaluate :** $\displaystyle\int \frac{1}{e^x + 5}\, dx.$

**Solution :** Let $I = \displaystyle\int \frac{1}{e^x + 5}\, dx$

$= \displaystyle\int \frac{e^{-x}\, dx}{e^{-x}(e^x + 5)}$

$= \displaystyle\int \frac{e^{-x}\, dx}{1 + 5e^{-x}}$

Put $1 + 5e^{-x} = t \quad \therefore \; -5e^{-x}\, dx = dt$

$\therefore \; e^{-x}\, dx = \frac{-dt}{5}$

$\therefore \; I = \displaystyle\int \frac{1}{t} \cdot \left(\frac{-dt}{5}\right) = -\frac{1}{5}\int \frac{1}{t}\, dt$

$= -\frac{1}{5} \log |t| + c$

$= -\frac{1}{5} \log |1 + 5\, e^{-x}| + c.$

**5. Evaluate :** $\int \frac{e^x\,dx}{e^{2x}(1+3e^x)}$.

**Solution :**

Let $I = \int \frac{e^x\,dx}{e^{2x}(1+3e^x)}$

Put $e^x = t \quad \therefore \; e^x\,dx = dt$

$$\therefore \; I = \int \frac{1}{t^2(1+3t)}\,dt$$

$$= \int \frac{(1-9t^2)+9t^2}{t^2(1+3t)}\,dt$$

$$= \int \left[\frac{1-9t^2}{t^2(1+3t)} + \frac{9t^2}{t^2(1+3t)}\right]dt$$

$$= \int \left[\frac{(1-3t)(1+3t)}{t^2(1+3t)} + \frac{9}{1+3t}\right]dt$$

$$= \int \left[\frac{1-3t}{t^2} + \frac{9}{1+3t}\right]dt$$

$$= \int \left(\frac{1}{t^2} - \frac{3}{t} + \frac{9}{1+3t}\right)dt$$

$$= \int t^{-2}\,dt - 3\int \frac{1}{t}\,dt + 9\int \frac{1}{1+3t}\,dt$$

$$= \frac{t^{-1}}{-1} - 3\log|t| + 9\,\frac{\log|1+3t|}{3} + c$$

$$= -e^{-x} - 3\log|e^x| + 3\log|1+3e^x| + c$$

$$= -e^{-x} - 3x\log e + 3\log|1+3e^x| + c$$

$$= -e^{-x} - 3x + 3\log|1+3e^x| + c \qquad \ldots [\because \; \log e = 1]$$

**6. Evaluate :** $\int (3x+2)\sqrt{2x+1}\,dx$.

**Solution :** Let $I = \int (3x+2)\sqrt{2x+1}\,dx$

Put $2x+1 = t \quad \therefore \; 2dx = dt$

$\therefore \; dx = \frac{dt}{2}$ and $x = \frac{t-1}{2}$

$$\therefore \; I = \int \left[3\left(\frac{t-1}{2}\right)+2\right]\sqrt{t}\cdot\frac{dt}{2}$$

$$= \frac{1}{2}\int \left(\frac{3t-3+4}{2}\right)\sqrt{t}\,dt$$

$$= \frac{1}{4}\int (3t+1)\sqrt{t}\,dt$$

$$= \frac{1}{4}\int (3t\sqrt{t}+\sqrt{t})\,dt$$

$$= \frac{3}{4}\int t^{\frac{3}{2}}dt + \frac{1}{4}\int t^{\frac{1}{2}}\,dt$$

$$= \frac{3}{4}\cdot\left[\frac{t^{\frac{5}{2}}}{\left(\frac{5}{2}\right)}\right] + \frac{1}{4}\cdot\left[\frac{t^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right] + c$$

$$= \frac{3}{10}(2x+1)^{\frac{5}{2}} + \frac{1}{6}(2x+1)^{\frac{3}{2}} + c.$$

**7. Evaluate :** $\int \frac{5x^2+4x+7}{(2x+3)^{\frac{3}{2}}}\,dx$.

**Solution :** Let $I = \int \frac{5x^2+4x+7}{(2x+3)^{\frac{3}{2}}}\,dx$

Put $2x+3 = t \quad \therefore \; 2dx = dt \quad \therefore \; dx = \frac{dt}{2}$

Also, $x = \frac{t-3}{2}$

$$\therefore \; I = \int \frac{5\left(\frac{t-3}{2}\right)^2 + 4\left(\frac{t-3}{2}\right)+7}{t^{\frac{3}{2}}}\cdot\frac{dt}{2}$$

$$= \frac{1}{2}\int \frac{5\left(\frac{t^2-6t+9}{4}\right)+2(t-3)+7}{t^{\frac{3}{2}}}\,dt$$

$$= \frac{1}{2}\int \frac{5t^2-30t+45+8t-24+28}{4t^{\frac{3}{2}}}\,dt$$

$$= \frac{1}{8}\int \frac{5t^2-22t+49}{t^{\frac{3}{2}}}\,dt$$

$$= \frac{1}{8}\int (5t^{\frac{1}{2}} - 22t^{-\frac{1}{2}} + 49t^{-\frac{3}{2}})\,dt$$

$$= \frac{5}{8}\int t^{\frac{1}{2}}dt - \frac{22}{8}\int t^{-\frac{1}{2}}dt + \frac{49}{8}\int t^{-\frac{3}{2}}\,dt$$

$$= \frac{5}{8}\cdot\frac{t^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} - \frac{11}{4}\cdot\frac{t^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} + \frac{49}{8}\cdot\frac{t^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)} + c$$

$$= \frac{5}{12}(2x+3)^{\frac{3}{2}} - \frac{11}{2}(2x+3)^{\frac{1}{2}} - \frac{49}{4}(2x+3)^{-\frac{1}{2}} + c$$

$$= \frac{5}{12}(2x+3)^{\frac{3}{2}} - \frac{11}{2}\sqrt{2x+3} - \frac{49}{4\sqrt{2x+3}} + c.$$

## EXAMPLES FOR PRACTICE 5.2

**Evaluate the following integrals :**

1. (1) $\int \frac{x^2}{1+x^6}\,dx$ (2) $\int \frac{dx}{x+x^{-3}}$

   (3) $\int (4x+2)\,\sqrt[3]{x^2+x+1}\,dx$

(4) $\int \frac{x+1}{\sqrt{2+2x+x^2}} dx$ (5) $\int \frac{x^3}{\sqrt{x^2+a^2}} dx$

(6) $\int \frac{1}{2x+x^{-n}} dx$ (7) $\int \frac{4x-6}{(x^2-3x+5)^{3/2}} dx$

(8) $\int \frac{x^{n-1}}{\sqrt{1+x^n}} dx$ (9) $\int \frac{3x^2}{\sqrt{1+x^3}} dx$

(10) $\int \frac{1}{3x+5x^{-n}} dx$ (11) $\int \frac{x^7}{(1+x^4)^2} dx.$

**2.** (1) $\int (2x+1)\sqrt{x-4}\, dx$ (2) $\int (5-3x)(2-3x)^{-\frac{1}{2}} dx$

(3) $\int \frac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}} dx$ [**Hint :** Put $x=t^6$]

(4) $\int (4-5x)(3-x)^{-\frac{3}{2}} dx$ (5) $\int \frac{4x}{(x^2+1)^2} dx.$

**3.** (1) $\int \frac{e^{5x}}{\sqrt{e^{5x}+1}} dx$ (2) $\int \frac{1}{1-e^x} dx$

(3) $\int \frac{1}{(e^x+e^{-x})^2} dx$ (4) $\int \frac{10x^9+10^x \cdot \log 10}{10^x+x^{10}} dx$

(5) $\int \frac{1}{1+e^{-x}} dx$ (6) $\int \frac{e^{2x}-1}{e^{2x}+1} dx.$

**4.** (1) $\int \frac{(\log x)^n}{x} dx$ (2) $\int \frac{1}{x \log x \cdot \log(\log x)} dx$

(3) $\int \frac{x+2^x \cdot \log 2}{x^2+2^{x+1}} dx$ (4) $\int 7^{x\log x} \cdot (1+\log x) dx$

(5) $\int \frac{x+1}{x(x+\log x)} dx$ (6) $\int \frac{1}{x(3+\log x)} dx.$

## Answers

**1.** (1) $\frac{1}{3}\tan^{-1}(x^3)+c$ (2) $\frac{1}{4}\log(x^4+1)+c$

(3) $\frac{3}{2}(x^2+x+1)^{4/3}+c$ (4) $\sqrt{2+2x+x^2}+c$

(5) $\frac{1}{3}(x^2+a^2)^{3/2}-a^2\sqrt{x^2+a^2}+c$

(6) $\frac{1}{2(n+1)}\log|2x^{n+1}+1|+c$

(7) $\frac{-4}{\sqrt{x^2+3x+5}}+c$ (8) $\frac{2}{n}\sqrt{1+x^n}+c$

(9) $2\sqrt{1+x^3}+c$ (10) $\frac{1}{3(n+1)}\log|3x^{n+1}+5|+c$

(11) $\frac{1}{4}\left[\log|1+x^4|+\frac{1}{(1+x^4)}\right]+c.$

**2.** (1) $\frac{4}{5}(x-4)^{\frac{5}{2}}+6(x-4)^{\frac{3}{2}}+c$

(2) $-2\sqrt{2-3x}-\frac{2}{9}(2-3x)^{\frac{3}{2}}+c$

(3) $2\sqrt{x}-3(\sqrt[3]{x})+6(\sqrt[6]{x})-6\log|\sqrt[6]{x}+1|+c$

(4) $\frac{-22}{\sqrt{3-x}}-10\sqrt{3-x}+c$

(5) $\frac{-2}{x^2+1}+c.$

**3.** (1) $\frac{2}{5}\sqrt{e^{5x}+1}+c$ (2) $-\log|e^{-x}-1|+c$

(3) $\frac{-1}{2(e^{2x}+1)}+c$ (4) $\log|10^x+x^{10}|+c$

(5) $\log|e^x+1|+c$ (6) $\log|e^x+e^{-x}|+c.$

**4.** (1) $\frac{1}{n+1}(\log x)^{n+1}+c$ (2) $\log|\log(\log x)|+c$

(3) $\frac{1}{2}\log|x^2+2^{x+1}|+c$ (4) $\frac{7^{x\log x}}{\log 7}+c$

(5) $\log|x+\log x|+c$ (6) $\log|3+\log x|+c.$

## 5.3 : INTEGRALS OF THE TYPE $\int \frac{ae^x+b}{ce^x+d} dx$, where $a, b, c, d \in R$

In integral of the type $\int \frac{ae^x+b}{ce^x+d} dx$,

Put, Numerator $= A$(Denominator) $+$

$B\left[\frac{d}{dx}(\text{Denominator})\right]$

where $A$ and $B$ are constants.

Obtain the values of $A$ and $B$ by equaling the coefficient of $e^x$ and constant term on both the sides. Then

$$\int \frac{ae^x+b}{ce^x+d} dx = \int \frac{A(ce^x+d)+B\cdot\frac{d}{dx}(ce^x+d)}{ce^x+d} dx$$

$$= A\int 1 dx + B\int \frac{ce^x}{ce^x+d} dx$$

$$= A\cdot x + B\log|ce^x+d|+c.$$

**EXERCISE 5.3** | **Textbook page 123**

**Evaluate the following :**

**1.** $\int \frac{3e^{2t}+5}{4e^{2t}-5} dt$

**Solution :** Let $I=\int \frac{3e^{2t}+5}{4e^{2t}-5}\,dt$

Put, Numerator $= A$(Denominator) $+$

$$B\left[\frac{d}{dx}\text{(Denominator)}\right]$$

$\therefore\ 3e^{2t}+5=A(4e^{2t}-5)+B\left[\frac{d}{dt}(4e^{2t}-5)\right]$

$\qquad = A(4e^{2t}-5)+B[4e^{2t}\times 2-0]$

$\therefore\ 3e^{2t}+5=(4A+8B)e^{2t}-5A$

Equating the coefficient of $e^{2t}$ and constant on both sides, we get

$4A+8B=3$ $\qquad\qquad$ ... (1)

and $-5A=5$ $\quad \therefore\ A=-1$

$\therefore$ from (1), $4(-1)+8B=3$

$\therefore\ 8B=7 \quad \therefore\ B=\frac{7}{8}$

$\therefore\ 3e^{2t}+5=-(4e^{2t}-5)+\frac{7}{8}(8e^{2t})$

$$\therefore\ I=\int\left[\frac{-(4e^{2t}-5)+\frac{7}{8}(8e^{2t})}{4e^{2t}-5}\right]dt$$

$$=\int\left[-1+\frac{\frac{7}{8}(8e^{2t})}{4e^{2t}-5}\right]dt$$

$$=-\int 1dt+\frac{7}{8}\int\frac{8e^{2t}}{4e^{2t}-5}\,dt$$

$$=-t+\frac{7}{8}\log|4e^{2t}-5|+c$$

$$\ldots\left[\because \int\frac{f'(x)}{f(x)}dx=\log|f(x)|+c\right].$$

**2.** $\int \frac{20-12e^x}{3e^x-4}\,dx$

**Solution :** Let $I=\int \frac{20-12e^x}{3e^x-4}\,dx$

Put, Numerator $= A$(Denominator) $+$

$$B\left[\frac{d}{dx}\text{(Denominator)}\right]$$

$\therefore\ 20-12e^x=A(3e^x-4)+B\left[\frac{d}{dx}(3e^x-4)\right]$

$\qquad = A(3e^x-4)+B(3e^x-0)$

$\therefore\ 20-12e^x=(3A+3B)e^x-4A$

Equating the coefficient of $e^x$ and constant on both sides, we get

$3A+3B=-12$ $\qquad\qquad$ ... (1)

and $-4A=20$ $\quad \therefore\ A=-5$

$\therefore$ from (1), $3(-5)+3B=-12$

$\therefore\ 3B=3 \quad \therefore\ B=1$

$\therefore\ 20-12e^x=-5(3e^x-4)+(3e^x)$

$$\therefore\ I=\int\left[\frac{-5(3e^x-4)+(3e^x)}{3e^x-4}\right]dx$$

$$=\int\left(-5+\frac{3e^x}{3e^x-4}\right)dx$$

$$=-5\int 1\,dx+\int\frac{3e^x}{3e^x-4}\,dx$$

$$=-5x+\log|3e^x-4|+c$$

$$\ldots\left[\because \int\frac{f'(x)}{f(x)}dx=\log|f(x)|+c\right]$$

[**Note :** *Answer in the textbook is incorrect.*]

**3.** $\int \frac{3e^x+4}{2e^x-8}\,dx$

**Solution :** Let $I=\int \frac{3e^x+4}{2e^x-8}\,dx$

Put, Numerator $= A$(Denominator) $+$

$$B\left[\frac{d}{dx}\text{(Denominator)}\right]$$

$\therefore\ 3e^x+4=A(2e^x-8)+B\left[\frac{d}{dx}(2e^x-8)\right]$

$\qquad = A(2e^x-8)+B(2e^x-0)$

$\therefore\ 3e^x+4=(2A+2B)e^x-8A$

Equating the coefficient of $e^x$ and constant on both sides, we get

$2A+2B=3$ $\qquad\qquad$ ... (1)

and $-8A=4$ $\quad \therefore\ A=-\frac{1}{2}$

$\therefore$ from (1), $2\left(-\frac{1}{2}\right)+2B=3$

$\therefore\ 2B=4 \quad \therefore\ B=2$

$\therefore\ 3e^x+4=-\frac{1}{2}(2e^x-8)+2(2e^x)$

$$\therefore\ I=\int\left[\frac{-\frac{1}{2}(2e^x-8)+2(2e^x)}{2e^x-8}\right]dx$$

$$=\int\left[-\frac{1}{2}+\frac{2(2e^x)}{2e^x-8}\right]dx$$

$$= -\frac{1}{2}\int 1\,dx + 2\int \frac{2e^x}{2e^x - 8}\,dx$$

$$= -\frac{1}{2}x + 2\log|2e^x - 8| + c$$

$$\ldots \left[\because \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + c\right]$$

**4.** $\int \frac{2e^x + 5}{2e^x + 1}\,dx$

**Solution :** *Refer to the solution of Q. 3.*

**Ans.** $5x - 4\log|2e^x + 1| + c.$

[**Note :** *Answer in the textbook is incorrect.*]

## EXAMPLES FOR PRACTICE 5.3

**Evaluate the following :**

**1.** $\int \frac{3e^x - 4}{4e^x + 5}\,dx$ **2.** $\int \frac{3e^x + 4}{2e^x - 8}\,dx$

**3.** $\int \frac{4e^x - 25}{2e^x - 5}\,dx$ **4.** $\int \frac{3e^{2x} + 1}{3e^{2x} - 1}\,dx.$

**Answers**

**1.** $-\frac{4}{5}x + \frac{31}{20}\log|4e^x + 5| + c$

**2.** $-\frac{x}{2} + 2\log|2e^x - 8| + c$

**3.** $5x - 3\log|2e^x - 5| + c$

**4.** $-x + \log|3e^{2x} - 1| + c.$

## 5.4 : SOME STANDARD RESULTS

**Standard Formulae :**

**1.** $\int \frac{1}{x^2 - a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$

**2.** $\int \frac{1}{a^2 - x^2}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$

**3.** $\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \log|x + \sqrt{x^2 + a^2}| + c$

**4.** $\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \log|x + \sqrt{x^2 - a^2}| + c.$

### 5.4.1 Integrals of the form $\int \frac{P(x)}{Q(x)}\,dx$, where degree $P(x) \geqslant$ degree $Q(x)$

In this type of integral, we divide $P(x)$ by $Q(x)$ and get the quotient $q(x)$ and remainder $r(x)$.

Use, Dividend = quotient × $Q(x)$ + remainder

$\therefore\ P(x) = q(x) \times Q(x) + r(x)$

$$\therefore\ \frac{P(x)}{Q(x)} = \frac{q(x) \times Q(x) + r(x)}{Q(x)}$$

$$= q(x) + \frac{r(x)}{Q(x)}$$

$$\therefore\ \int \frac{P(x)}{Q(x)}\,dx = \int q(x)\,dx + \int \frac{r(x)}{Q(x)}\,dx$$

By using standard integrals, we can evaluate this integral.

### 5.4.2 Integrals of the form $\int \frac{1}{ax^2 + bx + c}\,dx$ and $\int \frac{1}{\sqrt{ax^2 + bx + c}}\,dx$

**1. Integrals of the form** $\int \frac{1}{\sqrt{ax^2 + bx + c}}\,dx$ **:**

In this case, we express $ax^2 + bx + c$ in any one of the forms $a^2 + x^2$, $a^2 - x^2$, $x^2 - a^2$ by method of completion of square.

i.e. $ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + \frac{c}{a} - \frac{b^2}{4a^2}\right)$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right].$$

**2. Integrals of the form** $\int \frac{1}{\sqrt{ax^2 + bx + c}}\,dx$ **:**

In this case, we express $ax^2 + bx + c$ in any one of the forms $\sqrt{a^2 + x^2}$, $\sqrt{x^2 - a^2}$ by method of completion of square.

i.e. $\sqrt{ax^2 + bx + c} = \sqrt{a\left(x^2 + \frac{bx}{a} + \frac{c}{a}\right)}$

$$= \sqrt{a}\sqrt{x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + \frac{c}{a} - \frac{b^2}{4a^2}}$$

$$= \sqrt{a}\sqrt{\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}}.$$



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### 5.4.3 Integrals reducible to the form

$$\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx \quad \text{and} \quad \int \frac{1}{ax^2+bx+c}\,dx$$

In these type of examples, we use the proper substitution so that it is converted into the form

$$\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx \quad \text{or} \quad \int \frac{1}{ax^2+bx+c}\,dx.$$

**Remark :** Students are reminded that if $x$ is replaced by $ax+b$ in integrals of standard functions, then in standard formula we have to replace $x$ by $ax+b$ and divide the obtained answer by $a$.

### 5.4.4 Integrals of the form

$$\int \frac{px+q}{ax^2+bx+c}\,dx \quad \text{and} \quad \int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$$

**1. Integral of the form $\int \frac{px+q}{ax^2+bx+c}\,dx$ :**

To evaluate these types of integrals, we use the following steps :

**Step 1 :** Write Numerator $= A\left[\frac{d}{dx}\text{(Denominator)}\right]+B$

i.e. $px+q = A\left[\frac{d}{dx}(ax^2+bx+c)\right]+B$

where $A$ and $B$ are constants.

**Step 2 :** Obtain the values of $A$ and $B$ by equating the coefficients of same powers of $x$ on both the sides.

**Step 3 :** Replace $px+q$ by $A(2ax+b)+B$ in the given integral to get

$$\int \frac{px+q}{ax^2+bx+c}\,dx = A\int \frac{2ax+b}{ax^2+bx+c}\,dx + B\int \frac{1}{ax^2+bx+c}\,dx$$

**Step 4 :** Integrate the RHS of step 3.

**2. Integral of the form $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$ :**

To evaluate these types of integrals, we use the following steps :

**Step 1 :** Write numerator $= A\left[\frac{d}{dx}\text{(Denominator)}\right]+B$

i.e. $px+q = A\left[\frac{d}{dx}(ax^2+bx+c)\right]+B$

where $A$ and $B$ are constants.

**Step 2 :** Obtain the values of $A$ and $B$ by equating the coefficients of same powers of $x$ on both the sides.

**Step 3 :** Replace $px+q$ by $A(2ax+b)+B$ in the given integral to get

$$\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx = A\int \frac{2ax+b}{\sqrt{ax^2+bx+c}}\,dx + B\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx$$

**Step 4 :** Integrate the RHS of step 3.

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|---|---|

**Evaluate the following :**

**1.** $\int \frac{1}{4x^2-1}\,dx$

**Solution :** $\int \frac{1}{4x^2-1}\,dx = \frac{1}{4}\int \frac{1}{x^2-(1/4)}\,dx$

$$= \frac{1}{4}\int \frac{1}{x^2-\left(\frac{1}{2}\right)^2}\,dx$$

$$= \frac{1}{4}\times \frac{1}{2\left(\frac{1}{2}\right)}\log\left|\frac{x-\frac{1}{2}}{x+\frac{1}{2}}\right|+c$$

$$= \frac{1}{4}\log\left|\frac{2x-1}{2x+1}\right|+c.$$

**2.** $\int \frac{1}{x^2+4x-5}\,dx$

**Solution :** $\int \frac{1}{x^2+4x-5}\,dx$

$$= \int \frac{1}{(x^2+4x+4)-4-5}\,dx$$

$$= \int \frac{1}{(x+2)^2-(3)^2}\,dx$$

$$= \frac{1}{2\times 3}\log\left|\frac{x+2-3}{x+2+3}\right|+c$$

$$= \frac{1}{6}\log\left|\frac{x-1}{x+5}\right|+c.$$

**3.** $\int \frac{1}{4x^2-20x+17}\,dx$

**Solution :** $\int \frac{1}{4x^2-20x+17}\,dx$

$$= \frac{1}{4}\int \frac{1}{x^2-5x+\frac{17}{4}}\,dx$$

$$= \frac{1}{4}\int \frac{1}{\left(x^2-5x+\frac{25}{4}\right)-\frac{25}{4}+\frac{17}{4}}\,dx$$

$$= \frac{1}{4}\int \frac{1}{\left(x-\frac{5}{2}\right)^2 - (\sqrt{2})^2}\,dx$$

$$= \frac{1}{4} \times \frac{1}{2\sqrt{2}} \log \left| \frac{x - \frac{5}{2} - \sqrt{2}}{x - \frac{5}{2} + \sqrt{2}} \right| + c$$

$$= \frac{1}{8\sqrt{2}} \log \left| \frac{2x - 5 - 2\sqrt{2}}{2x - 5 + 2\sqrt{2}} \right| + c.$$

**4.** $\displaystyle\int \frac{x}{4x^4 - 20x^2 - 3}\,dx$

**Solution :** Let $I = \displaystyle\int \frac{x}{4x^4 - 20x^2 - 3}\,dx$

Put $x^2 = t$ $\quad \therefore\ 2x\,dx = dt$

$\therefore\ x\,dx = \frac{dt}{2}$

$$\therefore\ I = \int \frac{1}{4t^2 - 20t - 3} \cdot \frac{dt}{2}$$

$$= \frac{1}{2} \times \frac{1}{4} \int \frac{1}{t^2 - 5t - \frac{3}{4}}\,dt$$

$$= \frac{1}{8} \int \frac{1}{\left(t^2 - 5t + \frac{25}{4}\right) - \frac{25}{4} - \frac{3}{4}}\,dt$$

$$= \frac{1}{8} \int \frac{1}{\left(t - \frac{5}{2}\right)^2 - (\sqrt{7})^2}\,dt$$

$$= \frac{1}{8} \times \frac{1}{2\sqrt{7}} \log \left| \frac{t - \frac{5}{2} - \sqrt{7}}{t - \frac{5}{2} + \sqrt{7}} \right| + c$$

$$= \frac{1}{16\sqrt{7}} \log \left| \frac{2t - 5 - 2\sqrt{7}}{2t - 5 + 2\sqrt{7}} \right| + c$$

$$= \frac{1}{16\sqrt{7}} \log \left| \frac{2x^2 - 5 - 2\sqrt{7}}{2x^2 - 5 + 2\sqrt{7}} \right| + c.$$

[**Note :** *Answer in the textbook is incorrect.*]

**5.** $\displaystyle\int \frac{x^3}{16x^8 - 25}\,dx$

**Solution :** Let $I = \displaystyle\int \frac{x^3}{16x^8 - 25}\,dx$

Put $x^4 = t$ $\quad \therefore\ 4x^3 dx = dt$

$\therefore\ x^3 dx = \frac{dt}{4}$

$$\therefore\ I = \int \frac{1}{16t^2 - 25} \cdot \frac{dt}{4}$$

$$= \frac{1}{4} \times \frac{1}{16} \int \frac{1}{t^2 - \frac{25}{16}}\,dt$$

$$= \frac{1}{64} \int \frac{1}{t^2 - \left(\frac{5}{4}\right)^2}\,dt$$

$$= \frac{1}{64} \times \frac{1}{2 \times \frac{5}{4}} \log \left| \frac{t - \frac{5}{4}}{t + \frac{5}{4}} \right| + c$$

$$= \frac{1}{160} \log \left| \frac{4t - 5}{4t + 5} \right| + c$$

$$= \frac{1}{160} \log \left| \frac{4x^4 - 5}{4x^4 + 5} \right| + c.$$

[**Note :** *Answer in the textbook is incorrect.*]

**6.** $\displaystyle\int \frac{1}{a^2 - b^2x^2}\,dx$

**Solution :** $\displaystyle\int \frac{1}{a^2 - b^2x^2}\,dx = \frac{1}{b^2} \int \frac{1}{\frac{a^2}{b^2} - x^2}\,dx$

$$= \frac{1}{b^2} \int \frac{1}{\left(\frac{a}{b}\right)^2 - x^2}\,dx$$

$$= \frac{1}{b^2} \times \frac{1}{2\left(\frac{a}{b}\right)} \log \left| \frac{\frac{a}{b} + x}{\frac{a}{b} - x} \right| + c$$

$$= \frac{1}{2ab} \log \left| \frac{a + bx}{a - bx} \right| + c.$$

**7.** $\displaystyle\int \frac{1}{7 + 6x - x^2}\,dx$

**Solution :** $\displaystyle\int \frac{1}{7 + 6x - x^2}\,dx$

$$= \int \frac{1}{7 - (x^2 - 6x + 9) + 9}\,dx$$

$$= \int \frac{1}{(4)^2 - (x - 3)^2}\,dx$$

$$= \frac{1}{2 \times 4} \log \left| \frac{4 + x - 3}{4 - x + 3} \right| + c$$

$$= \frac{1}{8} \log \left| \frac{1 + x}{7 - x} \right| + c.$$

**8.** $\int \frac{1}{\sqrt{3x^2+8}}\,dx$

**Solution :** $\int \frac{1}{\sqrt{3x^2+8}}\,dx = \int \frac{1}{\sqrt{(\sqrt{3}x)^2+(\sqrt{8})^2}}\,dx$

$$= \frac{\log\left|\sqrt{3}x+\sqrt{(\sqrt{3}x)^2+(\sqrt{8})^2}\right|}{\sqrt{3}}+c$$

$$= \frac{1}{\sqrt{3}}\log\left|\sqrt{3}x+\sqrt{3x^2+8}\right|+c.$$

**9.** $\int \frac{1}{\sqrt{x^2+4x+29}}\,dx$

**Solution :** $\int \frac{1}{\sqrt{x^2+4x+29}}\,dx$

$$= \int \frac{1}{\sqrt{(x^2+4x+4)+25}}\,dx$$

$$= \int \frac{1}{\sqrt{(x+2)^2+(5)^2}}\,dx$$

$$= \log\left|(x+2)+\sqrt{(x+2)^2+(5)^2}\right|+c$$

$$= \log\left|(x+2)+\sqrt{x^2+4x+29}\right|+c.$$

**10.** $\int \frac{1}{\sqrt{3x^2-5}}\,dx$

**Solution :** $\int \frac{1}{\sqrt{3x^2-5}}\,dx = \int \frac{1}{\sqrt{(\sqrt{3}x)^2-(\sqrt{5})^2}}\,dx$

$$= \frac{\log\left|\sqrt{3}x+\sqrt{(\sqrt{3}x)^2-(\sqrt{5})^2}\right|}{\sqrt{3}}+c$$

$$= \frac{1}{\sqrt{3}}\log\left|\sqrt{3}x+\sqrt{3x^2-5}\right|+c.$$

**11.** $\int \frac{1}{\sqrt{x^2-8x-20}}\,dx$

**Solution :** $\int \frac{1}{\sqrt{x^2-8x-20}}\,dx$

$$= \int \frac{1}{\sqrt{(x^2-8x+16)-16-20}}\,dx$$

$$= \int \frac{1}{\sqrt{(x-4)^2-(6)^2}}\,dx$$

$$= \log\left|(x-4)+\sqrt{(x-4)^2-(6)^2}\right|+c$$

$$= \log\left|(x-4)+\sqrt{x^2-8x-20}\right|+c.$$

## ADDITIONAL SOLVED PROBLEMS – 5 (B)

**1. Evaluate :** $\int \frac{1}{3-10x-25x^2}\,dx.$

**Solution :** $\int \frac{1}{3-10x-25x^2}\,dx$

$$= \frac{1}{25}\int \frac{1}{\frac{3}{25}-\frac{10x}{25}-x^2}\,dx$$

$$= \frac{1}{25}\int \frac{1}{\frac{3}{25}-\left(x^2+\frac{2x}{5}+\frac{1}{25}\right)+\frac{1}{25}}\,dx$$

$$= \frac{1}{25}\int \frac{1}{\left(\frac{2}{5}\right)^2-\left(x+\frac{1}{5}\right)^2}\,dx$$

$$= \frac{1}{25}\times\frac{1}{2(2/5)}\log\left|\frac{\frac{2}{5}+x+\frac{1}{5}}{\frac{2}{5}-x-\frac{1}{5}}\right|+c$$

$$= \frac{1}{20}\log\left|\frac{2+5x+1}{2-5x-1}\right|+c$$

$$= \frac{1}{20}\log\left|\frac{3+5x}{1-5x}\right|+c$$

$$= -\frac{1}{20}\log\left|\frac{5x-1}{5x+3}\right|+c.$$

**2. Evaluate :** $\int \frac{1}{x[6(\log x)^2+7\log x+2]}\,dx.$

**Solution :** Let $I = \int \frac{1}{x[6(\log x)^2+7\log x+2]}\,dx$

Put $\log x = t$ $\quad\therefore \frac{1}{x}dx = dt$

$$\therefore I = \int \frac{1}{6t^2+7t+2}\,dt$$

$$= \frac{1}{6}\int \frac{1}{t^2+\frac{7}{6}t+\frac{2}{6}}\,dt$$

$$= \frac{1}{6}\int \frac{1}{\left(t^2+\frac{7}{6}t+\frac{49}{144}\right)-\frac{49}{144}+\frac{2}{6}}\,dt$$

$$= \frac{1}{6}\int \frac{1}{\left(t+\frac{7}{12}\right)^2-\left(\frac{1}{12}\right)^2}\,dt$$

$$=\frac{1}{6}\times\frac{1}{2\times\frac{1}{12}}\cdot\log\left|\frac{t+\frac{7}{12}-\frac{1}{12}}{t+\frac{7}{12}+\frac{1}{12}}\right|+c$$

$$=\log\left|\frac{12t+6}{12t+8}\right|+c$$

$$=\log\left|\frac{6\log x+3}{6\log x+4}\right|+c.$$

**3. Evaluate :** $\int\frac{1}{\sqrt{3x^2+5x+7}}\,dx.$

**Solution :** $\int\frac{dx}{\sqrt{3x^2+5x+7}}$

$$=\frac{1}{\sqrt{3}}\int\frac{dx}{\sqrt{x^2+(5/3)x+(7/3)}}$$

$$=\frac{1}{\sqrt{3}}\int\frac{dx}{\sqrt{\left(x^2+\frac{5x}{3}+\frac{25}{36}\right)+\left(\frac{7}{3}-\frac{25}{36}\right)}}$$

$$=\frac{1}{\sqrt{3}}\int\frac{dx}{\sqrt{\left(x+\frac{5}{6}\right)^2+\left(\frac{\sqrt{59}}{6}\right)^2}}$$

$$=\frac{1}{\sqrt{3}}\log\left|\left(x+\frac{5}{6}\right)+\sqrt{\left(x+\frac{5}{6}\right)^2+\left(\frac{\sqrt{59}}{6}\right)^2}\right|+c$$

$$=\frac{1}{\sqrt{3}}\log\left|\left(x+\frac{5}{6}\right)+\sqrt{x^2+\frac{5}{3}x+\frac{7}{3}}\right|+c.$$

**4. Evaluate :** $\int\frac{1}{\sqrt{(x-a)(x-b)}}\,dx.$

**Solution :** Let $I=\int\frac{1}{\sqrt{(x-a)(x-b)}}\,dx$

$(x-a)(x-b)=x^2-ax-bx+ab$

$$=\left[x^2-(a+b)x+\left(\frac{a+b}{2}\right)^2\right]+ab-\left(\frac{a+b}{2}\right)^2$$

$$=\left(x-\frac{a+b}{2}\right)^2+\left(\frac{4ab-a^2-b^2-2ab}{4}\right)$$

$$=\left(x-\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2$$

$$\therefore\ I=\int\frac{dx}{\sqrt{\left(x-\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2}}$$

$$=\log\left|\left(x-\frac{a+b}{2}\right)+\sqrt{\left(x-\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2}\right|+c$$

$$=\log\left|\left(x-\frac{a+b}{2}\right)+\sqrt{(x-a)(x-b)}\right|+c.$$

**5. Evaluate :** $\int\frac{1}{x^{2/3}\sqrt{x^{2/3}-4}}\,dx.$

**Solution :** Let $I=\int\frac{1}{x^{2/3}\sqrt{x^{2/3}-4}}\,dx$

$$=\int\frac{x^{-2/3}}{\sqrt{(x^{1/3})^2-4}}\,dx$$

Put $x^{\frac{1}{3}}=t\quad\therefore\ \frac{1}{3}x^{-\frac{2}{3}}dx=dt$

$\therefore\ x^{-\frac{2}{3}}dx=3\cdot dt$

$$\therefore\ I=\int\frac{1}{\sqrt{t^2-4}}\cdot 3dt=3\int\frac{1}{\sqrt{t^2-4}}\,dt$$

$$=3\log|t+\sqrt{t^2-4}|+c$$

$$=3\log|x^{\frac{1}{3}}+\sqrt{x^{2/3}-4}|+c.$$

**6. Evaluate :** $\int\frac{x-1}{3x^2-4x-3}\,dx.$

**Solution :** Let $I=\int\frac{x-1}{3x^2-4x-3}\,dx$

Let $x-1=A\left[\frac{d}{dx}(3x^2-4x-3)\right]+B$

$$=A(6x-4)+B=6Ax+(-4A+B)$$

Comparing the coefficient of $x$ and constant on both sides, we get

$6A=1$ and $-4A+B=-1$

$\therefore\ A=\frac{1}{6}$ and $-4\left(\frac{1}{6}\right)+B=-1\quad\therefore\ B=-\frac{1}{3}$

$$\therefore\ I=\int\frac{\frac{1}{6}(6x-4)-\frac{1}{3}}{3x^2-4x-3}\,dx$$

$$=\frac{1}{6}\int\frac{6x-4}{3x^2-4x-3}\,dx-\frac{1}{3}\int\frac{1}{3x^2-4x-3}\,dx$$

$$=\frac{1}{6}I_1-\frac{1}{3}I_2$$

$I_1$ is of the type $\int\frac{f'(x)}{f(x)}\,dx=\log|f(x)|+c$

$\therefore\ I_1=\log|3x^2-4x-3|+c_1$

$$I_2=\int\frac{1}{3x^2-4x-3}\,dx=\frac{1}{3}\int\frac{1}{x^2-\frac{4}{3}x-1}\,dx$$

$$= \frac{1}{3}\int \frac{1}{\left(x^2 - \frac{4}{3}x + \frac{4}{9}\right) - \frac{4}{9} - 1}\,dx$$

$$= \frac{1}{3}\int \frac{1}{\left(x - \frac{2}{3}\right)^2 - \left(\frac{\sqrt{13}}{3}\right)^2}\,dx$$

$$= \frac{1}{3} \times \frac{1}{2 \times \frac{\sqrt{13}}{3}} \log \left| \frac{x - \frac{2}{3} - \frac{\sqrt{13}}{3}}{x - \frac{2}{3} + \frac{\sqrt{13}}{3}} \right| + c_2$$

$$= \frac{1}{2\sqrt{13}} \log \left| \frac{3x - 2 - \sqrt{13}}{3x - 2 + \sqrt{13}} \right| + c_2$$

$$\therefore\ I = \frac{1}{6}\log|3x^2 - 4x - 3| - \frac{1}{6\sqrt{13}} \log \left| \frac{3x - 2 - \sqrt{13}}{3x - 2 + \sqrt{13}} \right| + c,$$

where $c = c_1 + c_2$.

**7. Evaluate :** $\int \frac{x^2 + 5x + 3}{x^2 + 3x + 2}\,dx$.

**Solution :** Let $I = \int \frac{x^2 + 5x + 3}{x^2 + 3x + 2}\,dx$

$$= \int \frac{(x^2 + 3x + 2) + (2x + 1)}{x^2 + 3x + 2}\,dx$$

$$= \int \left[1 + \frac{2x + 1}{x^2 + 3x + 2}\right] dx$$

$$= \int 1\,dx + \int \frac{2x + 1}{x^2 + 3x + 2}\,dx$$

Let $2x + 1 = A\left[\frac{d}{dx}(x^2 + 3x + 2)\right] + B$

$$= A(2x + 3) + B = 2Ax + (3A + B)$$

Comparing the coefficient of $x$ and constant on both sides, we get

$2A = 2$ and $3A + B = 1$

$\therefore\ A = 1$ and $3(1) + B = 1 \quad \therefore\ B = -2$

$$\therefore\ I = \int 1\,dx + \int \frac{(2x + 3) - 2}{x^2 + 3x + 2}\,dx$$

$$= \int 1\,dx + \int \frac{2x + 3}{x^2 + 3x + 2}\,dx - 2\int \frac{1}{x^2 + 3x + 2}\,dx$$

$$= \int 1\,dx + \int \frac{2x + 3}{x^2 + 3x + 2}\,dx - 2\int \frac{1}{\left(x^2 + 3x + \frac{9}{4}\right) - \frac{9}{4} + 2}\,dx$$

$$= \int 1\,dx + \int \frac{2x + 3}{x^2 + 3x + 2}\,dx - 2\int \frac{1}{\left(x + \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\,dx$$

$$= x + \log|x^2 + 3x + 2| - 2 \times \frac{1}{2\left(\frac{1}{2}\right)} \log \left| \frac{x + \frac{3}{2} - \frac{1}{2}}{x + \frac{3}{2} + \frac{1}{2}} \right| + c$$

$$\ldots \left[\because\ \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + c\right]$$

$$= x + \log|x^2 + 3x + 2| - 2\log\left|\frac{2x + 2}{2x + 4}\right| + c$$

$$= x + \log|x^2 + 3x + 2| - 2\log\left|\frac{x + 1}{x + 2}\right| + c.$$

**8. Evaluate :** $\int \frac{x}{\sqrt{x^2 + 6x + 10}}\,dx$.

**Solution :** Let $I = \int \frac{x}{\sqrt{x^2 + 6x + 10}}\,dx$

Let $x = A\left[\frac{d}{dx}(x^2 + 6x + 10)\right] + B$

$$= A(2x + 6) + B = 2Ax + (6A + B)$$

Comparing the coefficient of $x$ and constant on both sides, we get

$2A = 1$ and $6A + B = 0$

$\therefore\ A = \frac{1}{2}$ and $6\left(\frac{1}{2}\right) + B = 0 \quad \therefore\ B = -3$

$$\therefore\ I = \int \frac{\frac{1}{2}(2x + 6) - 3}{\sqrt{x^2 + 6x + 10}}\,dx$$

$$= \frac{1}{2}\int \frac{2x + 6}{\sqrt{x^2 + 6x + 10}}\,dx - 3\int \frac{1}{\sqrt{x^2 + 6x + 10}}\,dx$$

$$= \frac{1}{2}I_1 - 3I_2.$$

In $I_1$, put $x^2 + 6x + 10 = t \quad \therefore\ (2x + 6)\,dx = dt$.

$$\therefore\ I_1 = \int \frac{dt}{\sqrt{t}} = \int t^{-\frac{1}{2}}\,dt$$

$$= \frac{t^{\frac{1}{2}}}{(1/2)} + c_1 = 2\sqrt{x^2 + 6x + 10} + c_1$$

$$I_2 = \int \frac{1}{\sqrt{(x^2 + 6x + 9) + 1}}\,dx$$

$$= \int \frac{1}{\sqrt{(x + 3)^2 + (1)^2}}\,dx$$

$$= \log\left|(x + 3) + \sqrt{(x + 3)^2 + (1)^2}\right| + c_2$$

$$= \log\left|(x + 3) + \sqrt{x^2 + 6x + 10}\right| + c_2$$

$$\therefore\ I = \sqrt{x^2 + 6x + 10} - 3\log\left|(x + 3) + \sqrt{x^2 + 6x + 10}\right| + c,$$

where $c = c_1 + c_2$.

**9. Evaluate :** $\int \frac{2x+1}{\sqrt{x^2+4x+3}}\,dx.$

**Solution :** Let $I = \int \frac{2x+1}{\sqrt{x^2+4x+3}}\,dx$

$= \int \frac{(2x+4)-3}{\sqrt{x^2+4x+3}}\,dx$

$= \int \frac{2x+4}{\sqrt{x^2+4x+3}}\,dx - 3\int \frac{1}{\sqrt{x^2+4x+3}}\,dx$

$= I_1 - 3I_2$

In $I_1$, put $x^2+4x+3=t$

$\therefore\ (2x+4)\,dx = dt$

$\therefore\ I_1 = \int \frac{1}{\sqrt{t}}\,dt = \int t^{-\frac{1}{2}}dt$

$= \frac{t^{\frac{1}{2}}}{(1/2)} + c_1 = 2\sqrt{x^2+4x+3} + c_1$

$I_2 = \int \frac{1}{\sqrt{(x^2+4x+4)-1}}\,dx$

$= \int \frac{1}{\sqrt{(x+2)^2-(1)^2}}\,dx$

$= \log|(x+2)+\sqrt{(x+2)^2-(1)^2}| + c_2$

$= \log|(x+2)+\sqrt{x^2+4x+3}| + c_2$

$\therefore\ I = 2\sqrt{x^2+4x+3} - 3\log|(x+2)+\sqrt{x^2+4x+3}| + c,$ where $c = c_1 + c_2$.

**10. Evaluate :** $\int \sqrt{\frac{x-5}{x-7}}\,dx.$

**Solution :** Let $I = \int \sqrt{\frac{x-5}{x-7}}\,dx = \int \sqrt{\frac{x-5}{x-7} \times \frac{x-5}{x-5}}\,dx$

$= \int \frac{x-5}{\sqrt{x^2-12x+35}}\,dx$

Let $x-5 = A\left[\frac{d}{dx}(x^2-12x+35)\right] + B$

$= A(2x-12) + B$

$\therefore\ x-5 = 2Ax + (-12A+B)$

Comparing the coefficient of $x$ and constant on both the sides, we get

$2A = 1$ and $-12A + B = -5$

$\therefore\ A = \frac{1}{2}$ and $-12\left(\frac{1}{2}\right) + B = -5 \quad \therefore\ B = 1$

$\therefore\ x-5 = \frac{1}{2}(2x-12) + 1$

$\therefore\ I = \int \frac{\frac{1}{2}(2x-12)+1}{\sqrt{x^2-12x+35}}\,dx$

$= \frac{1}{2}\int \frac{2x-12}{\sqrt{x^2-12x+35}}\,dx + \int \frac{1}{\sqrt{x^2-12x+35}}\,dx$

$= \frac{1}{2}I_1 + I_2$

In $I_1$, put $x^2-12x+35 = t \quad \therefore\ (2x-12)dx = dt$

$\therefore\ I = \int \frac{1}{\sqrt{t}}\,dt = \int t^{-\frac{1}{2}}dt$

$= \frac{t^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} + c_1 = 2\sqrt{x^2-12x+35} + c_1$

$I_2 = \int \frac{1}{\sqrt{(x^2-12x+36)-1}}\,dx$

$= \int \frac{1}{\sqrt{(x-6)^2-(1)^2}}\,dx$

$= \log|(x-6)+\sqrt{(x-6)^2-(1)^2}| + c_2$

$= \log|(x-6)+\sqrt{x^2-12x+35}| + c_2$

$\therefore\ I = \sqrt{x^2-12x+35} + \log|(x-6)+\sqrt{x^2-12x+35}| + c,$ where $c = c_1 + c_2$.

## EXAMPLES FOR PRACTICE 5.4

Evaluate the following integrals :

**1.** (1) $\int \frac{1}{4-9x^2}\,dx$ (2) $\int \frac{1}{9x^2-4}\,dx$

(3) $\int \frac{1}{4x^2-1}\,dx.$

**2.** (1) $\int \frac{1}{5x^2-4x-1}\,dx$ (2) $\int \frac{1}{2x^2+x-1}\,dx$

(3) $\int \frac{1}{1+x-x^2}\,dx$ (4) $\int \frac{1}{3x^2+13x-10}\,dx$

(5) $\int \frac{1}{7+6x-x^2}\,dx$ (6) $\int \frac{1}{9x^2+6x-8}\,dx.$

**3.** (1) $\int \frac{1}{4e^x-9e^{-x}}\,dx$ (2) $\int \frac{x^2}{x^6-4x^3-12}\,dx$

(3) $\int \frac{e^x}{e^{2x}+6e^x+5}\,dx.$

**4.** (1) $\int \frac{1}{\sqrt{9x^2+25}}\,dx$ (2) $\int \frac{1}{\sqrt{4x^2-9}}\,dx$

(3) $\int \frac{1}{x\sqrt{(\log x)^2-5}}\,dx.$

**5.** (1) $\int \frac{dx}{\sqrt{x^2+4x+5}}$ (2) $\int \frac{dx}{\sqrt{2x^2-4x+7}}$

(3) $\int \frac{dx}{\sqrt{x^2+8x+25}}$ (4) $\int \frac{dx}{\sqrt{3x^2+5x+7}}$

(5) $\int \frac{dx}{\sqrt{x^2-4x-5}}$ (6) $\int \frac{dx}{\sqrt{3x^2+4x-7}}$

(7) $\int \frac{dx}{\sqrt{x^2+6x+5}}$ (8) $\int \frac{dx}{\sqrt{3x^2-4x+2}}$

(9) $\int \frac{1}{\sqrt{(x-2)(x-3)}}\,dx$ (10) $\int \frac{1}{\sqrt{2x^2+3x+5}}\,dx$.

**6.** (1) $\int \frac{x^2}{\sqrt{x^6+4x^3+13}}$ (2) $\int \frac{5a^x}{\sqrt{3a^{2x}-a^x+3}}\,dx$

(3) $\int \frac{dx}{x\sqrt{9(\log x)^2-1}}$ (4) $\int \frac{x^2}{\sqrt{x^6+2x^3+3}}\,dx.$

**7.** (1) $\int \frac{2x}{x^2-6x+6}\,dx$ (2) $\int \frac{4x+3}{x^2+5x+1}\,dx$

(3) $\int \frac{x-1}{3+4x-3x^2}\,dx$ (4) $\int \frac{3x-5}{5x^2-8x-11}\,dx$

(5) $\int \frac{x^3+x+1}{x^2-1}\,dx.$

**8.** (1) $\int \sqrt{\frac{x+1}{x+3}}\,dx$ (2) $\int \sqrt{\frac{1+x}{x}}\,dx$

(3) $\int \frac{3x+2}{\sqrt{2x^2+2x+1}}\,dx$ (4) $\int \frac{5x+2}{\sqrt{3x^2+4x+5}}\,dx$

(5) $\int \frac{2x+1}{\sqrt{x^2+2x+1}}\,dx$ (6) $\int \sqrt{\frac{x+1}{x+2}}\,dx.$

## Answers

**1.** (1) $\frac{1}{12}\log\left|\frac{2+3x}{2-3x}\right|+c$ (2) $\frac{1}{12}\log\left|\frac{3x-2}{3x+2}\right|+c$

(3) $\frac{1}{4}\log\left|\frac{2x-1}{2x+1}\right|+c.$

**2.** (1) $\frac{1}{6}\log\left|\frac{5x-5}{5x+1}\right|+c$ (2) $\frac{1}{3}\log\left|\frac{2x-1}{2x+2}\right|+c$

(3) $\frac{1}{\sqrt{5}}\log\left|\frac{\sqrt{5}-1+2x}{\sqrt{5}+1-2x}\right|+c$

(4) $\frac{1}{17}\log\left|\frac{3x-2}{3x+15}\right|+c$ (5) $\frac{1}{8}\log\left|\frac{1+x}{7-x}\right|+c$

(6) $\frac{1}{18}\log\left|\frac{3x-2}{3x+4}\right|+c.$

**3.** (1) $\frac{1}{12}\log\left|\frac{2e^x-3}{2e^x+3}\right|+c$ (2) $\frac{1}{24}\log\left|\frac{x^3-6}{x^3+2}\right|+c$

(3) $\frac{1}{4}\log\left|\frac{e^x+1}{e^x+5}\right|+c.$

**4.** (1) $\frac{1}{3}\log\left|x+\sqrt{x^2+\frac{25}{9}}\right|+c$ (2) $\frac{1}{2}\log\left|x+\sqrt{x^2-\frac{9}{4}}\right|+c$

(3) $\log|\log x+\sqrt{(\log x)^2-5}|+c.$

**5.** (1) $\log|(x+2)+\sqrt{x^2+4x+5}|+c$

(2) $\frac{1}{\sqrt{2}}\log\left|(x-1)+\sqrt{x^2-2x+\frac{7}{2}}\right|+c$

(3) $\log|(x+4)+\sqrt{x^2+8x+25}|+c$

(4) $\frac{1}{\sqrt{3}}\log\left|\left(x+\frac{5}{6}\right)+\sqrt{x^2+\frac{5x}{3}+\frac{7}{3}}\right|+c$

(5) $\log|(x-2)+\sqrt{x^2-4x-5}|+c$

(6) $\frac{1}{\sqrt{3}}\log\left|\left(x+\frac{2}{3}\right)+\sqrt{x^2+\frac{4x}{3}-\frac{7}{3}}\right|+c$

(7) $\log|(x+3)+\sqrt{x^2+6x+5}|+c$

(8) $\frac{1}{\sqrt{3}}\log\left|\left(x-\frac{2}{3}\right)+\sqrt{x^2-\frac{4x}{3}+\frac{2}{3}}\right|+c$

(9) $\log\left|\left(x-\frac{5}{2}\right)+\sqrt{x^2-5x+6}\right|+c$

(10) $\frac{1}{\sqrt{2}}\log\left|\left(x+\frac{3}{4}\right)+\sqrt{x^2+\frac{3}{2}x+\frac{5}{2}}\right|+c.$

**6.** (1) $\frac{1}{3}\log|(x^3+2)+\sqrt{x^6+4x^3+13}|+c$

(2) $\frac{5}{\sqrt{3}\log a}\log\left|\left(a^x-\frac{1}{6}\right)+\sqrt{a^{2x}-\frac{a^x}{3}+1}\right|+c$

(3) $\frac{1}{3}\log|3\log x+\sqrt{9(\log x)^2-1}|+c$

(4) $\frac{1}{3}\log|(x^3+1)+\sqrt{x^6+2x^3+3}|+c.$

**7.** (1) $\log|x^2-6x+6|+\sqrt{3}\log\left|\frac{x-3-\sqrt{3}}{x-3+\sqrt{3}}\right|+c$

(2) $2\log|x^2+5x+1|-\frac{7}{\sqrt{21}}\log\left|\frac{2x+5-\sqrt{21}}{2x+5+\sqrt{21}}\right|+c$

(3) $-\frac{1}{6}\log|3+4x-3x^2|-\frac{1}{6\sqrt{13}}\log\left|\frac{\sqrt{13}-2+3x}{\sqrt{13}+2-3x}\right|+c$

(4) $\frac{3}{10}\log|5x^2-8x-11|-\frac{13}{10\sqrt{71}}\log\left|\frac{5x-4-\sqrt{71}}{5x-4+\sqrt{71}}\right|+c$

(5) $\frac{x^2}{2}+\log|x^2-1|+\frac{1}{2}\log\left|\frac{x-1}{x+1}\right|+c.$

**8.** (1) $\sqrt{x^2+4x+3}-\log|(x+2)+\sqrt{x^2+4x+3}|+c$

(2) $\sqrt{x^2+x}+\frac{1}{2}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x}\right|+c$

(3) $\frac{3}{2}\sqrt{2x^2+2x+1}+\frac{1}{2\sqrt{2}}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+\frac{1}{2}}\right|+c$

(4) $\frac{5}{3}\sqrt{3x^2+4x-5}-\frac{4}{3\sqrt{3}}\log\left|\left(x+\frac{2}{3}\right)+\sqrt{x^2+\frac{4x}{3}+\frac{5}{3}}\right|+c$

(5) $2\sqrt{x^2+2x+1}-\log|x+1|+c$

(6) $\sqrt{x^2+3x+2}-\frac{1}{2}\log\left|\left(x+\frac{3}{2}\right)+\sqrt{x^2+3x+2}\right|+c.$

## 5.5 : INTEGRATION BY PARTS

A function to be integrated is called an *integrand*. If the integrand is a product of two functions, one of which is the derivative of some function (i.e. whose integral is known), then we use the rule of integration by parts which is given below.

**Result :** If $u$ and $v$ are functions of $x$, then

$$\int uv\,dx = u\int v\,dx-\int\left(\frac{du}{dx}\cdot\int v\,dx\right)dx.$$

**Rule for the proper choice of the first and second function :**

When the integrand is a product of two functions, out of which the second can be easily integrated (i.e. whose integral is known), we can use the rule of integration by parts. Hence, we should make a proper choice of the first function and second function.

Let us denote the different kinds of functions as :

$L$ : Logarithmic

$A$ : Algebraic

$E$ : Exponential

The order in which the two functions are to be taken as the first function and the second function, is according to the serial order of the letters of the word *LAE*.

**Remark :** A logarithmic function can be integrated by using $\int uv dx$, taking $v=1$.

### Integral of the type $\int e^x[f(x)+f'(x)]dx$ :

**Result :** If $f(x)$ is derivable function of $x$, then $\int e^x[f(x)+f'(x)]\,dx=e^x\cdot f(x)+c.$

**Remark :** This result can be used when the multiple of $e^x$ can be expressed as $f(x)+f'(x)$.

### Integrals of the type $\int\sqrt{x^2+a^2}\,dx$ and $\int\sqrt{x^2-a^2}\,dx$ :

1. $\int\sqrt{x^2+a^2}\,dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log|x+\sqrt{x^2+a^2}|+c$

2. $\int\sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log|x+\sqrt{x^2-a^2}|+c.$

### Integrals of the form $\int\sqrt{ax^2+bx+c}\,dx$ :

In this case, we express $ax^2+bx+c$ in any one of the forms $\sqrt{x^2+a^2}$, $\sqrt{x^2-a^2}$ by method of completion of square.

i.e. $ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$

$=a\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}+\frac{c}{a}-\frac{b^2}{4a^2}\right)$

$=a\left[\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right].$

**Note :** Students are reminded that if $x$ is replaced by $ax+b$ in integrals of standard functions, then in standard formula we have to replace $x$ by $ax+b$ and divide the obtained answer by $a$.

### Itegral of the form $\int(px+q)\sqrt{ax^2+bx+c}\,dx$ :

In this case, we write $px+q=A\left[\frac{d}{dx}(ax^2+bx+c)\right]+B$ where $A$ and $B$ are constants.

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|---|---|

**Evaluate the following :**

**1.** $\int x\log x\,dx$

**Solution :** $\int x\log x\,dx=\int(\log x)\cdot x\,dx$

$=(\log x)\int x\,dx-\int\left[\frac{d}{dx}(\log x)\int x\,dx\right]dx$

$=(\log x)\cdot\frac{x^2}{2}-\int\frac{1}{x}\cdot\frac{x^2}{2}\,dx$

$=\frac{1}{2}x^2\log x-\frac{1}{2}\int x\,dx$

$=\frac{x^2}{2}\log x-\frac{1}{2}\cdot\frac{x^2}{2}+c=\frac{x^2}{2}\log x-\frac{x^2}{4}+c.$

**2.** $\int x^2 e^{4x} dx$

**Solution :**

$$\int x^2 e^{4x}\,dx = x^2 \int e^{4x}\,dx - \int\left[\frac{d}{dx}(x^2)\int e^{4x}\,dx\right]dx$$

$$= x^2 \cdot \frac{e^{4x}}{4} - \int 2x \cdot \frac{e^{4x}}{4}\,dx$$

$$= \frac{1}{4}x^2 e^{4x} - \frac{1}{2}\int x\,e^{4x}\,dx$$

$$= \frac{1}{4}x^2 e^{4x} - \frac{1}{2}\left[x\int e^{4x}\,dx - \int\left\{\frac{d}{dx}(x)\int e^{4x}\,dx\right\}dx\right]$$

$$= \frac{1}{4}x^2 e^{4x} - \frac{1}{2}\left[x \cdot \frac{e^{4x}}{4} - \int 1 \cdot \frac{e^{4x}}{4}\,dx\right]$$

$$= \frac{1}{4}x^2 e^{4x} - \frac{1}{8}x \cdot e^{4x} + \frac{1}{8}\int e^{4x}\,dx$$

$$= \frac{1}{4}x^2 e^{4x} - \frac{1}{8}x\,e^{4x} + \frac{1}{8}\cdot\frac{e^{4x}}{4} + c$$

$$= \frac{1}{4}e^{4x}\left[x^2 - \frac{x}{2} + \frac{1}{8}\right] + c.$$

**3.** $\int x^2 e^{3x} dx$

**Solution :** *Refer to the solution of Q. 2.*

**Ans.** $\frac{1}{3}x^2 e^{3x} - \frac{2}{9}x e^{3x} + \frac{2}{27}e^{3x} + c.$

**4.** $\int x^3 e^{x^2} dx$

**Solution :** Let $I = \int x^3 e^{x^2}\,dx = \int x^2 e^{x^2} \cdot x\,dx$

Put $x^2 = t$ $\quad\therefore\ 2x\,dx = dt$

$\therefore\ x\,dx = \frac{dt}{2}$

$$\therefore\ I = \int t e^t \cdot \frac{dt}{2} = \frac{1}{2}\int t e^t dt$$

$$= \frac{1}{2}\left[t\int e^t dt - \int\left\{\frac{d}{dt}(t)\int e^t\,dt\right\}dt\right]$$

$$= \frac{1}{2}[te^t - \int 1 \cdot e^t dt]$$

$$= \frac{1}{2}[te^t - e^t] + c$$

$$= \frac{1}{2}(t-1)e^t + c$$

$$= \frac{1}{2}(x^2-1)e^{x^2} + c.$$

**5.** $\int e^x\left(\frac{1}{x} - \frac{1}{x^2}\right) dx$

**Solution :** Let $I = \int e^x\left(\frac{1}{x} - \frac{1}{x^2}\right) dx$

Put $f(x) = \frac{1}{x}$. Then $f'(x) = -\frac{1}{x^2}$

$$\therefore\ I = \int e^x - [f(x) + f'(x)]\,dx$$

$$= e^x \cdot f(x) + c = e^x \cdot \frac{1}{x} + c.$$

**6.** $\int e^x \cdot \frac{x}{(x+1)^2}\,dx$

**Solution :**

Let $I = \int e^x \cdot \frac{x}{(x+1)^2}\,dx$

$$= \int e^x\left[\frac{(x+1)-1}{(x+1)^2}\right]dx$$

$$= \int e^x\left[\frac{1}{x+1} - \frac{1}{(x+1)^2}\right]dx$$

Put $f(x) = \frac{1}{x+1}$

Then $f'(x) = \frac{d}{dx}(x+1)^{-1} = -1(x+1)^{-2} \cdot \frac{d}{dx}(x+1)$

$$= \frac{-1}{(x+1)^2} \times (1+0) = \frac{-1}{(x+1)^2}$$

$$\therefore\ I = \int e^x[f(x) + f'(x)]\,dx$$

$$= e^x \cdot f(x) + c = e^x \cdot \frac{1}{x+1} + c.$$

**7.** $\int e^x \cdot \frac{x-1}{(x+1)^3}\,dx$

**Solution :** Let $I = \int e^x \cdot \frac{x-1}{(x+1)^3}\,dx$

$$= \int e^x\left[\frac{(x+1)-2}{(x+1)^3}\right]dx$$

$$= \int e^x\left[\frac{1}{(x+1)^2} - \frac{2}{(x+1)^3}\right]dx$$

Put $f(x) = \frac{1}{(x+1)^2}$

Then $f'(x) = \frac{d}{dx}(x+1)^{-2} = -2(x+1)^{-3} \cdot \frac{d}{dx}(x+1)$

$$= \frac{-2}{(x+1)^3} \times (1+0) = \frac{-2}{(x+1)^3}$$

$$\therefore\ I = \int e^x\,[f(x) + f'(x)]\,dx$$

$$= e^x \cdot f(x) + c = e^x \cdot \frac{1}{(x+1)^2} + c.$$

8. $\int e^x\left[(\log x)^2+\frac{2\log x}{x}\right]dx$

**Solution :** Let $I=\int e^x\left[(\log x)^2+\frac{2\log x}{x}\right]dx$

Put $f(x)=(\log x)^2$

Then $f'(x)=\frac{d}{dx}(\log x)^2=2\log x\cdot\frac{d}{dx}(\log x)$

$$=2\log x\times\frac{1}{x}=\frac{2\log x}{x}$$

$\therefore\ I=\int e^x[f(x)+f'(x)]dx$

$=e^x\cdot f(x)+c=e^x\cdot(\log x)^2+c.$

9. $\int\left[\frac{1}{\log x}-\frac{1}{(\log x)^2}\right]dx$

**Solution :** Let $I=\int\left[\frac{1}{\log x}-\frac{1}{(\log x)^2}\right]dx$

Put $\log x=t \qquad \therefore\ x=e^t$

$\therefore\ dx=e^t\,dt$

$\therefore\ I=\int\left(\frac{1}{t}-\frac{1}{t^2}\right)e^t\,dt$

Let $f(t)=\frac{1}{t}$. Then $f'(t)=-\frac{1}{t^2}$

$\therefore\ I=\int e^t[f(t)+f'(t)]dt$

$=e^t\cdot f(t)+c=e^t\times\frac{1}{t}+c$

$=\frac{x}{\log x}+c.$

10. $\int\frac{\log x}{(1+\log x)^2}dx$

**Solution :** Let $I=\int\frac{\log x}{(1+\log x)^2}dx$

Put $\log x=t \qquad \therefore\ x=e^t$

$\therefore\ dx=e^t\,dt$

$\therefore\ I=\int\frac{t}{(1+t)^2}\cdot e^t\,dt$

$=\int e^t\left[\frac{(1+t)-1}{(1+t)^2}\right]dt$

$=\int e^t\left[\frac{1}{1+t}-\frac{1}{(1+t)^2}\right]dt$

Let $f(t)=\frac{1}{1+t}$.

$\therefore\ f'(t)=\frac{d}{dt}(1+t)^{-1}=-1(1+t)^{-2}(0+1)$

$=\frac{-1}{(1+t)^2}$

$\therefore\ I=\int e^t[f(t)+f'(t)]dt$

$=e^t\cdot f(t)+c=e^t\times\frac{1}{1+t}+c=\frac{x}{1+\log x}+c.$

## ADDITIONAL SOLVED PROBLEMS – 5 (C)

1. **Evaluate :** $\int x\cdot2^{-3x}dx.$

**Solution :**

$\int x\cdot2^{-3x}dx=x\int2^{-3x}dx-\int\left[\frac{d}{dx}(x)\int2^{-3x}dx\right]dx$

$=x\cdot\frac{2^{-3x}}{\log2}\times\frac{1}{(-3)}-\int1\cdot\frac{2^{-3x}}{\log2}\times\frac{1}{(-3)}dx$

$=-\frac{1}{3}\cdot\frac{x\cdot2^{-3x}}{\log2}+\frac{1}{3\log2}\int2^{-3x}dx$

$=-\frac{1}{3}\cdot\frac{x\cdot2^{-3x}}{\log2}+\frac{1}{3\log2}\cdot\frac{2^{-3x}}{\log2}\times\frac{1}{(-3)}+c$

$=-\frac{x\cdot2^{-3x}}{3\cdot\log2}-\frac{2^{-3x}}{9\cdot(\log2)^2}+c.$

2. **Evaluate :** $\int\frac{\log(\log x)}{x}dx.$

**Solution :** Let $I=\int\frac{\log(\log x)}{x}dx=\int\log(\log x)\cdot\frac{1}{x}dx$

Put $\log x=t \qquad \therefore\ \frac{1}{x}dx=dt$

$\therefore\ I=\int\log t\,dt=\int(\log t)\cdot1\,dt$

$=(\log t)\int1\,dt-\int\left[\frac{d}{dt}(\log t)\int1\,dt\right]dt$

$=(\log t)\,t-\int\frac{1}{t}\times t\,dt$

$=t\log t-\int1\,dt$

$=t\log t-t+c$

$=t(\log t-1)+c$

$=(\log x)[\log(\log x)-1]+c.$

3. **Evaluate :** $\int e^x.\left[\frac{x^2+1}{(x+1)^2}\right]dx.$

**Solution :** Let $I=\int e^x.\frac{x^2+1}{(x+1)^2}dx$

$=\int e^x.\left[\frac{(x^2-1)+2}{(x+1)^2}\right]dx$

$=\int e^x.\left[\frac{x^2-1}{(x+1)^2}+\frac{2}{(x+1)^2}\right]dx$

$=\int e^x.\left[\frac{x-1}{x+1}+\frac{2}{(x+1)^2}\right]dx$

Put $f(x)=\frac{x-1}{x+1}$

$\therefore\ f'(x)=\frac{d}{dx}\left(\frac{x-1}{x+1}\right)$

$=\frac{(x+1)\cdot\frac{d}{dx}(x-1)-(x-1)\frac{d}{dx}(x+1)}{(x+1)^2}$

$=\frac{(x+1)(1)-(x-1)(1)}{(x+1)^2}$

$=\frac{x+1-x+1}{(x+1)^2}=\frac{2}{(x+1)^2}$

$\therefore\ I=\int e^x[f(x)+f'(x)]\,dx$

$=e^x\cdot f(x)+c=e^x\left(\frac{x-1}{x+1}\right)+c.$

**4. Evaluate : $\int\sqrt{4x^2+5}\,dx$.**

**Solution :** $\int\sqrt{4x^2+5}\,dx=\int 2\sqrt{x^2+\frac{5}{4}}\,dx$

$=2\int\sqrt{x^2+\left(\frac{\sqrt5}{2}\right)^2}\,dx$

$=2\left[\frac{x}{2}\sqrt{x^2+\frac{5}{4}}+\frac{(5/4)}{2}\log\left|x+\sqrt{x^2+\frac{5}{4}}\right|\right]+c_1$

$=x\cdot\frac{\sqrt{4x^2+5}}{2}+\frac{5}{4}\log\left|\frac{2x+\sqrt{4x^2+5}}{2}\right|+c_1$

$=\frac{x}{2}\sqrt{4x^2+5}+\frac{5}{4}[\log|2x+\sqrt{4x^2+5}|-\log 2]+c_1$

$=\frac{x}{2}\sqrt{4x^2+5}+\frac{5}{4}\log|2x+\sqrt{4x^2+5}|-\frac{5}{4}\log 2+c_1$

$=\frac{x}{2}\sqrt{4x^2+5}+\frac{5}{4}\log|2x+\sqrt{4x^2+5}|+c,$

where $c=c_1-\frac{5}{4}\log 2$.

**5. Evaluate : $\int\sqrt{9x^2-4}\,dx$.**

**Solution :** $\int\sqrt{9x^2-4}\,dx=\int 3\sqrt{x^2-\frac{4}{9}}\,dx$

$=3\int\sqrt{x^2-\left(\frac{2}{3}\right)^2}\,dx$

$=3\left[\frac{x}{2}\sqrt{x^2-\frac{4}{9}}-\frac{(4/9)}{2}\log\left|x+\sqrt{x^2-\frac{4}{9}}\right|\right]+c_1$

$=\frac{3x}{2}\cdot\frac{\sqrt{9x^2-4}}{3}-\frac{2}{3}\log\left|\frac{3x+\sqrt{9x^2-4}}{3}\right|+c_1$

$=\frac{x}{2}\sqrt{9x^2-4}-\frac{2}{3}[\log|3x+\sqrt{9x^2-4}|-\log 3]+c_1$

$=\frac{x}{2}\sqrt{9x^2-4}-\frac{2}{3}\log|3x+\sqrt{9x^2-4}|+\frac{2}{3}\log 3+c_1$

$=\frac{x}{2}\sqrt{9x^2-4}-\frac{2}{3}\log|3x+\sqrt{9x^2-4}|+c,$

where $c=c_1+\frac{2}{3}\log 3$.

**6. Evaluate : $\int\sqrt{2x^2+3x+4}\,dx$.**

**Solution :** $\int\sqrt{2x^2+3x+4}\,dx$

$=\sqrt2\int\sqrt{x^2+\frac{3}{2}x+2}\,dx$

$=\sqrt2\int\sqrt{\left(x^2+\frac{3}{2}x+\frac{9}{16}\right)-\frac{9}{16}+2}\,dx$

$=\sqrt2\int\sqrt{\left(x+\frac{3}{4}\right)^2+\left(\frac{\sqrt{23}}{4}\right)^2}\,dx$

$=\sqrt2\left[\frac{(x+3/4)}{2}\sqrt{\left(x+\frac{3}{4}\right)^2+\left(\frac{\sqrt{23}}{4}\right)^2}+\right.$

$\left.\frac{\left(\frac{\sqrt{23}}{4}\right)^2}{2}\log\left|\left(x+\frac{3}{4}\right)+\sqrt{\left(x+\frac{3}{4}\right)^2+\left(\frac{\sqrt{23}}{4}\right)^2}\right|\right]+c_1$

$=\sqrt2\left[\frac{(4x+3)}{8}\sqrt{x^2+\frac{3x}{2}+2}+\right.$

$\left.\frac{23}{32}\log\left|\left(x+\frac{3}{4}\right)+\sqrt{x^2+\frac{3x}{2}+2}\right|\right]+c_1$

$=\frac{(4x+3)}{8}\sqrt{2x^2+3x+4}+$

$\frac{23\sqrt2}{32}\log\left|\left(\frac{4x+3}{4}\right)+\frac{\sqrt{2x^2+3x+4}}{\sqrt2}\right|+c_1$

$=\frac{(4x+3)}{8}\sqrt{2x^2+3x+4}+$

$\frac{23\sqrt2}{32}\left[\log\left|\left(\frac{4x+3}{2\sqrt2}\right)+\sqrt{2x^2+3x+4}\right|-\log\sqrt2\right]+c_1$

$=\frac{(4x+3)}{8}\sqrt{2x^2+3x+4}+$

$\frac{23\sqrt2}{32}\log\left|\frac{(4x+3)}{2\sqrt2}+\sqrt{2x^2+3x+4}\right|-\frac{23\sqrt2}{32}\log\sqrt2+c_1$

$=\frac{(4x+3)}{8}\sqrt{2x^2+3x+4}+$

$\frac{23\sqrt2}{32}\log\left|\frac{(4x+3)}{2\sqrt2}+\sqrt{2x^2+3x+4}\right|+c,$

where $c=c_1-\frac{23\sqrt2}{32}\log\sqrt2$.

**Remark :** Answer can also be written as :

$$\frac{(4x+3)}{8}\sqrt{2x^2+3x+4}+$$
$$\frac{23\sqrt{2}}{32}\log\left|\left(x+\frac{3}{4}\right)+\sqrt{x^2+\frac{3x}{2}+2}\right|+c.$$

**7. Evaluate : $\int e^x\sqrt{e^{2x}+1}\,dx$.**

**Solution :** Let $I=\int e^x\sqrt{e^{2x}+1}\,dx=\int\sqrt{e^{2x}+1}\cdot e^x\,dx$

Put $e^x=t \quad \therefore \ e^x\,dx=dt$

$$\therefore \ I=\int\sqrt{t^2+1}\,dt$$
$$=\frac{t}{2}\sqrt{t^2+1}+\frac{1}{2}\log|t+\sqrt{t^2+1}|+c$$
$$=\frac{e^x\sqrt{e^{2x}+1}}{2}+\frac{1}{2}\log|e^x+\sqrt{e^{2x}+1}|+c.$$

**8. Evaluate : $\int(3x-2)\sqrt{x^2+x+1}\,dx$.**

**Solution :** Let $I=\int(3x-2)\sqrt{x^2+x+1}\,dx$

Let $3x-2=A\left[\frac{d}{dx}(x^2+x+1)\right]+B$

$\therefore \ 3x-2=A(2x+1)+B=2Ax+(A+B)$

Comparing the coefficient of $x$ and constant on both sides, we get

$2A=3$ and $A+B=-2$

$\therefore \ A=\frac{3}{2}$ and $\frac{3}{2}+B=-2 \quad \therefore \ B=-\frac{7}{2}$

$$\therefore \ 3x-2=\frac{3}{2}(2x+1)-\frac{7}{2}$$
$$\therefore \ I=\int\left[\frac{3}{2}(2x+1)-\frac{7}{2}\right]\sqrt{x^2+x+1}\,dx$$
$$=\frac{3}{2}\int(2x+1)\sqrt{x^2+x+1}\,dx-\frac{7}{2}\int\sqrt{x^2+x+1}\,dx$$
$$=\frac{3}{2}\int(2x+1)\sqrt{x^2+x+1}\,dx-\frac{7}{2}\int\sqrt{\left(x^2+x+\frac{1}{4}\right)+\frac{3}{4}}\,dx$$
$$=\frac{3}{2}I_1-\frac{7}{2}\int\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dx$$

In $I_1$, put $x^2+x+1=t$

$\therefore \ (2x+1)dx=dt$

$$\therefore \ I_1=\int\sqrt{t}\,dt=\int t^{\frac{1}{2}}dt$$
$$=\frac{t^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}=\frac{2}{3}(x^2+x+1)^{\frac{3}{2}}$$

$$\therefore \ I=(x^2+x+1)^{\frac{3}{2}}-\frac{7}{2}\left[\frac{(x+\frac{1}{2})}{2}\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}+\right.$$
$$\left.\frac{\left(\frac{\sqrt{3}}{2}\right)^2}{2}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right|\right]+c$$
$$=(x^2+x+1)^{\frac{3}{2}}-\frac{7}{2}\left[\frac{(2x+1)}{4}\sqrt{x^2+x+1}+\right.$$
$$\left.\frac{3}{8}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right|\right]+c$$
$$=(x^2+x+1)^{\frac{3}{2}}-\frac{7}{8}(2x+1)\sqrt{x^2+x+1}-$$
$$\frac{21}{16}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right|+c.$$

## EXAMPLES FOR PRACTICE 5.5

**Evaluate the following integrals :**

**1.** $\int x^3\log x\,dx$ **2.** $\int xe^x\,dx$

**3.** $\int xe^{-2x}\,dx$ **4.** $\int\log x\,dx$

**5.** $\int(\log x)^2dx$ **6.** $\int e^x(x^2+2x+7)\,dx$

**7.** $\int e^x\left(\frac{x\log x+1}{x}\right)dx$ **8.** $\int e^x\cdot\frac{(x+3)}{(x+4)^2}\,dx$

**9.** $\int e^x\cdot\frac{2+x}{(3+x)^2}\,dx$ **10.** $\int e^x\cdot\frac{x^2-x+1}{(x^2+1)^{\frac{3}{2}}}\,dx$

**11.** $\int e^x\frac{(1-x)^2}{(x^2+1)^2}\,dx$ **12.** $\int e^{3x}\left(\log 2x+\frac{1}{3x}\right)dx$

**13.** $\int\frac{e^x}{x}\cdot[x(\log x)^2+2\log x]\,dx$

**14.** $\int\sqrt{4x^2-5}\,dx$ **15.** $\int\sqrt{x^2+4x+5}\,dx$

**16.** $\int\sqrt{3x^2+4x+1}\,dx$ **17.** $\int\frac{\sqrt{1+(\log x)^2}}{x}\,dx$

**18.** $\int(x-5)\sqrt{x^2-1}\,dx$ **19.** $\int(x+1)\sqrt{x^2+x+1}\,dx$

**20.** $\int(2x+3)\sqrt{3x^2+2x-5}\,dx$.

### Answers

**1.** $\frac{1}{4}x^4\log x-\frac{1}{16}x^4+c$ **2.** $(x-1)e^x+c$

**3.** $-\frac{1}{2}xe^{-2x}-\frac{1}{4}e^{-2x}+c$ **4.** $x(\log x-1)+c$

**5.** $x[(\log x)^2-2\log x+2]+c$

6. $e^x(x^2+7)+c$

7. $e^x \cdot \log x + c$

8. $e^x \cdot \frac{1}{x+4}+c$

9. $\frac{e^x}{3+x}+c$

10. $\frac{e^x}{\sqrt{x^2+1}}+c$

11. $\frac{e^x}{1+x^2}+c$

12. $\frac{1}{3}e^{3x}\cdot\log(2x)+c$. Put $3x=t$

13. $e^x\cdot(\log x)^2+c$

14. $\frac{x}{2}\sqrt{4x^2-5}-\frac{5}{4}\log|2x+\sqrt{4x^2-5}|+c$

15. $\frac{(x+2)}{2}\sqrt{x^2+4x+5}+\frac{1}{2}\log|(x+2)+\sqrt{x^2+4x+5}|+c$

16. $\frac{(3x+2)}{6}\sqrt{3x^2+4x+1}-$

$$\frac{1}{6\sqrt{3}}\log\left|\frac{3x+2}{3}+\sqrt{\frac{3x^2+4x+1}{3}}\right|+c$$

17. $\frac{(\log x)\sqrt{1+(\log x)^2}}{2}+$

$$\frac{1}{2}\log|(\log x)+\sqrt{1+(\log x)^2}|+c$$

18. $\frac{1}{3}(x^2-1)^{\frac{3}{2}}-\frac{5}{2}x\sqrt{x^2-1}+\frac{5}{2}\log|x+\sqrt{x^2-1}|+c$

19. $\frac{1}{3}(x^2+x+1)^{\frac{3}{2}}+\frac{(2x+1)}{8}\sqrt{x^2+x+1}+$

$$\frac{3}{16}\log\left|\left(\frac{2x+1}{2}\right)+\sqrt{x^2+x+1}\right|+c$$

20. $\frac{2}{9}(3x^2+2x-5)^{\frac{3}{2}}+\frac{7}{18}(3x+1)\sqrt{3x^2+2x-5}-$

$$\frac{56}{9\sqrt{3}}\log\left|\frac{(3x+1)}{3}+\sqrt{\frac{3x^2+2x-5}{3}}\right|+c.$$

## 5.6 : INTEGRATION BY METHOD OF PARTIAL FRACTIONS

If $P(x)$ and $Q(x)$ are two polynomials, then the function given by $f(x)=P(x)/Q(x)$ is a rational function. Here we consider two cases :

**Case (i) :** Degree of $P(x)$ < degree of $Q(x)$

**Case (ii) :** Degree of $P(x) \geqslant$ degree of $Q(x)$

In case (ii), we divide $P(x)$ by $Q(x)$ and get the quotient $\phi(x)$ and remainder $R(x)$ whose degree is less than the degree of $Q(x)$. Hence, we can express $f(x)$ as

$$f(x)=\phi(x)+\frac{R(x)}{Q(x)}$$

Since, $\phi(x)$ is a polynomial, its integral can be found easily.

Thus in both the cases, we are interested in finding the integral of a rational function in which the degree of the numerator is less than the degree of the denominator.

We are going to consider the case when the polynomial in the denominator can be factorized into distinct linear factors. In this case we can express the rational function into its partial fractions. Let us know about partial fractions.

Observe that,

$$\frac{2}{x-1}+\frac{1}{x+3}=\frac{2(x+3)+(x-1)}{(x-1)(x+3)}=\frac{3x+5}{x^2+2x-3}$$

i.e. $\frac{3x+5}{x^2+2x-3}=\frac{2}{x-1}+\frac{1}{x+3}$

Here, $\frac{2}{x-1}$ and $\frac{1}{x+3}$ are called **partial fractions** of the rational function $\frac{3x+5}{x^2+2x-3}$.

The general method of obtaining partial fractions of a rational function is explained through an example as follows :

**1. Distinct Linear Factors :** If the denominator consists of distinct linear factors, i.e. each factor occurring only once, then corresponding to each factor $ax+b$, there corresponds a partial fraction $\frac{A}{ax+b}$ and $\frac{P(x)}{Q(x)}$ can be written as a sum of partial fractions. If there are $n$ distinct linear factors, there will be $n$ partial fractions.

**For example :** $\frac{3x+5}{x^2+2x-3}=\frac{3x+5}{(x-1)(x+3)}=\frac{A}{x-1}+\frac{B}{x+3}$,

where $A$ and $B$ are constants.

Multiplying throughout by $(x-1)(x+3)$, we get

$3x+5=A(x+3)+B(x-1)$ ... (1)

Now, we have to find the values of $A$ and $B$. Here we note that equality (1) is true for all values of $x$. So we choose such values of $x$, which will make all the terms except one on the RHS of (1) equal to zero.

This choice of values is readily suggested by the factors in the denominator. We equate each factor to zero and choose each such value of $x$. Thus we put

$x-1=0$ and $x+3=0$ and get $x=1$ and $x=-3$.

Putting $x=1$ in (1), we get

$8=A(4)+B(0) \quad \therefore \ A=2$

Putting $x=-3$ in (1), we get

$-4=A(0)+B(-4) \quad \therefore \ B=1$

$$\therefore \ \frac{3x+5}{x^2+2x-3}=\frac{2}{x-1}+\frac{1}{x+3}.$$

### 2. Repeated Linear Factors :

In the rational function $\frac{P(x)}{Q(x)}$ if $Q(x)=(x-a)^k(x-a_1)(x-a_2)\ldots(x-a_r)$, then we assume,

$$\frac{P(x)}{Q(x)}=\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\frac{A_3}{(x-a)^3}+\ldots+\frac{A_k}{(x-a)^k}+\frac{B_1}{x-a_1}+\frac{B_2}{x-a_2}+\ldots+\frac{B_r}{x-a_r},$$

where $A_1, A_2, A_3, \ldots, A_k, B_1, B_2, \ldots B_r$ are constants.

### 3. Non-repeated Quadratic Factors :

If the denominator contains non-repeated quadratic factors, then corresponding to each quadratic factor $ax^2+bx+c$, we assume the partial fraction $\frac{Ax+B}{ax^2+bx+c}$, where $A$ and $B$ are constants.

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|---|---|

**Evaluate :**

**1.** $\int \frac{2x+1}{(x+1)(x-2)}\,dx$

**Solution :** Let $I=\int \frac{2x+1}{(x+1)(x-2)}\,dx$

Let $\frac{2x+1}{(x+1)(x-2)}=\frac{A}{x+1}+\frac{B}{x-2}$

$\therefore \ 2x+1=A(x-2)+B(x+1)$

Put $x+1=0$, i.e. $x=-1$, we get

$2(-1)+1=A(-3)+B(0) \quad \therefore \ A=\frac{1}{3}$

Put $x-2=0$, i.e. $x=2$, we get

$2(2)+1=A(0)+B(3) \quad \therefore \ B=\frac{5}{3}$

$$\therefore \ \frac{2x+1}{(x+1)(x-2)}=\frac{(1/3)}{x+1}+\frac{(5/3)}{x-2}$$

$$\therefore \ I=\int\left[\frac{(1/3)}{x+1}+\frac{(5/3)}{x-2}\right]dx$$

$$=\frac{1}{3}\int\frac{1}{x+1}\,dx+\frac{5}{3}\int\frac{1}{x-2}\,dx$$

$$=\frac{1}{3}\log|x+1|+\frac{5}{3}\log|x-2|+c.$$

**2.** $\int \frac{2x+1}{x(x-1)(x-4)}\,dx$

**Solution :** Let $I=\int \frac{2x+1}{x(x-1)(x-4)}\,dx$

Let $\frac{2x+1}{x(x-1)(x-4)}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x-4}$

$\therefore \ 2x+1=A(x-1)(x-4)+Bx(x-4)+Cx(x-1)$

Put $x=0$, we get

$2(0)+1=A(-1)(-4)+B(0)(-4)+C(0)(-1)$

$\therefore \ 1=4A \quad \therefore \ A=\frac{1}{4}$

Put $x-1=0$, i.e. $x=1$, we get

$2(1)+1=A(0)(-3)+B(1)(-3)+C(1)(0)$

$\therefore \ 3=-3B \quad \therefore \ B=-1$

Put $x-4=0$, i.e $x=4$, we get

$2(4)+1=A(3)(0)+B(4)(0)+C(4)(3)$

$\therefore \ 9=12C \quad \therefore \ C=\frac{3}{4}$

$$\therefore \ \frac{2x+1}{x(x-1)(x-4)}=\frac{\left(\frac{1}{4}\right)}{x}+\frac{(-1)}{x-1}+\frac{\left(\frac{3}{4}\right)}{x-4}$$

$$\therefore \ I=\int\left[\frac{\left(\frac{1}{4}\right)}{x}+\frac{(-1)}{x-1}+\frac{\left(\frac{3}{4}\right)}{x-4}\right]dx$$

$$=\frac{1}{4}\int\frac{1}{x}\,dx-\int\frac{1}{x-1}\,dx+\frac{3}{4}\int\frac{1}{x-4}\,dx$$

$$=\frac{1}{4}\log|x|-\log|x-1|+\frac{3}{4}\log|x-4|+c.$$

**3.** $\int \frac{x^2+x-1}{x^2+x-6}\,dx$

**Solution :** Let $I=\int\frac{x^2+x-1}{x^2+x-6}\,dx$

$$=\int\frac{(x^2+x-6)+5}{x^2+x-6}\,dx$$

$$=\int\left[1+\frac{5}{x^2+x-6}\right]dx$$

$$=\int 1\,dx+5\int\frac{1}{x^2+x-6}\,dx$$

Let $\frac{1}{x^2+x-6}=\frac{1}{(x+3)(x-2)}=\frac{A}{x+3}+\frac{B}{x-2}$

$\therefore\ 1 = A(x-2) + B(x+3)$

Put $x+3=0$, i.e. $x=-3$, we get

$1 = A(-5) + B(0) \qquad \therefore\ A = \frac{-1}{5}$

Put $x-2=0$, i.e. $x=2$, we get

$1 = A(0) + B(5) \qquad \therefore\ B = \frac{1}{5}$

$\therefore\ \frac{1}{x^2+x-6} = \frac{(-1/5)}{x+3} + \frac{(1/5)}{x-2}$

$\therefore\ I = \int 1\,dx + 5\int\left[\frac{(-1/5)}{x+3} + \frac{(1/5)}{x-2}\right]dx$

$= \int 1\,dx - \int \frac{1}{x+3}\,dx + \int \frac{1}{x-2}\,dx$

$= x - \log|x+3| + \log|x-2| + c.$

**4.** $\int \frac{x}{(x-1)^2(x+2)}\,dx$

**Solution :** Let $I = \int \frac{x}{(x-1)^2(x+2)}\,dx$

Let $\frac{x}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$

$\therefore\ x = A(x-1)(x+2) + B(x+2) + C(x-1)^2$

Put $x-1=0$, i.e. $x=1$, we get

$1 = A(0)(3) + B(3) + C(0) \qquad \therefore\ B = \frac{1}{3}$

Put $x+2=0$, i.e. $x=-2$, we get

$-2 = A(-3)(0) + B(0) + C(9) \qquad \therefore\ C = -\frac{2}{9}$

Put $x=-1$, we get,

$-1 = A(-2)(1) + B(1) + C(4)$

But $B = \frac{1}{3}$ and $C = -\frac{2}{9}$

$\therefore\ -1 = -2A + \frac{1}{3} - \frac{8}{9}$

$\therefore\ 2A = -\frac{5}{9} + 1 = \frac{4}{9} \qquad \therefore\ A = \frac{2}{9}$

$\therefore\ \frac{x}{(x-1)^2(x+2)} = \frac{(2/9)}{x-1} + \frac{(1/3)}{(x-1)^2} + \frac{(-2/9)}{x+2}$

$\therefore\ I = \int\left[\frac{(2/9)}{x-1} + \frac{(1/3)}{(x-1)^2} + \frac{(-2/9)}{x+2}\right]dx$

$= \frac{2}{9}\int\frac{1}{x-1}dx + \frac{1}{3}\int(x-1)^{-2}dx - \frac{2}{9}\int\frac{1}{x+2}dx$

$= \frac{2}{9}\log|x-1| + \frac{1}{3}\cdot\frac{(x-1)^{-1}}{-1} - \frac{2}{9}\log|x+2| + c$

$= \frac{2}{9}\log\left|\frac{x-1}{x+2}\right| - \frac{1}{3(x-1)} + c.$

**5.** $\int \frac{3x-2}{(x+1)^2(x+3)}\,dx$

**Solution :** Let $I = \int \frac{3x-2}{(x+1)^2(x+3)}\,dx$

Let $\frac{3x-2}{(x+1)^2(x+3)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+3}$

$\therefore\ 3x-2 = A(x+1)(x+3) + B(x+3) + C(x+1)^2$

Put $x+1=0$, i.e. $x=-1$, we get

$3(-1)-2 = A(0)(2) + B(2) + C(0)$

$\therefore\ -5 = 2B \qquad \therefore\ B = -\frac{5}{2}$

Put $x+3=0$, i.e. $x=-3$, we get

$3(-3)-2 = A(-2)(0) + B(0) + C(4)$

$\therefore\ -11 = 4C \qquad \therefore\ C = -\frac{11}{4}$

Put $x=0$, we get

$3(0)-2 = A(1)(3) + B(3) + C(1)$

$\therefore\ -2 = 3A + 3B + C$

But $B = -\frac{5}{2}$ and $C = -\frac{11}{4}$

$\therefore\ -2 = 3A + 3\left(-\frac{5}{2}\right) - \frac{11}{4}$

$\therefore\ 3A = -2 + \frac{15}{2} + \frac{11}{4} = \frac{-8+30+11}{4} = \frac{33}{4}$

$\therefore\ A = \frac{11}{4}$

$\therefore\ \frac{3x-2}{(x+1)^2(x+3)} = \frac{\left(\frac{11}{4}\right)}{x+1} + \frac{\left(-\frac{5}{2}\right)}{(x+1)^2} + \frac{\left(-\frac{11}{4}\right)}{(x+3)}$

$\therefore\ I = \int\left[\frac{\left(\frac{11}{4}\right)}{x+1} + \frac{\left(-\frac{5}{2}\right)}{(x+1)^2} + \frac{\left(-\frac{11}{4}\right)}{x+3}\right]dx$

$= \frac{11}{4}\int\frac{1}{x+1}\,dx - \frac{5}{2}\int(x+1)^{-2}\,dx - \frac{11}{4}\int\frac{1}{x+3}\,dx$

$= \frac{11}{4}\log|x+1| - \frac{5}{2}\cdot\frac{(x+1)^{-1}}{-1} - \frac{11}{4}\log|x+3| + c$

$= \frac{11}{4}\log\left|\frac{x+1}{x+3}\right| + \frac{5}{2(x+1)} + c.$

**6.** $\int \frac{1}{x(x^5+1)}\,dx$

**Solution :** Let $I = \int \frac{1}{x(x^5+1)}\,dx$

$= \int \frac{x^4}{x^5(x^5+1)}\,dx$

Put $x^5 = t$. Then $5x^4\,dx = dt$

$\therefore\ x^4\,dx = \frac{dt}{5}$

$$\therefore\ I = \int \frac{1}{t(t+1)}\cdot\frac{dt}{5}$$

$$= \frac{1}{5}\int \frac{(t+1)-t}{t(t+1)}\,dt$$

$$= \frac{1}{5}\int\left(\frac{1}{t}-\frac{1}{t+1}\right)dt$$

$$= \frac{1}{5}\left[\int\frac{1}{t}dt - \int\frac{1}{t+1}dt\right]$$

$$= \frac{1}{5}[\log|t| - \log|t+1|] + c$$

$$= \frac{1}{5}\log\left|\frac{t}{t+1}\right| + c = \frac{1}{5}\log\left|\frac{x^5}{x^5+1}\right| + c.$$

**7.** $\displaystyle\int \frac{1}{x(x^n+1)}\,dx$

**Solution :** Let $I = \displaystyle\int \frac{1}{x(x^n+1)}\,dx$

$$= \int \frac{x^{n-1}}{x^n(x^n+1)}\,dx$$

Put $x^n = t$ $\quad\therefore\ nx^{n-1}\,dx = dt$

$\therefore\ x^{n-1}\,dx = \frac{dt}{n}$

$$\therefore\ I = \int \frac{1}{t(t+1)}\cdot\frac{dt}{n}$$

$$= \frac{1}{n}\int \frac{(t+1)-t}{t(t+1)}\,dt$$

$$= \frac{1}{n}\int\left(\frac{1}{t}-\frac{1}{t+1}\right)dt$$

$$= \frac{1}{n}\left[\int\frac{1}{t}dt - \int\frac{1}{t+1}dt\right]$$

$$= \frac{1}{n}[\log|t| - \log|t+1|] + c$$

$$= \frac{1}{n}\log\left|\frac{t}{t+1}\right| + c = \frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right| + c.$$

**8.** $\displaystyle\int \frac{5x^2+20x+6}{x^3+2x^2+x}\,dx$

**Solution :** Let $I = \displaystyle\int \frac{5x^2+20x+6}{x^3+2x^2+x}\,dx$

$$= \int \frac{5x^2+20x+6}{x(x^2+2x+1)}\,dx$$

$$= \int \frac{5x^2+20x+6}{x(x+1)^2}\,dx$$

Let $\dfrac{5x^2+20x+6}{x(x+1)^2} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$

$\therefore\ 5x^2+20x+6 = A(x+1)^2 + Bx(x+1) + Cx$

Put $x = 0$, we get

$0+0+6 = A(1) + B(0)(1) + C(0) \qquad \therefore\ A = 6$

Put $x+1 = 0$, i.e. $x = -1$, we get

$5(1) + 20(-1) + 6 = A(0) + B(-1)(0) + C(-1)$

$\therefore\ -9 = -C \qquad \therefore\ C = 9$

Put $x = 1$, we get

$5(1) + 20(1) + 6 = A(4) + B(1)(2) + C(1)$

But $A = 6$ and $C = 9$

$\therefore\ 31 = 24 + 2B + 9 \qquad \therefore\ B = -1$

$$\therefore\ \frac{5x^2+20x+6}{x(x+1)^2} = \frac{6}{x} - \frac{1}{x+1} + \frac{9}{(x+1)^2}$$

$$\therefore\ I = \int\left[\frac{6}{x} - \frac{1}{x+1} + \frac{9}{(x+1)^2}\right]dx$$

$$= 6\int\frac{1}{x}dx - \int\frac{1}{x+1}dx + 9\int(x+1)^{-2}\,dx$$

$$= 6\log|x| - \log|x+1| + 9.\frac{(x+1)^{-1}}{-1} + c$$

$$= 6\log|x| - \log|x+1| - \frac{9}{x+1} + c.$$

## ADDITIONAL SOLVED PROBLEMS – 5 (D)

**1. Evaluate :** $\displaystyle\int \frac{1}{x^3(1-x)}\,dx.$

**Solution :** Let $I = \displaystyle\int \frac{1}{x^3(1-x)}\,dx$

$$= \int \frac{(1-x^3)+x^3}{x^3(1-x)}\,dx$$

$$= \int\left[\frac{1-x^3}{x^3(1-x)} + \frac{1}{1-x}\right]dx$$

$$= \int\left[\frac{(1-x)(1+x+x^2)}{x^3(1-x)} + \frac{1}{1-x}\right]dx$$

$$= \int\left(\frac{1+x+x^2}{x^3} + \frac{1}{1-x}\right)dx$$

$$= \int\left(\frac{1}{x^3} + \frac{1}{x^2} + \frac{1}{x} + \frac{1}{1-x}\right)dx$$

$$= \int x^{-3}dx + \int x^{-2}dx + \int\frac{1}{x}\,dx + \int\frac{1}{1-x}\,dx$$

$$= \frac{x^{-2}}{-2} + \frac{x^{-1}}{-1} + \log|x| + \frac{\log|1-x|}{-1} + c$$

$$= \frac{-1}{2x^2} - \frac{1}{x} + \log|x| - \log|1-x| + c$$

$$= \log\left|\frac{x}{1-x}\right| - \frac{1}{2x^2} - \frac{1}{x} + c.$$

**2. Evaluate :** $\int \frac{1+\log x}{x(2+\log x)(3+\log x)}\, dx.$

**Solution :** Let $I = \int \frac{1+\log x}{x(2+\log x)(3+\log x)}\, dx$

$$= \int \frac{1+\log x}{(2+\log x)(3+\log x)} \cdot \frac{1}{x}\, dx$$

Put $\log x = t$ $\quad \therefore \frac{1}{x}\, dx = dt$

$$\therefore I = \int \frac{1+t}{(2+t)(3+t)}\, dt$$

Let $\frac{1+t}{(2+t)(3+t)} = \frac{A}{2+t} + \frac{B}{3+t}$

$\therefore 1+t = A(3+t) + B(2+t)$

Put $2+t=0$, i.e. $t=-2$, we get

$1-2 = A(1) + B(0) \qquad \therefore A = -1$

Put $3+t=0$, i.e. $t=-3$, we get

$1-3 = A(0) + B(-1) \qquad \therefore B = 2$

$$\therefore \frac{1+t}{(2+t)(3+t)} = \frac{-1}{2+t} + \frac{2}{3+t}$$

$$\therefore I = \int\left(\frac{-1}{2+t} + \frac{2}{3+t}\right) dt$$

$$= -\int \frac{1}{2+t}\, dt + 2\int \frac{1}{3+t}\, dt$$

$$= -\log|2+t| + 2\log|3+t| + c$$

$$= -\log|2+\log x| + 2\log|3+\log x| + c.$$

## EXAMPLES FOR PRACTICE 5.6

**Evaluate the following integrals :**

**1.** $\int \frac{3x-2}{(x-2)(x+1)}\, dx$ **2.** $\int \frac{dx}{x^2-5x+6}$

**3.** $\int \frac{5x+2}{x^2-3x+2}\, dx$ **4.** $\int \frac{2x-1}{(x-1)(x+2)\,(x-3)}\, dx$

**5.** $\int \frac{1}{x(x-2)(x-4)}\, dx$ **6.** $\int \frac{x^2+2}{(x-1)(x+2)(x+3)}\, dx$

**7.** $\int \frac{x^2+1}{x^2-4x+3}\, dx$ **8.** $\int \frac{x-x^2}{x^2-2x-3}\, dx$

**9.** $\int \frac{dx}{(x-1)^2(x+1)}$ **10.** $\int \frac{x^2}{(x+1)(x+2)^2}\, dx$

**11.** $\int \frac{3x+1}{(x-2)^2(x+2)}\, dx$ **12.** $\int \frac{x^2+x-1}{(x+1)(x^2+3x+2)}\, dx$

**13.** $\int \frac{x^2+x+1}{(x-1)^3}\, dx$ **14.** $\int \frac{x^3-4x^2+3x+11}{x^2+5x+6}\, dx$

**15.** $\int \frac{\log x}{x(1+\log x)(2+\log x)}\, dx.$

### Answers

**1.** $\frac{5}{3}\log|x+1| + \frac{4}{3}\log|x-2| + c$

**2.** $\log\left|\frac{x-3}{x-2}\right| + c$

**3.** $12\log|x-2| - 7\log|x-1| + c$

**4.** $-\frac{1}{6}\log|x-1| - \frac{1}{3}\log|x+2| + \frac{1}{2}\log|x-3| + c$

**5.** $\frac{1}{8}\log|x| - \frac{1}{4}\log|x-2| + \frac{1}{8}\log|x-4| + c$

**6.** $\frac{1}{4}\log|x-1| - 2\log|x+2| + \frac{11}{4}\log|x+3| + c$

**7.** $x + 5\log|x-3| - \log|x-1| + c$

**8.** $-x - \frac{3}{2}\log|x-3| + \frac{1}{2}\log|x+1| + c$

**9.** $\frac{1}{4}\log\left|\frac{x+1}{x-1}\right| - \frac{1}{2(x-1)} + c$

**10.** $\log|x+1| + \frac{4}{x+2} + c$

**11.** $\frac{5}{16}\log\left|\frac{x-2}{x+2}\right| - \frac{7}{4(x-2)} + c$

**12.** $\frac{1}{x+1} + \log|x+2| + c$

**13.** $\log|x-1| - \frac{3}{x-1} - \frac{3}{2(x-1)^2} + c$

**14.** $\frac{x^2}{2} + x - 9\log|x-2| + 11\log|x-3| + c$

**15.** $2\log|2+\log x| - \log|1+\log x| + c.$

**ACTIVITY** | **Textbook page 136**

**Evaluate :** $\int \frac{x-1}{(x-3)(x-2)}\,dx.$

**Solution :** $\frac{x-1}{(x-3)(x-2)} = \frac{[A]}{(x-3)} + \frac{[B]}{(x-2)}$ ... (1)

$x - 1 = A(x-2) + B(x-3)$ ... (2)

$\boxed{1}x + \boxed{-1} = \boxed{A+B}x + \boxed{-2A-3B}$ ... (3)

**Note :** Two polynomials are equal if corresponding coefficients are equal. For linear functions, this means that $ax + b = cx + d$ for all $x$ exactly when $a = c$ and $b = d$.

Alternately, you can evaluate equation (2) for various values of $x$ to get equations relating $A$ and $B$. Some values of $x$ will be more helpful than others.

$\boxed{1} = \boxed{A+B}$

$\boxed{-1} = \boxed{-2A-3B}$

Continue solving for the constants $A$ and $B$.

$A = \boxed{2}$, $B = \boxed{-1}$

$\therefore \frac{x-1}{(x-3)(x-2)} = \frac{\boxed{2}}{(x-3)} + \frac{\boxed{-1}}{(x-2)}$

$\therefore \int \frac{x-1}{(x-3)(x-2)}\,dx = \int \frac{\boxed{2}}{(x-3)}\,dx + \int \frac{\boxed{-1}}{(x-2)}\,dx$

$I = \boxed{2\log|x-3|} + \boxed{-\log|x-2|} + c$

$= 2\log|x-3| - \log|x-2| + c.$

## MISCELLANEOUS EXERCISE – 5

(Textbook pages 137 to 139)

**(I) Choose the correct alternative from the following :**

1. The value of $\int \frac{dx}{\sqrt{1-x}}$ is

(a) $2\sqrt{1-x}+c$ (b) $-2\sqrt{1-x}+c$

(c) $\sqrt{x}+c$ (d) $x+c$

2. $\int \sqrt{1+x^2}\,dx = \ldots\ldots\ldots$

(a) $\frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\log(x+\sqrt{1+x^2})+c$

(b) $\frac{2}{3}(1+x^2)^{3/2}+c$

(c) $\frac{1}{3}(1+x^2)+c$

(d) $\frac{(x)}{\sqrt{1+x^2}}+c$

3. $\int x^2(3)^{x^3}\,dx = \ldots\ldots\ldots$

(a) $(3)^{x^3}+c$ (b) $\frac{(3)^{x^3}}{3\cdot\log 3}+c$

(c) $\log 3(3)^{x^3}+c$ (d) $x^3(3)^{x^3}$

4. $\int \frac{x+2}{2x^2+6x+5}\,dx = p\int \frac{4x+6}{2x^2+6x+5}\,dx + \frac{1}{2}\int \frac{dx}{2x^2+6x+5}$, then $p = \ldots\ldots\ldots$

(a) $\frac{1}{3}$ (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) 2

5. $\int \frac{dx}{(x-x^2)} = \ldots\ldots\ldots$

(a) $\log x - \log(1-x)+c$ (b) $\log(1-x^2)+c$

(c) $-\log x + \log(1-x)+c$ (d) $\log(x-x^2)+c$

6. $\int \frac{dx}{(x-8)(x+7)} = \ldots\ldots\ldots$

(a) $\frac{1}{15}\log\left|\frac{x+2}{x-1}\right|+c$ (b) $\frac{1}{15}\log\left|\frac{x+8}{x+7}\right|+c$

(c) $\frac{1}{15}\log\left|\frac{x-8}{x+7}\right|+c$ (d) $(x-8)(x-7)+c$

7. $\int \left(x+\frac{1}{x}\right)^3 dx = \ldots\ldots\ldots$

(a) $\frac{1}{4}\left(x+\frac{1}{x}\right)^4+c$

(b) $\frac{x^4}{4}+\frac{3x^2}{2}+3\log x-\frac{1}{2x^2}+c$

(c) $\frac{x^4}{4}+\frac{3x^2}{2}+3\log x+\frac{1}{x^2}+c$

(d) $(x-x^{-1})^3+c$

8. $\int \left(\frac{e^{2x}+e^{-2x}}{e^x}\right)dx = \ldots\ldots\ldots$

(a) $e^x - \frac{1}{3e^{3x}}+c$ (b) $e^x+\frac{1}{3e^{3x}}+c$

(c) $e^{-x}+\frac{1}{3e^{3x}}+c$ (d) $e^{-x}-\frac{1}{3e^{3x}}+c$

9. $\int(1-x)^{-2}dx = \ldots\ldots\ldots$

(a) $(1+x)^{-1}+c$ (b) $(1-x)^{-1}+c$

(c) $(1-x)^{-1}-1+c$ (d) $(1-x)^{-1}+1+c$

10. $\int\frac{(x^3+3x^2+3x+1)}{(x+1)^5}dx = \ldots\ldots\ldots$

(a) $\frac{-1}{x+1}+c$ (b) $\left(\frac{-1}{x+1}\right)^5+c$

(c) $\log(x+1)+c$ (d) $\log|x+1|^5+c$

**Answers**

1. (b) $-2\sqrt{1-x}+c$

2. (a) $\frac{x}{2}\sqrt{1+x^2}+\frac{1}{2}\log(x+\sqrt{1+x^2})+c$

3. (b) $\frac{(3)^{x^3}}{3\cdot\log 3}+c$ [Hint : Put $x^3=t$]

4. (c) $\frac{1}{4}$

$\left[\text{Hint : } \int\frac{x+2}{2x^2+6x+5}dx = \int\frac{\frac{1}{4}(4x+6)+\frac{1}{2}}{2x^2+6x+5}dx.\right]$

5. (a) $\log x-\log(1-x)+c$

6. (c) $\frac{1}{15}\log\left|\frac{x-8}{x+7}\right|+c$

7. (b) $\frac{x^4}{4}+\frac{3x^2}{2}+3\log x-\frac{1}{2x^2}+c$

$\left[\text{Hint : } \left(x+\frac{1}{x}\right)^3 = x^3+3x+\frac{3}{x}+\frac{1}{x^3}.\right]$

8. (a) $e^x-\frac{1}{3e^{3x}}+c$

9. (b) $(1-x)^{-1}+c$

10. (a) $\frac{-1}{x+1}+c$

[Hint : $x^3+3x^2+3x+1=(x+1)^3$]

**(II) Fill in the blanks :**

1. $\int\frac{5(x^6+1)}{x^2+1}dx = x^5+\ldots\ldots\ldots x^3+5x+c.$

2. $\int\frac{x^2+x-6}{(x-2)(x-1)}dx = x+\ldots\ldots+c.$

3. If $f'(x)=\frac{1}{x}+x$ and $f(1)=\frac{5}{2}$, then

$f(x)=\log x+\frac{x^2}{2}+\ldots\ldots\ldots$

4. To find the value of $\int\frac{(1+\log x)\,dx}{x}$, the proper substitution is $\ldots\ldots\ldots$

5. $\int\frac{1}{x^3}[\log x^x]^2dx = p(\log x)^3+c$, then $p=\ldots\ldots\ldots$

**Answers**

1. $-\frac{5}{3}$ [Hint : $x^6+1=(x^2+1)(x^4-x^2+1)$]

2. $4\log|x-1|$ [Hint : $x^2+x-6=(x+3)(x-2)$]

3. 2

[Hint : $f(x)=\int\left(\frac{1}{x}+x\right)dx = \log|x|+\frac{x^2}{2}+c$

$\therefore f(1)=\log 1+\frac{1}{2}+c=\frac{5}{2}$ $\therefore c=2$

$\therefore f(x)=\log|x|+\frac{x^2}{2}+2$]

4. $1+\log x=t$

5. $\frac{1}{3}$ $\left[\text{Hint : } \frac{1}{x^3}(\log x^x)^2 = \frac{1}{x^3}(x\log x)^2 = \frac{(\log x)^2}{x}.\right]$

**(III) State whether each of the following is *True* or *False* :**

1. The proper substitution for $\int x(x^x)^x(2\log x+1)dx$ is $(x^x)^x=t$.

2. If $\int x\,e^{2x}\,dx$ is equal to $e^{2x}f(x)+c$ where $c$ is constant of integration, then $f(x)$ is $\frac{(2x-1)}{2}$.

3. If $\int x\,f(x)\,dx=\frac{f(x)}{2}$, then $f(x)=e^{x^2}$.

4. If $\int\frac{(x-1)\,dx}{(x+1)(x-2)} = A\log|x+1|+B\log|x-2|$, then $A+B=1$.

5. For $\int\frac{x-1}{(x+1)^3}e^x dx = e^x f(x)+c$, $f(x)=(x+1)^2$.

**Answers**

1. True 2. False 3. True 4. True 5. False.

**(IV) Solve the following :**

1. Evaluate :

(i) $\int\frac{5x^2-6x+3}{2x-3}dx$

**Solution :** Let $I=\int \frac{5x^2-6x+3}{2x-3}\,dx$

$$2x-3\overline{)5x^2-6x+3}\left(\frac{5}{2}x+\frac{3}{4}\right.$$
$$\begin{array}{r} 5x^2-\frac{15}{2}x \\ -\quad + \\ \hline \frac{3}{2}x+3 \\ \frac{3}{2}x-\frac{9}{4} \\ -\quad + \\ \hline \frac{21}{4} \end{array}$$

$\therefore\ 5x^2-6x+3=\left(\frac{5}{2}x+\frac{3}{4}\right)(2x-3)+\frac{21}{4}$

$$\therefore\ I=\int\left[\frac{\left(\frac{5}{2}x+\frac{3}{4}\right)(2x-3)+\frac{21}{4}}{2x-3}\right]dx$$
$$=\int\left[\frac{5}{2}x+\frac{3}{4}+\frac{\left(\frac{21}{4}\right)}{2x-3}\right]dx$$
$$=\frac{5}{2}\int x\,dx+\frac{3}{4}\int 1\,dx+\frac{21}{4}\int\frac{1}{2x-3}\,dx$$
$$=\frac{5}{2}\cdot\frac{x^2}{2}+\frac{3}{4}x+\frac{21}{4}\cdot\frac{\log|2x-3|}{2}+c$$
$$=\frac{5x^2}{4}+\frac{3x}{4}+\frac{21}{8}\log|2x-3|+c.$$

**(ii)** $\int(5x+1)^{\frac{4}{9}}dx$

**Solution :** $\int(5x+1)^{\frac{4}{9}}dx=\frac{(5x+1)^{\frac{4}{9}+1}}{\frac{4}{9}+1}\times\frac{1}{5}+c$

$$=\frac{9}{65}(5x+1)^{\frac{13}{9}}+c.$$

**(iii)** $\int\frac{1}{2x+3}\,dx$

**Solution :** $\int\frac{1}{2x+3}\,dx=\frac{\log|2x+3|}{2}+c$

$=\frac{1}{2}\log|2x+3|+c.$

**(iv)** $\int\frac{x-1}{\sqrt{x+4}}\,dx$

**Solution :** $\int\frac{x-1}{\sqrt{x+4}}\,dx=\int\frac{(x+4)-5}{\sqrt{x+4}}\,dx$

$$=\int\left(\frac{x+4}{\sqrt{x+4}}-\frac{5}{\sqrt{x+4}}\right)dx$$
$$=\int\left(\sqrt{x+4}-\frac{5}{\sqrt{x+4}}\right)dx$$
$$=\int(x+4)^{\frac{1}{2}}dx-5\int(x+4)^{-\frac{1}{2}}dx$$
$$=\frac{(x+4)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}-5\cdot\frac{(x+4)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)}+c$$
$$=\frac{2}{3}(x+4)^{\frac{3}{2}}-10\sqrt{x+4}+c.$$

**(v) If $f'(x)=\sqrt{x}$ and $f(1)=2$, then find the value of $f(x)$.**

**Solution :** By the definition of integral

$f(x)=\int f'(x)dx=\int\sqrt{x}\,dx$

$$=\frac{x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}+c=\frac{2}{3}x^{\frac{3}{2}}+c \qquad \ldots (1)$$

$\therefore\ f(1)=\frac{2}{3}(1)^{\frac{3}{2}}+c=\frac{2}{3}+c$

But $f(1)=2$

$\therefore\ \frac{2}{3}+c=2 \qquad \therefore\ c=\frac{4}{3}$

$\therefore$ from (1), $f(x)=\frac{2}{3}x^{\frac{3}{2}}+\frac{4}{3}.$

**(vi)** $\int|x|\,dx$ **if** $x<0$

**Solution :** $\int|x|\,dx=\int -x\,dx \qquad \ldots [\because x<0]$

$$=-\int x\,dx=-\frac{x^2}{2}+c.$$

**2. Evaluate :**

**(i) Find the primitive of $\frac{1}{1+e^x}$.**

**Solution :** Let $I$ be the primitive of $\frac{1}{1+e^x}$.

Then $I=\int\frac{1}{1+e^x}\,dx$

$$=\int\frac{\left(\frac{1}{e^x}\right)}{\left(\frac{1+e^x}{e^x}\right)}dx=\int\frac{e^{-x}}{e^{-x}+1}\,dx$$
$$=-\int\frac{-e^{-x}}{e^{-x}+1}\,dx$$
$$=-\log|e^{-x}+1|+c$$

$$\ldots\left[\because \frac{d}{dx}(e^{-x}+1)=-e^{-x} \text{ and } \int\frac{f'(x)}{f(x)}\,dx=\log|f(x)|+c\right]$$

**(ii)** $\int \frac{ae^x + be^{-x}}{(ae^x - be^{-x})^2} dx$

**Solution :** Let $I = \int \frac{ae^x + be^{-x}}{(ae^x - be^{-x})^2} dx$

Put $ae^x - be^{-x} = t$

$\therefore \ (ae^x + be^{-x})\,dx = dt$

$\therefore \ I = \int \frac{1}{t^2} dt = \int t^{-2} dt$

$= \frac{t^{-1}}{-1} + c = \frac{-1}{t} + c$

$= \frac{-1}{ae^x + be^{-x}} + c.$

[**Note :** *Question is modified.*]

**(iii)** $\int \frac{1}{2x + 3x \log x} dx$

**Solution :** Let $I = \int \frac{1}{2x + 3x \log x} dx$

$= \int \frac{1}{(2 + 3 \log x)} \cdot \frac{1}{x} dx$

Put $2 + 3 \log x = t$ $\quad \therefore \ \frac{3}{x} dx = dt$

$\therefore \ \frac{1}{x} dx = \frac{dt}{3}$

$\therefore \ I = \int \frac{1}{t} \cdot \frac{dt}{3} = \frac{1}{3} \int \frac{1}{t} dt$

$= \frac{1}{3} \log |t| + c$

$= \frac{1}{3} \log |2 + 3 \log x| + c.$

**(iv)** $\int \frac{1}{\sqrt{x} + x} dx$

**Solution :** *Refer to the solution of Q. 9 of Exercise 5.2.*

**Ans.** $2 \log |1 + \sqrt{x}| + c.$

**(v)** $\int \frac{2e^x - 3}{4e^x + 1} dx$

**Solution :** *Refer to the solution of Q. 3 of Exercise 5.3.*

**Ans.** $-3x + \frac{7}{2} \log |4e^x + 1| + c.$

**3. Evaluate :**

**(i)** $\int \frac{dx}{\sqrt{4x^2 - 5}} dx$

**Solution :** $\int \frac{dx}{\sqrt{4x^2 - 5}} = \frac{1}{2} \int \frac{1}{\sqrt{x^2 - \frac{5}{4}}} dx$

$= \frac{1}{2} \log \left| x + \sqrt{x^2 - \frac{5}{4}} \right| + c.$

**(ii)** $\int \frac{dx}{3 - 2x - x^2} dx$

**Solution :** $\int \frac{dx}{3 - 2x - x^2} = \int \frac{dx}{3 - (x^2 + 2x + 1) + 1}$

$= \int \frac{1}{(2)^2 - (x + 1)^2} dx$

$= \frac{1}{2 \times 2} \log \left| \frac{2 + x + 1}{2 - x - 1} \right| + c$

$= \frac{1}{4} \log \left| \frac{3 + x}{1 - x} \right| + c.$

**(iii)** $\int \frac{dx}{9x^2 - 25}$

**Solution :** $\int \frac{dx}{9x^2 - 25} = \frac{1}{9} \int \frac{1}{x^2 - \frac{25}{9}} dx$

$= \frac{1}{9} \int \frac{1}{x^2 - \left(\frac{5}{3}\right)^2} dx$

$= \frac{1}{9} \times \frac{1}{2 \times \frac{5}{3}} \log \left| \frac{x - \frac{5}{3}}{x + \frac{5}{3}} \right| + c$

$= \frac{1}{30} \log \left| \frac{3x - 5}{3x + 5} \right| + c.$

**(iv)** $\int \frac{e^x}{\sqrt{e^{2x} + 4e^x + 13}} dx$

**Solution :** Let $I = \int \frac{e^x}{\sqrt{e^{2x} + 4e^x + 13}} dx$

Put $e^x dx$ $\quad \therefore \ e^x dx = dt$

$\therefore \ I = \int \frac{1}{\sqrt{t^2 + 4t + 13}} dt$

$= \int \frac{1}{\sqrt{(t^2 + 4t + 4) + 9}} dt$

$= \int \frac{1}{\sqrt{(t + 2)^2 + (3)^2}} dt$

$= \log |(t + 2) + \sqrt{(t + 2)^2 + (3)^2}| + c$

$= \log |(t + 2) + \sqrt{t^2 + 4t + 13}| + c$

$= \log |(e^x + 2) + \sqrt{e^{2x} + 4e^x + 13}| + c.$

**(v)** $\int \frac{dx}{x[(\log x)^2 + 4 \log x - 1]}$

**Solution :** Let $I = \int \frac{dx}{x[(\log x)^2 + 4 \log x - 1]}$

$= \int \frac{1}{(\log x)^2 + 4 \log x - 1} \cdot \frac{1}{x} dx$

Put $\log x = t$ $\quad \therefore \frac{1}{x}dx = dt$

$$\therefore I = \int \frac{1}{t^2+4t-1}dt$$

$$= \int \frac{1}{(t^2+4t+4)-5}dt$$

$$= \int \frac{1}{(t+2)^2-(\sqrt{5})^2}dt$$

$$= \frac{1}{2\sqrt{5}}\log\left|\frac{t+2-\sqrt{5}}{t+2+\sqrt{5}}\right|+c$$

$$= \frac{1}{2\sqrt{5}}\log\left|\frac{\log x+2-\sqrt{5}}{\log x+2+\sqrt{5}}\right|+c.$$

**(vi)** $\int \frac{dx}{5-16x^2}$

**Solution :** $\int \frac{dx}{5-16x^2} = \frac{1}{16}\int \frac{dx}{\frac{5}{16}-x^2}dx$

$$= \frac{1}{16}\int \frac{1}{\left(\frac{\sqrt{5}}{4}\right)^2-x^2}dx$$

$$= \frac{1}{16}\times\frac{1}{2\times\frac{\sqrt{5}}{4}}\log\left|\frac{\frac{\sqrt{5}}{4}+x}{\frac{\sqrt{5}}{4}-x}\right|+c$$

$$= \frac{1}{8\sqrt{5}}\log\left|\frac{\sqrt{5}+4x}{\sqrt{5}-4x}\right|+c.$$

**(vii)** $\int \frac{dx}{25x-x(\log x)^2}$

**Solution :** Let $I = \int \frac{dx}{25x-x(\log x)^2}$

$$= \int \frac{1}{25-(\log x)^2}\cdot\frac{1}{x}dx$$

Put $\log x = t$ $\quad \therefore \frac{1}{x}dx = dt$

$$\therefore I = \int \frac{1}{25-t^2}dt$$

$$= \frac{1}{2\times 5}\log\left|\frac{5+t}{5-t}\right|+c$$

$$= \frac{1}{10}\log\left|\frac{5+\log x}{5-\log x}\right|+c.$$

**(viii)** $\int \frac{e^x}{4e^{2x}-1}dx$

**Solution :** Let $I = \int \frac{e^x}{4e^{2x}-1}dx$

Put $e^x = t$ $\quad \therefore e^x dx = dt$

$$\therefore I = \int \frac{1}{4t^2-1}dt = \frac{1}{4}\int \frac{1}{t^2-\frac{1}{4}}dt$$

$$= \frac{1}{4}\int \frac{1}{t^2-\left(\frac{1}{2}\right)^2}dt$$

$$= \frac{1}{4}\times\frac{1}{2\times\frac{1}{2}}\log\left|\frac{t-\frac{1}{2}}{t+\frac{1}{2}}\right|+c$$

$$= \frac{1}{4}\log\left|\frac{2t-1}{2t+1}\right|+c$$

$$= \frac{1}{4}\log\left|\frac{2e^x-1}{2e^x+1}\right|+c.$$

[**Note :** *Answer in the textbook is incorrect.*]

**4. Evaluate :**

**(i)** $\int (\log x)^2 dx$

**Solution :** $\int (\log x)^2 dx = \int (\log x)^2\cdot 1\,dx$

$$= (\log x)^2\int 1dx - \int\left[\frac{d}{dx}(\log x)^2\cdot\int 1\,dx\right]dx$$

$$= (\log x)^2\cdot x - \int\left[2\log x\cdot\frac{d}{dx}(\log x)\times x\right]dx$$

$$= x(\log x)^2 - \int 2\log x\times\frac{1}{x}\times x\,dx$$

$$= x(\log x)^2 - 2\int(\log x)\cdot 1\,dx$$

$$= x(\log x)^2 - 2\left\{(\log x)\int 1\,dx - \left[\frac{d}{dx}(\log x)\int 1\,dx\right]dx\right\}$$

$$= x(\log x)^2 - 2\left\{(\log x)\cdot x - \int\frac{1}{x}\times x\,dx\right\}$$

$$= x(\log x)^2 - 2x\log x + 2\int 1\,dx$$

$$= x(\log x)^2 - 2x\log x + 2x + c.$$

**(ii)** $\int e^x\frac{1+x}{(2+x)^2}dx$

**Solution :** *Refer to the solution of Q. 6 of Exercise 5.5.*

**Ans.** $\frac{e^x}{2+x}+c.$

**(iii)** $\int xe^{2x}dx.$

**Solution :** $\int xe^{2x}dx = x\int e^{2x}dx - \int\left[\frac{d}{dx}(x)\int e^{2x}dx\right]dx$

$$= x\cdot\frac{e^{2x}}{2} - \int 1\cdot\frac{e^{2x}}{2}\,dx$$

$$= \frac{1}{2}xe^{2x} = \frac{1}{2}\int e^{2x}\,dx$$

$$= \frac{1}{2}xe^{2x} - \frac{1}{2}\cdot\frac{e^{2x}}{2} + c$$

$$= e^{2x}\left(\frac{x}{2} - \frac{1}{4}\right) + c$$

$$= \left(\frac{2x-1}{4}\right)e^{2x} + c.$$

**(iv)** $\int \log(x^2 + x)\,dx$

**Solution :** Let $I = \int \log(x^2+x)dx = \int[\log(x^2+x)]\cdot 1\,dx$

$$= [\log(x^2+x)]\int 1\,dx - \int\left[\frac{d}{dx}\{\log(x^2+x)\}\cdot\int dx\right]dx$$

$$= [\log(x^2+x)]\cdot x - \int\frac{1}{x^2+x}\cdot\frac{d}{dx}(x^2+x)\times x\,dx$$

$$= x\log(x^2+x) - \int\frac{1}{x(x+1)}\cdot(2x+1)\cdot x\,dx$$

$$= x\log(x^2+x) - \int\frac{2x+1}{x+1}\,dx$$

$$= x\log(x^2+x) - \int\frac{2(x+1)-1}{x+1}\,dx$$

$$= x\log(x^2+x) - \int\left(2 - \frac{1}{x+1}\right)dx$$

$$= x\log(x^2+x) - 2\int 1\,dx + \int\frac{1}{x+1}\,dx$$

$$= x\log(x^2+x) - 2x + \log|x+1| + c.$$

**(v)** $\int e^{\sqrt{x}}dx$

**Solution :** Let $I = \int e^{\sqrt{x}}dx$

Put $\sqrt{x} = t \qquad \therefore\ x = t^2$

$\therefore\ dx = 2t\,dt$

$\therefore\ I = \int e^t\cdot 2t\,dt = 2\int te^t\,dt$

$$= 2\left[t\int e^t dt - \int\left\{\frac{d}{dt}(t)\int e^t dt\right\}\right]dt$$

$$= 2[t\cdot e^t - \int 1\cdot e^t dt]$$

$$= 2[t\cdot e^t - e^t] + c$$

$$= 2(t-1)e^t + c$$

$$= 2(\sqrt{x}-1)\,e^{\sqrt{x}} + c.$$

**(vi)** $\int\sqrt{x^2+2x+5}\ dx$

**Solution :** $\int\sqrt{x^2+2x+5}\,dx$

$$= \int\sqrt{(x^2+2x+1)+4}\,dx$$

$$= \int\sqrt{(x+1)^2+(2)^2}\,dx$$

$$= \frac{(x+1)}{2}\sqrt{(x+1)^2+(2)^2} + \frac{(2)^2}{2}\log|(x+1)+\sqrt{(x+1)^2+(2)^2}| + c$$

$$= \frac{(x+1)}{2}\sqrt{x^2+2x+5} + 2\log|(x+1)+\sqrt{x^2+2x+5}| + c.$$

**(vii)** $\int\sqrt{x^2-8x+7}\ dx$

**Solution :** $\int\sqrt{x^2-8x+7}\,dx$

$$= \int\sqrt{(x^2-8x+16)-9}\,dx$$

$$= \int\sqrt{(x-4)^2-(3)^2}\,dx$$

$$= \frac{(x-4)}{2}\sqrt{(x-4)^2-(3)^2} - \frac{(3)^2}{2}\log|(x-4)+\sqrt{(x-4)^2-(3)^2}| + c$$

$$= \frac{(x-4)}{2}\sqrt{x^2-8x+7} - \frac{9}{2}\log|(x-4)+\sqrt{x^2-8x+7}| + c.$$

**5. Evaluate :**

**(i)** $\int\frac{3x-1}{2x^2-x-1}\,dx$

**Solution :** Let $I = \int\frac{3x-1}{2x^2-x-1}\,dx$

$$= \int\frac{3x-1}{(x-1)(2x+1)}\,dx$$

Let $\frac{3x-1}{(x-1)(2x+1)} = \frac{A}{x-1} + \frac{B}{2x+1}$

$\therefore\ 3x-1 = A(2x+1) + B(x-1)$

Put $x-1=0$, i.e. $x=1$, we get

$3(1)-1 = A(3)+B(0) \qquad \therefore\ 2 = 3A \qquad \therefore\ A = \frac{2}{3}$

Put $2x+1=0$, i.e. $x = -\frac{1}{2}$, we get

$3\left(-\frac{1}{2}\right) - 1 = A(0) + B\left(-\frac{3}{2}\right)$

$\therefore\ -\frac{5}{2} = -\frac{3}{2}B \qquad \therefore\ B = \frac{5}{3}$



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$$\therefore \frac{3x-1}{(x-1)(2x+1)}=\frac{\left(\frac{2}{3}\right)}{x-1}+\frac{\left(\frac{5}{3}\right)}{2x+1}$$

$$\therefore I=\int\left[\frac{\left(\frac{2}{3}\right)}{x-1}+\frac{\left(\frac{5}{3}\right)}{2x+1}\right]dx$$

$$=\frac{2}{3}\int\frac{1}{x-1}dx+\frac{5}{3}\int\frac{1}{2x+1}dx$$

$$=\frac{2}{3}\log|x-1|+\frac{5}{3}\cdot\frac{\log|2x+1|}{2}+c$$

$$=\frac{2}{3}\log|x-1|+\frac{5}{6}\log|2x+1|+c.$$

**(ii)** $\displaystyle\int\frac{2x^3-3x^2-9x+1}{2x^2-x-10}dx$

**Solution :** Let $I=\displaystyle\int\frac{2x^3-3x^2-9x+1}{2x^2-x-10}dx$

$$\begin{array}{r|l} 2x^2-x-10) & 2x^3-3x^2-9x+1\ (x-1 \\ & 2x^3-x^2-10x \\ & -\quad+\quad+ \\ \hline & -2x^2+x+1 \\ & -2x^2+x+10 \\ & +\quad-\quad- \\ \hline & -9 \end{array}$$

$$\therefore 2x^3-3x^2-9x+1=(x-1)(2x^2-x-10)-9$$

$$\therefore I=\int\left[\frac{(x-1)(2x^2-x-10)-9}{2x^2-x-10}\right]dx$$

$$=\int\left[(x-1)-\frac{9}{2x^2-x-10}\right]dx$$

$$=\int(x-1)\,dx-\frac{9}{2}\int\frac{1}{x^2-\frac{1}{2}x-5}dx$$

$$=\int x\,dx-\int 1\,dx-\frac{9}{2}\int\frac{1}{\left(x^2-\frac{1}{2}x+\frac{1}{16}\right)-\frac{1}{16}-5}dx$$

$$=\int x\,dx-\int 1\,dx-\frac{9}{2}\int\frac{1}{\left(x-\frac{1}{4}\right)^2-\left(\frac{9}{4}\right)^2}dx$$

$$=\frac{x^2}{2}-x-\frac{9}{2}\times\frac{1}{2\times\frac{9}{4}}\log\left|\frac{x-\frac{1}{4}-\frac{9}{4}}{x-\frac{1}{4}+\frac{9}{4}}\right|+c_1$$

$$=\frac{x^2}{2}-x-\log\left|\frac{x-\frac{5}{2}}{x+2}\right|+c_1$$

$$=\frac{x^2}{2}-x-\log\left|\frac{2x-5}{2(x+2)}\right|+c_1$$

$$=\frac{x^2}{2}-x+\log\left|\frac{2(x+2)}{2x-5}\right|+c_1$$

$$=\frac{x^2}{2}-x+\log\left|\frac{x+2}{2x-5}\right|+\log 2+c_1$$

$$=\frac{x^2}{2}-x+\log\left|\frac{x+2}{2x-5}\right|+c,\text{ where } c_1=\log 2+c_1$$

[**Note :** *Answer in the textbook is incorrect.*]

**(iii)** $\displaystyle\int\frac{1+\log x}{x(3+\log x)(2+3\log x)}dx$

**Solution :**

Let $I=\displaystyle\int\frac{1+\log x}{x(3+\log x)(2+3\log x)}dx$

$$=\int\frac{1+\log x}{(3+\log x)(2+3\log x)}\cdot\frac{1}{x}dx$$

Put $\log x=t$ $\quad\therefore \frac{1}{x}dx=dt$

$$\therefore I=\int\frac{1+t}{(3+t)(2+3t)}dt$$

Let $\dfrac{1+t}{(3+t)(2+3t)}=\dfrac{A}{3+t}+\dfrac{B}{2+3t}$

$\therefore 1+t=A(2+3t)+B(3+t)$

Put $3+t=0$, i.e. $t=-3$, we get

$1-3=A(-7)+B(0)$

$\therefore -2=-7A \quad \therefore A=\frac{2}{7}$

Put $2+3t=0$, i.e. $t=-\frac{2}{3}$, we get

$1-\frac{2}{3}=A(0)+B\left(\frac{7}{3}\right)$

$\therefore \frac{1}{3}=\frac{7}{3}B \quad \therefore B=\frac{1}{7}$

$$\therefore \frac{1+t}{(3+t)(2+3t)}=\frac{\left(\frac{2}{7}\right)}{3+t}+\frac{\left(\frac{1}{7}\right)}{2+3t}$$

$$\therefore\ I=\int\left[\frac{\left(\frac{2}{7}\right)}{3+t}+\frac{\left(\frac{1}{7}\right)}{2+3t}\right]dt$$

$$=\frac{2}{7}\int\frac{1}{3+t}\,dt+\frac{1}{7}\int\frac{1}{2+3t}\,dt$$

$$=\frac{2}{7}\log|3+t|+\frac{1}{7}\cdot\frac{\log|2+3t|}{3}+c$$

$$=\frac{2}{7}\log|3+\log x|+\frac{1}{21}\log|2+3\log x|+c.$$

## ACTIVITIES | Textbook pages 139 and 140

**1.** $\int\frac{1}{(x^2-5x+4)}2x\,dx.$

**Solution :** $\frac{2x}{\boxed{x-1}\boxed{x-4}}=\frac{C}{\boxed{x-1}}+\frac{D}{x-4}$

$\therefore\ 2x=C(x-4)+D(x-1)$

$\therefore\ C=\boxed{\frac{-2}{3}},\ D=\boxed{\frac{8}{3}}$

$$\therefore\ \int\frac{1}{(x-1)(x-4)}2x\,dx=\int\left[\frac{\boxed{\frac{-2}{3}}}{(x-1)}+\frac{\boxed{\frac{8}{3}}}{(x-4)}\right]dx$$

$$=\int\frac{\boxed{-\frac{2}{3}}}{(x-1)}dx+\int\frac{\boxed{\frac{8}{3}}}{(x-4)}dx$$

$$=\boxed{\frac{-2}{3}\log|x-1|}+\boxed{\frac{8}{3}\log|x-4|}+c$$

**2.** $\int x^{13/2}(1+x^{5/2})^{1/2}\,dx.$

**Solution :** $\int x^{\boxed{5}}x^{3/2}(1+x^{5/2})^{1/2}\,dx$

$=\int(x^{5/2})^2\,x^{3/2}(1+x^{5/2})^{1/2}\,dx$

Let $1+x^{5/2}=t$ $\quad\boxed{x^{\frac{3}{2}}}\,dx=\boxed{\frac{2}{5}}\,dt$

$$I=\frac{2}{5}\int(t-1)^2\,t^{1/2}\,dt$$

$$=\frac{2}{5}\int(t^2-2t+1)t^{1/2}\,dt$$

$$=\frac{2}{5}\left[\boxed{\int t^{5/2}}dt-\boxed{\int 2t^{3/2}}dt+\boxed{\int t^{1/2}}dt\right]$$

$$=\frac{2}{5}\left\{\boxed{\frac{t^{7/2}}{\left(\frac{7}{2}\right)}}-\boxed{2\cdot\frac{t^{5/2}}{\left(\frac{5}{2}\right)}}+\boxed{\frac{t^{3/2}}{\left(\frac{3}{2}\right)}}\right\}+c$$

$$=\frac{4}{35}(1+x^{5/2})^{7/2}-\frac{8}{25}(1+x^{5/2})^{5/2}+\frac{4}{15}(1+x^{5/2})^{3/2}+c.$$

**3.** $\int\frac{dx}{(x+2)(x^2+1)}=\dots\dots\dots$

$\int\frac{1}{x^2+1}dx=\tan^{-1}x+c$ **(Given)**

**Solution :** $\frac{1}{(x+2)(x^2+1)}=\frac{\boxed{A}}{(x+2)}=\frac{Bx+C}{(x^2+1)}$

$\therefore\ 1=A(x^2+1)+(Bx+C)(x+2)$

Put $x=-2$, we get, $A=\frac{1}{5}$

Now, comparing the coefficients of $x^2$ and constant term, we get

$0=A+B$

and $1=A+2C$

$\therefore\ B=\frac{-1}{5},\ C=\frac{2}{5}$

$$\frac{1}{(x+2)(x^2+1)}=\frac{\boxed{\frac{1}{5}}}{(x+2)}+\frac{\boxed{-\frac{1}{5}}x+\boxed{\frac{2}{5}}}{(x^2+1)}$$

$$I=\frac{1}{5}\int\frac{dx}{(x+2)}-\frac{1}{10}\int\frac{\boxed{2}x}{x^2+1}dx+\boxed{\frac{2}{5}}\int\frac{dx}{x^2+1}$$

$$=\boxed{\frac{1}{5}\log|x+2|}-\boxed{\frac{1}{10}\log|x^2+1|}+\boxed{\frac{2}{5}\tan^{-1}x}+c.$$

**4. If** $\int\frac{1}{x^5+x}dx=f(x)+c$**, then the value of** $\int\frac{x^4}{x+x^5}dx$ **is equal to .........**

**Solution :** $I=\int\left[\frac{x^4+1-\boxed{1}}{x+x^5}\right]dx$

$$=\int\frac{1}{x}dx-\int\frac{1}{x^5+x}dx$$

$$I=\boxed{\log|x|}-\boxed{f(x)+c}$$

$$I=\log x-f(x)+c_1,\ \text{where } c_1=-c.$$

## ACTIVITIES FOR PRACTICE

**1. Evaluate :** $\int\frac{3x+2}{2x+1}dx.$

**Solution :** $\int\frac{3x+2}{2x+1}dx=\int\frac{\frac{3}{2}(2x+1)+\boxed{\phantom{0}}}{2x+1}dx$

$$=\int\left[\frac{3}{2}+\frac{\boxed{\phantom{0}}}{2x+1}\right]dx$$

$= \frac{3}{2}\int 1dx + \square \int \frac{1}{2x+1}\,dx$

$= \frac{3}{2}x + \square \frac{\log|2x+1|}{\square} + c$

$= \frac{3}{2}x + \square \log|2x+1| + c.$

**2. Evaluate :** $\int \frac{x^{n-1}}{\sqrt{1+4x^n}}\,dx.$

**Solution :** Let $I = \int \frac{x^{n-1}}{\sqrt{1+4x^n}}\,dx$

Put $x^n = t \qquad \therefore\ nx^{n-1}dx = dt$

$\therefore\ x^{n-1}dx = \frac{dt}{\square}$

$\therefore\ I = \int \frac{1}{\sqrt{1+4t}}\cdot\frac{dt}{\square} = \frac{1}{\square}\int (1+4t)^{-\frac{1}{2}}dt$

$= \frac{1}{\square}\cdot\frac{(1+4t)^{\frac{1}{2}}}{\square} + c$

$= \frac{1}{\square}\cdot\sqrt{1+4x^n} + c.$

**3. Evaluate :** $\int x^5\cdot\sqrt{a^3+x^3}\,dx.$

**Solution :** Let $I = \int x^5\cdot\sqrt{a^3+x^3}\,dx$

$= \int x^3\cdot\sqrt{a^3+x^3}\cdot\square\,dx$

Put $a^3+x^3 = t \qquad \therefore\ 3x^2dx = dt$

$\therefore\ x^2dx = \frac{dt}{\square}$

$\therefore\ I = \int (t-a^3)\sqrt{t}\,\frac{dt}{\square}$

$= \frac{1}{\square}\int (t^{\frac{3}{2}} - a^3t^{\square})dt$

$= \frac{1}{\square}\int t^{\frac{3}{2}}\,dt - \frac{a^3}{\square}\int t^{\square}\,dt$

$= \frac{1}{\square}\cdot\frac{t^{\frac{5}{2}}}{\left(\frac{5}{2}\right)} - \frac{a^3}{\square}\cdot\frac{t^{\square}}{\square} + c$

$= \square(a^3+x^3)^{\frac{5}{2}} - \frac{2a^3}{\square}(a^3+x^3)^{\square} + c.$

**4. Evaluate :** $\int \frac{1}{\sqrt{x^2+4x+29}}\,dx.$

**Solution :** $\int \frac{1}{\sqrt{x^2+4x+29}}\,dx$

$= \int \frac{1}{\sqrt{(x^2+4x+4)+\square}}\,dx$

$= \int \frac{1}{\sqrt{(x+2)^2+(\square)^2}}\,dx$

$= \log|(x+2)+\square| + c.$

**5. Evaluate :** $\int x^2\log x\,dx.$

**Solution :** $\int x^2\log x\,dx = \int (\log x)\cdot x^2dx$

$= (\log x)\int x^2\,dx - \int\left[\frac{d}{dx}(\log x)\int x^2dx\right]dx$

$= (\log x)\cdot\frac{\square}{\square} - \int \frac{1}{\square}\cdot\frac{\square}{\square}\,dx$

$= \frac{1}{\square}x^3\log x - \frac{1}{\square}\int \square\,dx$

$= \frac{1}{\square}x^3\log x - \frac{1}{\square}\cdot\frac{\square}{3} + c$

$= \frac{x^3}{\square}(\square\log x + 1) + c.$

**6. Evaluate :** $\int e^x\cdot\frac{x-1}{(x+1)^3}\,dx.$

**Solution :** Let $I = \int e^x\cdot\frac{x-1}{(x+1)^3}\,dx$

$= \int e^x\left[\frac{(x+1)-\square}{(x+1)^3}\right]dx$

$= \int e^x\left[\frac{1}{(x+1)^2} - \frac{\square}{(x+1)^3}\right]dx$

Put $f(x) = \frac{1}{(x+1)^2}$

$\therefore\ f'(x) = \frac{d}{dx}(x+1)^{-2} = -2(x+1)^{\square}\cdot(1+0)$

$= \frac{-2}{(x+1)^{\square}}$

$\therefore\ I = \int e^x[f(x)+f'(x)]\,dx$

$= e^x\cdot f(x) + c = \frac{e^x}{\square} + c.$

**7. Evaluate :** $\int \sqrt{9x^2-4}\,dx.$

**Solution :** $\int \sqrt{9x^2-4}\,dx = 3\int\sqrt{x^2-\frac{4}{9}}\,dx$

$= 3\int\sqrt{x^2-\left(\frac{2}{3}\right)^2}\,dx$

$= 3\left|\frac{\square}{2}\sqrt{x^2-\frac{4}{9}}-\frac{\square}{2}\log|x+\square|\right|+c_1$

$= \frac{3\square}{2}\cdot\frac{\sqrt{9x^2-4}}{\square}-\frac{\square}{3}\log\left|\frac{3x+\square}{\square}\right|+c_1$

$= \frac{x}{\square}\cdot\sqrt{9x^2-4}-\frac{\square}{3}[\log|3x+\square|-\log 3]+c_1$

$= \frac{x}{\square}\cdot\sqrt{9x^2-4}-\frac{\square}{3}\log|3x+\square|+\frac{2}{3}\log 3+c_1$

$= \frac{x}{\square}\sqrt{9x^2-4}-\frac{\square}{3}\log|3x+\square|+c,$

where $c = c_1+\frac{2}{3}\log 3.$

---

**8. Evaluate :** $\int\frac{2x}{4-3x-x^2}\,dx.$

**Solution :** Let $I = \int\frac{2x}{(4+x)(1-x)}$

Let $\frac{2x}{4-3x-x^2} = \frac{2x}{(4+x)(1-x)} = \frac{A}{4+x}+\frac{B}{1-x}$

$\therefore\ 2x = A(1-x)+B(4+x)$

$\therefore\ A = \square,\ B = \square$

$\therefore\ \frac{2x}{4-3x-x^2} = \frac{\square}{4+x}+\frac{\square}{1-x}$

$\therefore\ I = \int\left(\frac{\square}{4+x}+\frac{\square}{1-x}\right)dx$

$= -\square\int\frac{1}{4+x}\,dx+\square\int\frac{1}{1-x}\,dx$

$= -\square\log|4+x|-\square\log\square+c.$

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## OBJECTIVE SECTION

### MULTIPLE CHOICE QUESTIONS

**Select and write the correct answer from the given alternatives in each of the following questions :**

**1.** $\int\frac{1}{\sqrt{1-2x}}\,dx = \ldots\ldots\ldots$

(a) $\sqrt{1-2x}+c$ (b) $-\sqrt{1-2x}+c$

(c) $(1-2x)^{\frac{3}{2}}+c$ (d) $-(1-2x)^{\frac{3}{2}}+c$

**2.** $\int\frac{1}{x(x-1)}\,dx = \ldots\ldots\ldots$

(a) $\log\left|\frac{x-1}{x}\right|+c$ (b) $\log\left|\frac{x}{x-1}\right|+c$

(c) $\log|x|+\log|x-1|+c$ (d) $\log|x+\sqrt{x^2-1}|+c$

**3.** $\int\frac{e^{5\log x}-e^{4\log x}}{e^{3\log x}-e^{2\log x}}\,dx$ is

(a) $\log|x^3-x^2|+c$ (b) $\frac{x^3}{3}+c$

(c) $\frac{x}{2}+c$ (d) $\log|x(x-1)|+c$

**4.** $\int(e^{2a\log x}+e^{3x\log a})\,dx$ is

(a) $x^{2a}+a^{3x}+c$ (b) $\frac{e^{2a\log x}}{x}+\frac{e^{3x\log a}}{3}+c$

(c) $\frac{x^{2a+1}}{2a+1}+\frac{a^{3x}}{3\log a}+c$ (d) $\frac{x^{2a+1}}{2a+1}+\frac{a^{3x}}{\log a}+c$

**5.** $\int e^{x\log a}\cdot e^x\,dx$ is

(a) $(ae)^x+c$ (b) $\frac{(ae)^x}{\log ae}+c$

(c) $(ae)^x\log ae+c$ (d) $\frac{ae^x}{\log ae}+c$

**6.** $\int\frac{4^{x+1}-7^{x-1}}{28^x}\,dx = \ldots\ldots\ldots$

(a) $\left(\frac{1}{7\log_e 4}\right)\cdot 4^{-x}-\left(\frac{4}{\log_e 7}\right)\cdot 7^{-x}+c$

(b) $\left(\frac{1}{7\log_e 4}\right)\cdot 4^{x}+\left(\frac{4}{\log_e 7}\right)\cdot 7^{x}+c$

(c) $\frac{4^{-x}}{\log_e 7}-\frac{7^{-x}}{\log_e 4}+c$

(d) $\left(\frac{4}{\log_e 7}\right)\cdot 7^{-x}-\left(\frac{1}{7\log_e 4}\right)\cdot 4^{-x}+c$

**7.** $\int\frac{2x-3}{\sqrt{4x+1}}\,dx$ is

(a) $\frac{1}{4}(4x+1)^{\frac{3}{2}}-\frac{7}{4}\sqrt{4x+1}+c$

(b) $\frac{1}{12}(4x+1)^{\frac{3}{2}}+\frac{7}{4}\sqrt{4x+1}+c$

(c) $\frac{1}{12}(4x+1)^{\frac{3}{2}}-\frac{7}{4}\sqrt{4x+1}+c$

(d) $\frac{1}{6}(4x+1)^{\frac{3}{2}}-\frac{7}{4}\sqrt{4x+1}+c$

**8.** $\int\frac{dx}{3x-2\sqrt{x}}$ is

(a) $\log|3x-2\sqrt{x}|+c$ (b) $\frac{2}{3}\log|3x|-\frac{3}{2}\log|x|+c$

(c) $\log|3x-2|+c$ (d) $\frac{2}{3}\log|3\sqrt{x}-2|+c$

9. $\int \frac{dx}{x+x^n}$ is

(a) $\frac{1}{n}\log|x^n|+c$ (b) $\frac{1}{1-n}\log|x^{1-n}+1|+c$

(c) $\log|x^n+1|+c$ (d) $\frac{1}{n}\log|x^n+1|+c$

10. $\int \frac{2x}{(2x+1)^2}dx = \ldots\ldots\ldots$

(a) $\frac{1}{2}\log|2x+1|+\frac{1}{2(2x+1)}+c$

(b) $\frac{1}{2}\log|2x+1|-\frac{1}{2(2x+1)}+c$

(c) $2\log|2x+1|+\frac{1}{2(2x+1)}+c$

(d) $\frac{1}{2}\log|2x+1|+\frac{2}{2x+1}+c$

11. $\int \frac{1}{x}\cdot\log x\,dx = \ldots\ldots\ldots$

(a) $\log(\log x)+c$ (b) $\frac{1}{2}(\log x)^2+c$

(c) $2\log x+c$ (d) $\log x+c$

12. $\int e^{3\log x}(x^4+1)^{-1}dx = \ldots\ldots\ldots$

(a) $\frac{1}{4}\log(x^4+1)+c$ (b) $-\log(x^4+1)+c$

(c) $\log(x^4+1)+c$ (d) $-\frac{1}{4}\log(x^4+1)+c$

13. $\int \frac{4x+1}{\sqrt{2x^2+x-3}}dx$ is

(a) $\sqrt{(2x^2+x-3)}+c$ (b) $\log|2x^2+x-3|+c$

(c) $2\sqrt{2x^2+x-3}+c$ (d) $\frac{1}{2}\log|2x^2+x-3|+c$

14. $\int \frac{dx}{x^2(1-3x)} = \ldots\ldots\ldots$

(a) $-\frac{1}{x}+3\log\left|\frac{x}{1-3x}\right|+c$ (b) $\log\left|\frac{x^2}{1-3x}\right|+c$

(c) $\log\left|\frac{x}{1-3x}\right|+c$ (d) $3\log\left|\frac{x}{1-3x}\right|+c$

15. If $\int \frac{1+x+x^2}{x^3(1+x)}dx = \frac{k}{x^2}+l\log\left|\frac{x}{1+x}\right|+c$, then

(a) $k=-\frac{1}{2}, l=1$ (b) $k=-\frac{1}{2}, l=-1$

(c) $k=\frac{1}{2}, l=-1$ (d) $k=\frac{1}{2}, l=1$

16. $\int \frac{dx}{x(x^2+1)} = \ldots\ldots\ldots$

(a) $\log x-\frac{1}{2}\log(x^2+1)+c$ (b) $\frac{1}{2}\log\left(\frac{x^2+1}{x^2}\right)+c$

(c) $\log x+\frac{1}{2}\log(x^2+1)+c$ (d) $\frac{1}{4}\log\left(\frac{x^2}{x^2+1}\right)+c$

17. $\int e^{3x}\left(\log 2x+\frac{1}{3x}\right)dx = \ldots\ldots\ldots$

(a) $e^{3x}\log(2x)+c$ (b) $e^{3x}\cdot\log x+c$

(c) $\frac{1}{3}e^{3x}\cdot\log(2x)+c$ (d) $\frac{1}{3}\cdot e^{3x}\cdot\log x+c$

18. If $\int \frac{4e^x+6e^{-x}}{9e^x-4e^{-x}}dx = Ax+B\log(9e^{2x}-4)+c$, then the values of $A$, $B$ are

(a) $-\frac{3}{2}, \frac{35}{36}$ (b) $\frac{3}{2}, \frac{35}{36}$

(c) $-\frac{3}{2}, \frac{-35}{36}$ (d) $\frac{3}{2}, \frac{-35}{36}$

19. $\int \frac{\log(x+2)-\log x}{x(x+2)}dx = \ldots\ldots\ldots$

(a) $\frac{1}{4}\left(\log\left|\frac{x+2}{x}\right|\right)^2+c$

(b) $[\log(x+2)-\log x]^2+c$

(c) $-\frac{1}{4}\left(\log\left|\frac{x+2}{x}\right|\right)^2+c$

(d) $-\frac{1}{2}\left(\log\left|\frac{x+2}{x}\right|\right)^2+c$

20. $\int x^3\cdot e^{x^2}dx = \ldots\ldots\ldots$

(a) $\frac{1}{2}(x^2+1)e^{x^2}+c$ (b) $(x^2+1)e^{x^2}+c$

(c) $\frac{1}{2}(x^2-1)e^{x^2}+c$ (d) $(x^2-1)e^{x^2}+c$

21. $\int \frac{x+2}{\sqrt{x^2-1}}dx$ is

(a) $2\sqrt{x^2-1}+c$

(b) $\log|\sqrt{x^2-1}|+c$

(c) $\log|x+\sqrt{x^2-1}|+c$

(d) $\sqrt{x^2-1}+2\log|x+\sqrt{x^2-1}|+c$

22. $\int \frac{dx}{x(x^7+3)} = \ldots\ldots\ldots$

(a) $\frac{1}{21}\log\left|\frac{x^7}{x^7+3}\right|+c$ (b) $\frac{1}{7}\log|x^7+3|+c$

(c) $\frac{1}{7}\log\left|\frac{x^7+3}{x^7}\right|+c$ (d) $\frac{1}{21}\log|x^7+3|+c$

23. $\int \frac{x^{11}+x^{12}+x^{13}}{x^{14}+x^{15}}\,dx=$ .........

(a) $\log|x^{14}+x^{15}|+c$ (b) $\log\left|\frac{x}{1+x}\right|-\frac{1}{2x^2}+c$

(c) $\log\left|\frac{x}{1+x}\right|+\frac{1}{2x^2}+c$ (d) $\log\left|\frac{1+x}{x}\right|-\frac{1}{2x^2}+c$

24. If $\int \frac{x}{(x^2-a^2)(x^2-b^2)}\,dx=k\log\left(\frac{x^2-a^2}{x^2-b^2}\right)+c$, then $k=$ .........

(a) $\frac{1}{a^2-b^2}$ (b) $\frac{1}{b^2-a^2}$

(c) $\frac{1}{2(a^2-b^2)}$ (d) $\frac{1}{2(b^2-a^2)}$

25. If $\int x^2 \log x\,dx=kx^3\log x+lx^3+c$, then

(a) $k=\frac{1}{3}, l=\frac{1}{9}$ (b) $k=-\frac{1}{3}, l=\frac{1}{9}$

(c) $k=-\frac{1}{3}, l=-\frac{1}{9}$ (d) $k=\frac{1}{3}, l=-\frac{1}{9}$

### Answers

1. (b) $-\sqrt{1-2x}+c$
2. (a) $\log\left|\frac{x-1}{x}\right|+c$
3. (b) $\frac{x^3}{3}+c$
4. (c) $\frac{x^{2a+1}}{2a+1}+\frac{a^{3x}}{3\log a}+c$
5. (b) $\frac{(ae)^x}{\log ae}+c$
6. (a) $\left(\frac{1}{7\log_e 4}\right)\cdot 4^{-x}-\left(\frac{4}{\log_e 7}\right)\cdot 7^{-x}+c$
7. (c) $\frac{1}{12}(4x+1)^{\frac{3}{2}}-\frac{7}{4}\sqrt{4x+1}+c$
8. (d) $\frac{2}{3}\log|3\sqrt{x}-2|+c$
9. (b) $\frac{1}{1-n}\log|x^{1-n}+1|+c$
10. (a) $\frac{1}{2}\log|2x+1|+\frac{1}{2(2x+1)}+c$
11. (b) $\frac{1}{2}(\log x)^2+c$
12. (a) $\frac{1}{4}\log(x^4+1)+c$
13. (c) $2\sqrt{2x^2+x-3}+c$
14. (a) $-\frac{1}{x}+3\log\left|\frac{x}{1-3x}\right|+c$
15. (a) $k=-\frac{1}{2}, l=1$
16. (a) $\log x-\frac{1}{2}\log(x^2+1)+c$
17. (c) $\frac{1}{3}e^{3x}\cdot\log(2x)+c$
18. (a) $-\frac{3}{2}, \frac{35}{36}$
19. (c) $-\frac{1}{4}\left(\log\left|\frac{x+2}{x}\right|\right)^2+c$
20. (c) $\frac{1}{2}(x^2-1)e^{x^2}+c$
21. (d) $\sqrt{x^2-1}+2\log|x+\sqrt{x^2-1}|+c$
22. (a) $\frac{1}{21}\log\left|\frac{x^7}{x^7+3}\right|+c$
23. (b) $\log\left|\frac{x}{1+x}\right|-\frac{1}{2x^2}+c$
24. (c) $\frac{1}{2(a^2-b^2)}$
25. (d) $k=\frac{1}{3}, l=-\frac{1}{9}$.

## TRUE OR FALSE

**State whether the following statements are *True* or *False* :**

1. $\int \frac{f'(x)}{\sqrt{f(x)}}\,dx=2\sqrt{f(x)}+c$.
2. $\int \frac{1}{\sqrt{2x-3}}\,dx=\frac{1}{2}\log|\sqrt{2x-3}|+c$.
3. $\int \log x\,dx=x(1+\log x)+c$.
4. $\int e^x\left(\frac{x-1}{x^2}\right)dx=\frac{e^x}{x}+c$.
5. $\int \frac{x}{(x+1)(x+2)}\,dx=\frac{1}{2}\log\left|\frac{(x+2)^2}{x+1}\right|+c$.
6. $\int (ax+b)^n dx=\frac{(ax+b)^{n+1}}{n+1}+c, (n\neq -1)$.
7. $\int \frac{5(x^6+1)}{x^2+1}\,dx=x^5-\frac{5}{3}x^3+5x+c$.
8. $\int \frac{dx}{\sqrt{x}+\sqrt{x-2}}=3\left[x^{\frac{3}{2}}-(x-2)^{\frac{3}{2}}\right]+c$.
9. $\int \frac{3x^2}{\sqrt{1+x^3}}\,dx=\sqrt{1+x^3}+c$.
10. $\int \sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}+\frac{a^2}{2}\log|x+\sqrt{x^2-a^2}|+c$.

## Answers

1. True 2. False 3. False 4. True 5. False
6. False 7. True 8. False 9. False 10. False.

## FILL IN THE BLANKS

Fill in the following blanks with an appropriate words/numbers :

1. $\int \frac{1}{x\sqrt{x}}\,dx = \ldots\ldots\ldots$
2. $\int \frac{f'(x)}{f(x)}\,dx = \ldots\ldots\ldots$
3. $\int \frac{1}{x+\sqrt{x}}\,dx = \ldots\ldots\ldots$
4. $\int \frac{3e^x - 4}{4e^x + 5}\,dx = \ldots\ldots\ldots$
5. $\int 2x\sqrt{1-x^2}\,dx = \ldots\ldots\ldots$
6. $\int \frac{1}{(e^x + e^{-x})^2}\,dx = \ldots\ldots\ldots$
7. $\int \frac{dx}{\sqrt{(x-1)(x-2)}} = \ldots\ldots\ldots$
8. $\int x^2 e^x\,dx = \ldots\ldots\ldots$
9. $\int \frac{xe^x}{(1+x)^2}\,dx = \ldots\ldots\ldots$
10. $\int \frac{e^x}{x+2}[1+(x+2)\log(x+2)]\,dx = \ldots\ldots\ldots$

## Answers

1. $-\frac{2}{\sqrt{x}} + c$
2. $\log|f(x)| + c$
3. $2\log|1+\sqrt{x}| + c$
4. $-\frac{4}{5}x + \frac{31}{20}\log|4e^x + 5| + c$
5. $-\frac{2}{3}(1-x^2)^{\frac{3}{2}} + c$
6. $-\frac{1}{2(e^{2x}+1)} + c$
7. $\log\left|\left(x-\frac{3}{2}\right) + \sqrt{x^2-3x+2}\right| + c$
8. $e^x(x^2 - 2x + 2) + c$
9. $\frac{e^x}{1+x} + c$
10. $e^x \cdot \log(x+2) + c.$

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# 6 DEFINITE INTEGRATION

## CHAPTER OUTLINE



## IMPORTANT FORMULAE

1. If $\int f(x)\,dx = g(x) + c$, then
$\int_a^b f(x)\,dx = [g(x)]_a^b = g(b) - g(a)$

2. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

3. $\int_a^b k \cdot f(x)\,dx = k\int_a^b f(x)\,dx$, where $k$ is a constant.

4. $\int_a^a f(x)\,dx = 0$

5. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

6. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

7. If $a < c < b$, then
$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

8. $\int_0^a f(x)\,dx = \int_0^a f(a - x)\,dx$

9. $\int_a^b f(x)\,dx = \int_a^b f(a + b - x)\,dx$

10. $\int_0^{2a} f(x)\,dx = \int_0^a [f(x) + f(2a - x)]\,dx$

11. $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$, if $f$ is an even function
$= 0$ , if $f$ is an odd function.

### INTRODUCTION

We have studied indefinite integration in chapter 5. In this chapter, we shall study the integrals of functions which are defined over an interval $[a, b]$, i.e. $\int_a^b f(x)\,dx$ (read as integral of $f(x)$ w.r.t. $x$ between $x = a$ and $x = b$).

## 6.1 : FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

The *Fundamental Theorem of Integral Calculus* provides an easy method of evaluating $\int_a^b f(x)\,dx$. The theorem is stated below and we accept it without proof.

*If a function $f$ is continuous on the closed interval $[a, b]$ and $\int f(x)\,dx = \phi(x)$, then*
*$\int_a^b f(x)\,dx = [\phi(x)]_a^b = \phi(b) - \phi(a)$.*

$a$ and $b$ are called the *limits of integration*, $a$ being the lower limit and $b$ the upper limit of integration.

Note that the indefinite integral $\int f(x)\,dx$ is a function of $x$ but the definite integral $\int_a^b f(x)\,dx$ is a real number. The definite integral does not contain the constant of integration. We can easily verify this as follows :

Let $\int f(x)\,dx = \phi(x) + c$. Then

$$\int_a^b f(x)\,dx = [\phi(x)+c]_a^b$$
$$= [\phi(b)+c] - [\phi(a)+c]$$
$$= \phi(b) - \phi(a)$$

**Results :**

If $f$ and $g$ are integrable functions of $x$ and $k$ is a constant, then

1. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
2. $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx.$

**EXERCISE 6.1** | **Textbook page 145**

**Evaluate the following definite integrals :**

**1.** $\int_4^9 \frac{1}{\sqrt{x}}\,dx$

**Solution :** $\int_4^9 \frac{1}{\sqrt{x}}\,dx = \int_4^9 x^{-\frac{1}{2}}\,dx$

$$= \left[\frac{x^{\frac{1}{2}}}{1/2}\right]_4^9 = 2[\sqrt{x}]_4^9$$
$$= 2(\sqrt{9} - \sqrt{4})$$
$$= 2(3-2) = 2.$$

**2.** $\int_{-2}^3 \frac{1}{x+5}\,dx$

**Solution :** $\int_{-2}^3 \frac{1}{x+5}\,dx$

$$= [\log|x+5|]_{-2}^3$$
$$= \log 8 - \log 3$$
$$= \log\left(\frac{8}{3}\right).$$

**3.** $\int_2^3 \frac{x}{x^2-1}\,dx$

**Solution :** $\int_2^3 \frac{x}{x^2-1}\,dx$

$$= \frac{1}{2}\int_2^3 \frac{2x}{x^2-1}\,dx$$
$$= \frac{1}{2}[\log|x^2-1|]_2^3 \quad \ldots \left[\because \frac{d}{dx}(x^2-1) = 2x \text{ and } \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + c\right]$$
$$= \frac{1}{2}[\log(9-1) - \log(4-1)] = \frac{1}{2}\log\left(\frac{8}{3}\right).$$

**4.** $\int_0^1 \frac{x^2+3x+2}{\sqrt{x}}\,dx$

**Solution :** $\int_0^1 \frac{x^2+3x+2}{\sqrt{x}}\,dx$

$$= \int_0^1 \left(\frac{x^2}{\sqrt{x}} + \frac{3x}{\sqrt{x}} + \frac{2}{\sqrt{x}}\right)dx$$
$$= \int_0^1 \left(x^{\frac{3}{2}} + 3x^{\frac{1}{2}} + 2x^{-\frac{1}{2}}\right)dx$$
$$= \left[\frac{x^{\frac{5}{2}}}{5/2} + 3\left(\frac{x^{\frac{3}{2}}}{3/2}\right) + 2\left(\frac{x^{\frac{1}{2}}}{1/2}\right)\right]_0^1$$
$$= \left[\frac{2}{5}x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + 4x^{\frac{1}{2}}\right]_0^1$$
$$= \left[\frac{2}{5}(1)^{\frac{5}{2}} + 2(1)^{\frac{3}{2}} + 4(1)^{\frac{1}{2}}\right] - (0+0+0)$$
$$= \frac{2}{5} + 2 + 4 = \frac{32}{5}.$$

**5.** $\int_2^3 \frac{x}{(x+2)(x+3)}\,dx$

**Solution :** Let $I = \int_2^3 \frac{x}{(x+2)(x+3)}\,dx$

Let $\frac{x}{(x+2)(x+3)} = \frac{A}{x+3} + \frac{B}{x+2}$

$\therefore x = A(x+2) + B(x+3)$

Put $x+3=0$, i.e. $x=-3$, we get

$-3 = A(-1) + B(0) \quad \therefore A = 3$

Put $x+2=0$, i.e. $x=-2$, we get

$-2 = A(0) + B(1) \quad \therefore B = -2$

$\therefore \frac{x}{(x+2)(x+3)} = \frac{3}{x+3} + \frac{(-2)}{x+2}$

$$\therefore I = \int_2^3 \left[\frac{3}{x+3} + \frac{(-2)}{x+2}\right]dx$$
$$= [3\log(x+3) - 2\log(x+2)]_2^3$$
$$= [3\log(3+3) - 2\log(3+2)] - [3\log(2+3) - 2\log(2+2)]$$
$$= 3\log 6 - 5\log 5 + 2\log 4$$
$$= \log 6^3 - \log 5^5 + \log 4^2$$

$= \log 216 - \log 3125 + \log 16$

$= \log\left(\frac{216 \times 16}{3125}\right) = \log\left(\frac{3456}{3125}\right).$

**6.** $\int_1^2 \frac{dx}{x^2+6x+5}$

**Solution :** $\int_1^2 \frac{dx}{x^2+6x+5}$

$= \int_1^2 \frac{dx}{(x^2+6x+9)-4}$

$= \int_1^2 \frac{1}{(x+3)^2-(2)^2}\,dx$

$= \frac{1}{2(2)}\left[\log\left|\frac{x+3-2}{x+3+2}\right|\right]_1^2 = \frac{1}{4}\left[\log\left|\frac{x+1}{x+5}\right|\right]_1^2$

$= \frac{1}{4}\left[\log\frac{3}{7} - \log\frac{2}{6}\right]$

$= \frac{1}{4}\log\left(\frac{3}{7}\times\frac{6}{2}\right) = \frac{1}{4}\log\left(\frac{9}{7}\right).$

**7. If $\int_0^a (2x+1)\,dx = 2$, find the real values of '$a$'.**

**Solution :** Let $I = \int_0^a (2x+1)\,dx$

$= \left[2\cdot\frac{x^2}{2} + x\right]_0^a$

$= a^2 + a - 0 = a^2 + a$

$\therefore$ $I = 2$ gives $a^2 + a = 2$

$\therefore$ $a^2 + a - 2 = 0$

$\therefore$ $(a+2)(a-1) = 0$

$\therefore$ $a+2=0$ or $a-1=0$

$\therefore$ $a = -2$ or $a = 1$.

**8. If $\int_1^a (3x^2+2x+1)\,dx = 11$, find '$a$'.**

**Solution :** Let $I = \int_1^a (3x^2+2x+1)\,dx$

$= \left[3\left(\frac{x^3}{3}\right) + 2\left(\frac{x^2}{2}\right) + x\right]_1^a$

$= [x^3 + x^2 + x]_1^a$

$= (a^3 + a^2 + a) - (1+1+1)$

$= a^3 + a^2 + a - 3$

$\therefore$ $I = 11$ gives $a^3 + a^2 + a - 3 = 11$

$\therefore$ $a^3 + a^2 + a - 14 = 0$

$\therefore$ $(a^3 - 8) + (a^2 + a - 6) = 0$

$\therefore$ $(a-2)(a^2+2a+4) + (a+3)(a-2) = 0$

$\therefore$ $(a-2)(a^2+2a+4+a+3) = 0$

$\therefore$ $(a-2)(a^2+3a+7) = 0$

$\therefore$ $a-2=0$ or $a^2+3a+7=0$

$\therefore$ $a = 2$ or $a = \frac{-3 \pm \sqrt{9-28}}{2}$

The latter two roots are not real.

$\therefore$ they are rejected.

$\therefore$ $a = 2$.

**9.** $\int_0^1 \frac{1}{\sqrt{1+x}+\sqrt{x}}\,dx$

**Solution :** $\int_0^1 \frac{1}{\sqrt{1+x}+\sqrt{x}}\,dx$

$= \int_0^1 \frac{1}{\sqrt{1+x}+\sqrt{x}} \times \frac{\sqrt{1+x}-\sqrt{x}}{\sqrt{1-x}-\sqrt{x}}\,dx$

$= \int_0^1 \frac{\sqrt{1+x}-\sqrt{x}}{1+x-x}\,dx$

$= \int_0^1 (\sqrt{1+x}-\sqrt{x})\,dx$

$= \int_0^1 (1+x)^{\frac{1}{2}}dx - \int_0^1 x^{\frac{1}{2}}dx$

$= \left[\frac{(1+x)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right]_0^1 - \left[\frac{x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right]_0^1$

$= \frac{2}{3}\left[(1+x)^{\frac{3}{2}}\right]_0^1 - \frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^1$

$= \frac{2}{3}(2^{\frac{3}{2}} - 1) - \frac{2}{3}(1-0)$

$= \frac{2}{3}(2^{\frac{3}{2}} - 1 - 1) = \frac{2}{3}(2\sqrt{2} - 2)$

$= \frac{4}{3}(\sqrt{2} - 1).$

**10.** $\int_1^2 \frac{3x}{9x^2-1}\,dx$

**Solution :**

Let $I = \int_1^2 \frac{3x}{9x^2-1}\,dx = \int_1^2 \frac{3x}{(3x)^2-1}\,dx$

Put $3x = t$ $\quad\therefore$ $3dx = dt$ $\quad\therefore$ $dx = \frac{dt}{3}$





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When $x=1$, $t=3\times 1=3$

When $x=2$, $t=3\times 2=6$

$$\therefore\ I=\int_3^6 \frac{t}{t^2-1}\cdot\frac{dt}{3}=\frac{1}{6}\int_3^6 \frac{2t}{t^2-1}\,dt$$

$$=\frac{1}{6}\Big[\log|t^2-1|\Big]_3^6 \qquad \ldots\left[\because \frac{d}{dt}(t^2-1)=2t\right]$$

$$=\frac{1}{6}[\log 35-\log 8]$$

$$=\frac{1}{6}\log\left(\frac{35}{8}\right).$$

*Alternative Method :*

$$\int_1^2 \frac{3x}{9x^2-1}\,dx$$

$$=\frac{1}{6}\int_1^2 \frac{18x}{9x^2-1}\,dx$$

$$=\frac{1}{6}\Big[\log|9x^2-1|\Big]_1^2$$

$$\ldots\left[\because \frac{d}{dx}(9x^2-1)=18x \text{ and } \int\frac{f'(x)}{f(x)}\,dx=\log|f(x)|\right]$$

$$=\frac{1}{6}[\log 35-\log 8]$$

$$=\frac{1}{6}\log\left(\frac{35}{8}\right).$$

**11.** $\displaystyle\int_1^3 \log x\,dx$

**Solution :** $\displaystyle\int_1^3 \log x\,dx=\int_1^3 (\log x)\cdot 1\,dx$

$$=[(\log x)]\int 1dx]_1^3-\int_1^3\left[\frac{d}{dx}(\log x)\int 1\,dx\right]dx$$

$$=[(\log x)x]_1^3-\int_1^3 \frac{1}{x}\times x\,dx$$

$$=(3\log 3-\log 1)-\int_1^3 1\,dx$$

$$=3\log 3-[x]_1^3 \qquad \ldots[\because \log 1=0]$$

$$=\log 3^3-(3-1)$$

$$=\log 27-2.$$

[**Note :** *Answer in the textbook is incorrect.*]

## ADDITIONAL SOLVED PROBLEMS – 6 (A)

**1. If $f(x)=a+bx+cx^2$, show that**

$$\int_0^1 f(x)\,dx=\frac{1}{6}\left[f(0)+4f\left(\frac{1}{2}\right)+f(1)\right].$$

**Solution :** $\displaystyle\int_0^1 f(x)\,dx=\int_0^1 (a+bx+cx^2)\,dx$

$$=a\int_0^1 1\,dx+b\int_0^1 x\,dx+c\int_0^1 x^2\,dx$$

$$=\left[ax+\frac{bx^2}{2}+\frac{cx^3}{3}\right]_0^1$$

$$=a+\frac{b}{2}+\frac{c}{3} \qquad \ldots(1)$$

Now, $f(0)=a+b(0)+c(0)^2=a$

$$f\left(\frac{1}{2}\right)=a+b\left(\frac{1}{2}\right)+c\left(\frac{1}{2}\right)^2=a+\frac{b}{2}+\frac{c}{4}$$

and $f(1)=a+b(1)+c(1)^2=a+b+c$

$$\therefore\ \frac{1}{6}\left[f(0)+4f\left(\frac{1}{2}\right)+f(1)\right]$$

$$=\frac{1}{6}\left[a+4\left(a+\frac{b}{2}+\frac{c}{4}\right)+(a+b+c)\right]$$

$$=\frac{1}{6}[a+4a+2b+c+a+b+c]$$

$$=\frac{1}{6}[6a+3b+2c]$$

$$=a+\frac{b}{2}+\frac{c}{3} \qquad \ldots(2)$$

$\therefore$ from (1) and (2),

$$\int_0^1 f(x)\,dx=\frac{1}{6}\left[f(0)+4f\left(\frac{1}{2}\right)+f(1)\right].$$

**2. Evaluate :** $\displaystyle\int_0^1 \frac{dx}{\sqrt{x^2-x+1}}$.

**Solution :** $\displaystyle\int_0^1 \frac{dx}{\sqrt{x^2-x+1}}=\int_0^1 \frac{dx}{\sqrt{\left(x^2-x+\frac{1}{4}\right)+\frac{3}{4}}}$

$$=\int_0^1 \frac{dx}{\sqrt{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}$$

$$=\left[\log\left|\left(x-\frac{1}{2}\right)+\sqrt{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right|\right]_0^1$$

$$=\left[\log\left|\left(x-\frac{1}{2}\right)+\sqrt{x^2-x+1}\right|\right]_0^1$$

$= \log \left|\frac{1}{2} + \sqrt{1-1+1}\right| - \log \left|-\frac{1}{2} + \sqrt{1}\right|$

$= \log \frac{3}{2} - \log \frac{1}{2} = \log 3.$

---

**3. Evaluate :** $\int_2^5 \frac{1}{x^2+4x-5}\,dx.$

**Solution :** $\int_2^5 \frac{1}{x^2+4x-5}\,dx = \int_2^5 \frac{1}{(x^2+4x+4)-9}\,dx$

$= \int_2^5 \frac{1}{(x+2)^2-(3)^2}\,dx$

$= \frac{1}{2\times 3}\left[\log\left|\frac{x+2-3}{x+2+3}\right|\right]_2^5$

$= \frac{1}{6}\left[\log\left|\frac{x-1}{x+5}\right|\right]_2^5$

$= \frac{1}{6}\left[\log\left(\frac{4}{10}\right) - \log\left(\frac{1}{7}\right)\right]$

$= \frac{1}{6}\log\left(\frac{4}{10}\times 7\right) = \frac{1}{6}\log\left(\frac{14}{5}\right).$

---

**4. Evaluate :** $\int_0^1 \frac{\log(1+x)}{1+x}\,dx.$

**Solution :** Let $I = \int_0^1 \frac{\log(1+x)}{1+x}\,dx$

$= \int_0^1 \log(1+x)\cdot\frac{1}{1+x}\,dx$

Put $\log(1+x) = t$

$\therefore \frac{1}{1+x}\,dx = dt$

When $x=0,\ t=\log 1 = 0$

When $x=1,\ t=\log 2$

$\therefore I = \int_0^{\log 2} t\,dt = \left[\frac{t^2}{2}\right]_0^{\log 2}$

$= \frac{1}{2}[(\log 2)^2 - 0] = \frac{1}{2}(\log 2)^2.$

---

**5. Evaluate :** $\int_0^2 x^2 e^x\,dx.$

**Solution :** $\int_0^2 x^2 e^x\,dx = [x^2 \int e^x\,dx]_0^2 - \int_0^2 \left[\frac{d}{dx}(x^2)\int e^x\,dx\right]dx$

$= [x^2 e^x]_0^2 - \int_0^2 2x\cdot e^x\,dx$

$= (4e^2 - 0) - 2\int_0^2 x\cdot e^x\,dx$

$= 4e^2 - 2\left[\{x\int e^x\,dx\}_0^2 - \int_0^2\left(\frac{d}{dx}(x)\int e^x dx\right)dx\right]$

$= 4e^2 - 2[\{xe^x\}_0^2 - \int_0^2 1\cdot e^x\,dx]$

$= 4e^2 - 2(2e^2 - 0) + 2\int_0^2 e^x\,dx$

$= 4e^2 - 4e^2 + 2[e^x]_0^2$

$= 2(e^2 - 1).$

---

**6. Evaluate :** $\int_2^3 \frac{dx}{x(x^3-1)}.$

**Solution :** Let $I = \int_2^3 \frac{dx}{x(x^3-1)} = \int_2^3 \frac{x^2 dx}{x^3(x^3-1)}$

Put $x^3 = t \quad \therefore\ 3x^2 dx = dt$

$\therefore\ x^2 dx = \frac{dt}{3}$

When $x=2,\ t=8$

When $x=3,\ t=27$

$\therefore\ I = \int_8^{27} \frac{1}{t(t-1)}\cdot\frac{dt}{3}$

$= \frac{1}{3}\int_8^{27} \frac{t-(t-1)}{t(t-1)}\,dt$

$= \frac{1}{3}\int_8^{27}\left(\frac{1}{t-1} - \frac{1}{t}\right)dt$

$= \frac{1}{3}[\log|t-1| - \log|t|]_8^{27}$

$= \frac{1}{3}\left[\log\left|\frac{t-1}{t}\right|\right]_8^{27}$

$= \frac{1}{3}\left[\log\left(\frac{26}{27}\right) - \log\left(\frac{7}{8}\right)\right]$

$= \frac{1}{3}\log\left(\frac{26}{27}\times\frac{8}{7}\right) = \frac{1}{3}\log\left(\frac{208}{189}\right).$

---

## EXAMPLES FOR PRACTICE 6.1

**1. Evaluate the following definite integrals :**

(1) $\int_0^1 \frac{dx}{\sqrt{x+3}}$

(2) $\int_0^1 \frac{dx}{\sqrt{x+1}-\sqrt{x}}$

(3) $\int_0^1 \frac{3x^3-4x^2+1}{\sqrt{x}}\,dx$

(4) $\int_0^1 \frac{1}{2x+5}\,dx$

(5) $\int_2^3 \frac{x^3+2x-2}{x^3(x-1)}\,dx$

(6) $\int_2^4 \frac{4x^2-2}{x^3-x}\,dx$

(7) $\int_0^5 \frac{2x\ dx}{(x-1)(x-2)}$

(8) $\int_0^2 \frac{1}{4+x-x^2}\,dx$

(9) $\int_0^4 \frac{dx}{\sqrt{x^2+2x+3}}$

(10) $\int_0^2 xe^{2x}\,dx$

(11) $\int_1^2 x \log x \, dx$ (12) $\int_1^2 \frac{\log x}{x^2} dx$

(13) $\int_1^e (\log x)^2 dx$ (14) $\int_1^2 \frac{1}{(x+1)(x+3)} dx$

(15) $\int_0^3 \frac{x}{(x+1)(x+2)} dx$ (16) $\int_1^3 \frac{1}{x(1+x^2)} dx.$

**2. Evaluate the following definite integrals :**

(1) $\int_{1/e}^{e} \frac{dx}{x(\log x)^{1/3}}$ (2) $\int_1^2 \frac{1}{x(2+\log x)^2} dx$

(3) $\int_0^1 x^3(1-x^2)^{\frac{1}{2}} dx$ (4) $\int_0^1 x^5\sqrt{1-x^2}\, dx$

(5) $\int_0^1 \frac{x^5}{\sqrt{1-x^2}} dx$ (6) $\int_0^{\log 2} \frac{e^x \, dx}{e^{2x}+4e^x+3}.$

**3.** (1) If $\int_a^b x^3 dx = 0$ and $\int_a^b x^2 dx = \frac{2}{3}$, find both $a$ and $b$.

(2) If $\int_0^k 4x^3 dx = 16$, find $k$.

(3) If $\int_1^4 (3x^2+bx+5)dx = 93$, find $b$.

(4) If $\int_a^2 (3x^2+2x+4)\,dx = 32$, find $a$.

(5) If $\int_0^1 (3x^2+2x+a)dx = 0$, find $a$.

### Answers

**1.** (1) $2(2-\sqrt{3})$ (2) $\frac{4\sqrt{2}}{3}$ (3) $\frac{44}{35}$

(4) $\frac{1}{2}\log\left(\frac{7}{5}\right)$ (5) $\log 2 + \frac{5}{36}$ (6) $\log 20$

(7) $4\log\left(\frac{3}{4}\right)$ (8) $\frac{1}{\sqrt{17}}\log\left(\frac{5+\sqrt{17}}{5-\sqrt{17}}\right)$

(9) $\log\left(\frac{5+3\sqrt{3}}{1+\sqrt{3}}\right)$ (10) $\frac{1}{4}(3e^4+1)$

(11) $2\log 2 - \frac{3}{4}$ (12) $\frac{1}{2}\log\left(\frac{e}{2}\right)$

(13) $e-2$ (14) $\frac{1}{2}\log\left(\frac{6}{5}\right)$

(15) $\log\left(\frac{25}{16}\right)$ (16) $\log 3 - \frac{1}{2}\log 5.$

**2.** (1) 0 (2) $\frac{\log 2}{\log(2e^2)}$ (3) $\frac{2}{15}$

(4) $\frac{8}{105}$ (5) $\frac{8}{15}$ (6) $\frac{1}{2}\log\left(\frac{6}{5}\right).$

**3.** (1) $a=-1, b=1$ (2) $k=2$ (3) $b=2$

(4) $a=-2$ (5) $a=-2.$

## 6.2 : PROPERTIES OF DEFINITE INTEGRALS

In this section, we shall state few properties of definite integrals (without proof) using which, we can evaluate definite integrals without finding the antiderivatives or primitives.

**Property 1 :** $\int_a^a f(x)\,dx = 0$

**Property 2 :** $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

**Property 3 :** $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

**Property 4 :** $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$ where $a < c < b$

**Property 5 :** $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

**Property 6 :** $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

**Property 7 :** $\int_0^{2a} f(x)\,dx = \int_0^a [f(x)+f(2a-x)]\,dx$

**Property 8 :** $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$, if $f$ is an even function

$= 0$ , if $f$ is an odd function

**Note :** $f$ is an even function if $f(-x)=f(x)$ and $f$ is an odd function if $f(-x)=-f(x)$.

**EXERCISE 6.2** | **Textbook page 148**

**Evaluate the following integrals :**

**1.** $\int_{-9}^{9} \frac{x^3}{4-x^2} dx$

**Solution :** Let $I = \int_{-9}^{9} \frac{x^3}{4-x^2} dx$

Let $f(x) = \frac{x^3}{4-x^2}$

$\therefore f(-x) = \frac{(-x)^3}{4-(-x)^2} = \frac{-x^3}{4+x^2} = -f(x)$

$\therefore f$ is an odd function.

$\therefore \int_{-9}^{9} f(x)\,dx = 0$

i.e. $\int_{-9}^{9} \frac{x^3}{4-x^2} dx = 0.$

**2.** $\int_0^a x^2(a-x)^{\frac{3}{2}}\,dx$

**Solution :** We use the property

$\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$\therefore \int_0^a x^2(a-x)^{\frac{3}{2}}dx = \int_0^a (a-x)^2(a-a+x)^{\frac{3}{2}}dx$

$= \int_0^a (a^2-2ax+x^2)x^{\frac{3}{2}}\,dx$

$= \int_0^a (a^2x^{\frac{3}{2}} - 2ax^{\frac{5}{2}} + x^{\frac{7}{2}})\,dx$

$= a^2\int_0^a x^{\frac{3}{2}}\,dx - 2a\int_0^a x^{\frac{5}{2}}\,dx + \int_0^a x^{\frac{7}{2}}\,dx$

$= a^2\left[\frac{x^{\frac{5}{2}}}{\left(\frac{5}{2}\right)}\right]_0^a - 2a\left[\frac{x^{\frac{7}{2}}}{\left(\frac{7}{2}\right)}\right]_0^a + \left[\frac{x^{\frac{9}{2}}}{\left(\frac{9}{2}\right)}\right]_0^a$

$= \frac{2a^2}{5}[a^{\frac{5}{2}}-0] - \frac{4a}{7}[a^{\frac{7}{2}}-0] + \frac{2}{9}[a^{\frac{9}{2}}-0]$

$= \frac{2}{5}a^{\frac{9}{2}} - \frac{4}{7}a^{\frac{9}{2}} + \frac{2}{9}a^{\frac{9}{2}} = \left(\frac{2}{5}-\frac{4}{7}+\frac{2}{9}\right)a^{\frac{9}{2}}$

$= \left(\frac{126-180+70}{315}\right)a^{\frac{9}{2}} = \frac{16}{315}a^{\frac{9}{2}}.$

---

**3.** $\int_1^3 \frac{\sqrt[3]{x+5}}{\sqrt[3]{x+5}+\sqrt[3]{9-x}}\,dx$

**Solution :** Let $I = \int_1^3 \frac{\sqrt[3]{x+5}}{\sqrt[3]{x+5}+\sqrt[3]{9-x}}\,dx$ ... (1)

We use the property, $\int_a^b f(x)dx = \int_a^b f(a+b-x)\,dx$

Hence in $I$, we replace $x$ by $1+3-x$.

$\therefore I = \int_1^3 \frac{\sqrt[3]{1+3-x+5}}{\sqrt[3]{1+3-x+5}+\sqrt[3]{9-1-3+x}}\,dx$

$= \int_1^3 \frac{\sqrt[3]{9-x}}{\sqrt[3]{9-x}+\sqrt[3]{x+5}}\,dx$ ... (2)

Adding (1) and (2), we get

$2I = \int_1^3 \frac{\sqrt[3]{x+5}}{\sqrt[3]{x+5}+\sqrt[3]{9-x}}\,dx + \int_1^3 \frac{\sqrt[3]{9-x}}{\sqrt[3]{9-x}+\sqrt[3]{x+5}}\,dx$

$= \int_1^3 \frac{\sqrt[3]{x+5}+\sqrt[3]{9-x}}{\sqrt[3]{x+5}+\sqrt[3]{9-x}}\,dx$

$= \int_1^3 1dx = [x]_1^3$

$= 3-1 = 2$

$\therefore I = 1$

Hence, $\int_1^3 \frac{\sqrt[3]{x+5}}{\sqrt[3]{x+5}+\sqrt[3]{9-x}}\,dx = 1.$

---

**4.** $\int_2^5 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{7-x}}\,dx$

**Solution :** Let $I = \int_2^5 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{7-x}}\,dx$ ... (1)

We use the property, $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx.$

Hence in $I$, we change $x$ by $2+5-x$.

$\therefore I = \int_2^5 \frac{\sqrt{2+5-x}}{\sqrt{2+5-x}+\sqrt{7-2-5+x}}\,dx$

$= \int_2^5 \frac{\sqrt{7-x}}{\sqrt{7-x}+\sqrt{x}}\,dx$ ... (2)

Adding (1) and (2), we get

$2I = \int_2^5 \frac{\sqrt{7-x}}{\sqrt{x}+\sqrt{7-x}}\,dx + \int_2^5 \frac{\sqrt{x}}{\sqrt{7-x}+\sqrt{x}}\,dx$

$= \int_2^5 \frac{\sqrt{x}+\sqrt{7-x}}{\sqrt{x}+\sqrt{7-x}}\,dx$

$= \int_2^5 1\,dx = [x]_2^5 = 5-2 = 3$

$\therefore I = \frac{3}{2}$

Hence, $\int_2^5 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{7-x}}\,dx = \frac{3}{2}.$

---

**5.** $\int_1^2 \frac{\sqrt{x}}{\sqrt{3-x}+\sqrt{x}}\,dx$

**Solution :** *Refer to the solution of Q. 4.*

**Ans.** $\frac{1}{2}$.

---

**6.** $\int_2^7 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{9-x}}\,dx$

**Solution :** *Refer to the solution of Q. 4.*

**Ans.** $\frac{5}{2}$.

---

**7.** $\int_0^1 \log\left(\frac{1}{x}-1\right)dx$

**Solution :** Let $I = \int_0^1 \log\left(\frac{1}{x}-1\right)dx = \int_0^1 \log\left(\frac{1-x}{x}\right)dx$

We use the property, $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$\therefore I = \int_0^1 \log\left[\frac{1-(1-x)}{1-x}\right]dx = \int_0^1 \log\left(\frac{x}{1-x}\right)dx$

$= \int_0^1 -\log\left(\frac{1-x}{x}\right)dx = -\int_0^1 \log\left(\frac{1-x}{x}\right)dx$

$\therefore\ I = -I$

$\therefore\ 2I = 0 \qquad \therefore\ I = 0$

Hence, $\int_0^1 \log\left(\frac{1}{x} - 1\right) dx = 0.$

**8.** $\int_0^1 x(1-x)^5 dx$

**Solution :** We use the property,

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$$

$$\therefore\ \int_0^1 x(1-x)^5 dx = \int_0^1 (1-x)(1-1+x)^5\,dx$$

$$= \int_0^1 (1-x)x^5 dx = \int_0^1 (x^5 - x^6)\,dx$$

$$= \int_0^1 x^5 dx - \int_0^1 x^6 dx$$

$$= \left[\frac{x^6}{6}\right]_0^1 - \left[\frac{x^7}{7}\right]_0^1$$

$$= \frac{1}{6}(1-0) - \frac{1}{7}(1-0)$$

$$= \frac{1}{6} - \frac{1}{7} = \frac{1}{42}.$$

[**Note :** *Answer in the textbook is incorrect.*]

## ADDITIONAL SOLVED PROBLEMS – 6 (B)

**1. Evaluate :** $\int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}}\,dx.$

**Solution :** Let $I = \int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}}\,dx$ ... (1)

We use the property, $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

Hence in $I$, we change $x$ by $a - x$.

$$\therefore\ I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x} + \sqrt{a-a+x}}\,dx$$

$$= \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x} + \sqrt{x}}\,dx \qquad \dots (2)$$

Adding (1) and (2), we get

$$2I = \int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}}\,dx + \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x} + \sqrt{x}}\,dx$$

$$= \int_0^a \frac{\sqrt{x} + \sqrt{a-x}}{\sqrt{x} + \sqrt{a-x}}\,dx$$

$$= \int_0^a 1\,dx = [x]_0^a = a - 0 = a \qquad \therefore\ I = \frac{a}{2}$$

Hence, $\int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}}\,dx = \frac{a}{2}.$

**2. Evaluate :** $\int_{-2}^{2} f(x)\,dx$**, where**

$$f(x) = \begin{cases} 3 - 4x, & x \leqslant 0 \\ 3 + 4x, & x \geqslant 0. \end{cases}$$

**Solution :** $\int_{-2}^{2} f(x)\,dx = \int_{-2}^{0} f(x)\,dx + \int_0^2 f(x)\,dx$

$$= \int_{-2}^{0} (3-4x)\,dx + \int_0^2 (3+4x)\,dx$$

$$= \left[3x - 4\cdot\frac{x^2}{2}\right]_{-2}^{0} + \left[3x + 4\cdot\frac{x^2}{2}\right]_0^2$$

$$= [0 - (-6-8)] + [6+8] - 0]$$

$$= 14 + 14 = 28.$$

**3. Evaluate :** $\int_{-2}^{2} \frac{x^2}{x^2-1}\,dx.$

**Solution :** Let $I = \int_{-2}^{2} \frac{x^2}{x^2-1}\,dx$

Let $f(x) = \frac{x^2}{x^2-1}$

Then $f(-x) = \frac{(-x)^2}{(-x)^2 - 1} = \frac{x^2}{x^2-1} = f(x)$

$\therefore\ f$ is an even function.

$$\therefore\ \int_{-2}^{2} f(x)\,dx = 2\int_0^2 f(x)\,dx$$

$$\therefore\ I = 2\int_0^2 \frac{x^2}{x^2-1}\,dx$$

$$= 2\int_0^2 \frac{(x^2-1)+1}{x^2-1}\,dx$$

$$= 2\int_0^2 \left(1 + \frac{1}{x^2-1}\right) dx$$

$$= 2\left[x + \frac{1}{2}\log\left|\frac{x-1}{x+1}\right|\right]_0^2$$

$$= 2\left[2 + \frac{1}{2}\log\frac{1}{3} - 0\right]$$

$$= 4 + \log\frac{1}{3}$$

Hence, $\int_{-2}^{2} \frac{x^2}{x^2-1}\,dx = 4 + \log\frac{1}{3}.$

**4. Evaluate :** $\int_{-a}^{a} x^3\sqrt{a^2-x^2}\,dx.$

**Solution :** Let $I=\int_{-a}^{a} x^3\sqrt{a^2-x^2}\,dx$

Let $f(x)=x^3\sqrt{a^2-x^2}$

$\therefore\ f(-x)=(-x)^3\sqrt{a^2-(-x)^2}=-x^3\sqrt{a^2-x^2}=-f(x)$

$\therefore\ f$ is an odd function.

$\therefore\ \int_{-a}^{a} f(x)\,dx=0$

i.e. $\int_{-a}^{a} x^3\sqrt{a^2-x^2}\,dx=0.$

## EXAMPLES FOR PRACTICE 6.2

**Evaluate the following definite integrals :**

**1.** (1) $\int_0^1 x(1-x)^{\frac{3}{2}}\,dx$ (2) $\int_0^2 x^2\sqrt{2-x}\,dx$

(3) $\int_0^3 x^2\sqrt{3-x}\,dx$ (4) $\int_0^2 x^2(2-x)^{\frac{5}{2}}\,dx$

(5) $\int_0^1 x^2(1-x)^{\frac{3}{2}}\,dx$ (6) $\int_0^1 x(1-x)^n\,dx.$

**2.** (1) $\int_0^5 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{5-x}}\,dx$ (2) $\int_0^3 \frac{\sqrt[3]{x+4}}{\sqrt[3]{x+4}+\sqrt[3]{7-x}}\,dx$

(3) $\int_0^4 \frac{\sqrt[3]{x+5}}{\sqrt[3]{x+5}+\sqrt[3]{9-x}}\,dx$

(4) $\int_0^a \frac{\sqrt[n]{x+c}}{\sqrt[n]{x+c}+\sqrt[n]{a+c-x}}\,dx.$

**3.** (1) $\int_3^9 \frac{\sqrt[3]{12-x}}{\sqrt[3]{x}+\sqrt[3]{12-x}}$

(2) $\int_1^2 \frac{\sqrt{x}-\sqrt{3-x}}{1+\sqrt{x(3-x)}}\,dx$

(3) $\int_4^7 \frac{(11-x)^2}{x^2+(11-x)^2}\,dx$

(4) $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx.$

**4.** (1) $\int_{-2}^{2} \frac{x^3}{4+x^6}\,dx$ (2) $\int_{-5}^{5} \frac{x^3+x}{5+x^2}\,dx$

(3) $\int_{-4}^{4} \frac{1+x^2}{1-x^2}\,dx.$

### Answers

**1.** (1) $\frac{4}{35}$ (2) $\frac{128\sqrt{2}}{105}$ (3) $\frac{144\sqrt{3}}{35}$ (4) $\frac{512\sqrt{2}}{693}$

(5) $\frac{16}{315}$ (6) $\frac{1}{(n+1)(n+2)}$.

**2.** (1) $\frac{5}{2}$ (2) $\frac{3}{2}$ (3) 2 (4) $\frac{a}{2}$.

**3.** (1) 3 (2) 0 (3) $\frac{3}{2}$ (4) $\frac{b-a}{2}$.

**4.** (1) 0 (2) 0 (3) $2\log\left(\frac{5}{3}\right)-8$.

## MISCELLANEOUS EXERCISE – 6

**(Textbook pages 148 to 150)**

**(I) Choose the correct alternative :**

**1.** $\int_{-9}^{9} \frac{x^3}{4-x^2}\,dx=$ .........

(a) 0 (b) 3 (c) 9 (d) $-9$

**2.** $\int_{-2}^{3} \frac{dx}{x+5}=$ .........

(a) $-\log\left(\frac{8}{3}\right)$ (b) $\log\left(\frac{8}{3}\right)$

(c) $\log\left(\frac{3}{8}\right)$ (d) $-\log\left(\frac{3}{8}\right)$

**3.** $\int_2^3 \frac{x}{x^2-1}\,dx=$ .........

(a) $\log\left(\frac{8}{3}\right)$ (b) $-\log\left(\frac{8}{3}\right)$

(c) $\frac{1}{2}\log\left(\frac{8}{3}\right)$ (d) $\frac{-1}{2}\log\frac{8}{3}$

**4.** $\int_4^9 \frac{dx}{\sqrt{x}}=$ .........

(a) 9 (b) 4 (c) 2 (d) 0

**5.** If $\int_0^a 3x^2\,dx=8$, then $a=$ .........

(a) 2 (b) 0 (c) $\frac{8}{3}$ (d) $a$

**6.** $\int_2^3 x^4\,dx=$ .........

(a) $\frac{1}{2}$ (b) $\frac{5}{2}$ (c) $\frac{5}{211}$ (d) $\frac{211}{5}$

**7.** $\int_0^2 e^x\,dx=$ .........

(a) $e-1$ (b) $1-e$ (c) $1-e^2$ (d) $e^2-1$

**8.** $\int_a^b f(x)\,dx=$ .........

(a) $\int_b^a f(x)\,dx$ (b) $-\int_a^b f(x)\,dx$

(c) $-\int_b^a f(x)\,dx$ (d) $\int_0^a f(x)\,dx$

9. $\int_{-7}^{7} \frac{x^3}{x^2+7} dx = \dots\dots\dots$

(a) 7 (b) 49 (c) 0 (d) $\frac{7}{2}$

10. $\int_{2}^{7} \frac{\sqrt{x}}{\sqrt{x}+\sqrt{9-x}} dx = \dots\dots\dots$

(a) $\frac{7}{2}$ (b) $\frac{5}{2}$ (c) 7 (d) 2

### Answers

1. (a) 0 2. (b) $\log\left(\frac{8}{3}\right)$ 3. (c) $\frac{1}{2}\log\left(\frac{8}{3}\right)$

4. (c) 2 5. (a) 2 6. (d) $\frac{211}{5}$ 7. (d) $e^2-1$

8. (c) $-\int_{b}^{a} f(x)\,dx$ 9. (c) 0 10. (b) $\frac{5}{2}$.

## (II) Fill in the blanks :

1. $\int_{0}^{2} e^x \, dx = \dots\dots\dots$

2. $\int_{2}^{3} x^4 \, dx = \dots\dots\dots$

3. $\int_{0}^{1} \frac{dx}{2x+5} = \dots\dots\dots$

4. If $\int_{0}^{a} 3x^2 \, dx = 8$, then $a = \dots\dots\dots$

5. $\int_{4}^{9} \frac{1}{\sqrt{x}} dx = \dots\dots\dots$

6. $\int_{2}^{3} \frac{x}{x^2-1} dx = \dots\dots\dots$

7. $\int_{-2}^{3} \frac{dx}{x+5} = \dots\dots\dots$

8. $\int_{-9}^{9} \frac{x^3}{4-x^2} dx = \dots\dots\dots$

### Answers

1. $e^2-1$ 2. $\frac{211}{5}$

3. $\frac{1}{2}\log\left(\frac{7}{5}\right)$

[Note : *Answer in the textbook is incorrect.*]

4. 2 5. 2 6. $\frac{1}{2}\log\left(\frac{8}{3}\right)$ 7. $\log\left(\frac{8}{3}\right)$ 8. 0.

## (III) State whether each of the following is *True* or *False* :

1. $\int_{a}^{b} f(x)\,dx = -\int_{b}^{a} f(x)\,dx.$

[Note : *Question is modified.*]

2. $\int_{a}^{b} f(x)\,dx = \int_{a}^{b} f(t)\,dt.$

3. $\int_{0}^{a} f(x)\,dx = \int_{a}^{0} f(a-x)\,dx.$

4. $\int_{a}^{b} f(x)\,dx = \int_{a}^{b} f(x-a-b)\,dx.$

5. $\int_{-5}^{5} \frac{x^3}{x^2+7} dx = 0.$

6. $\int_{1}^{2} \frac{\sqrt{x}}{\sqrt{3-x}+\sqrt{x}} dx = \frac{1}{2}.$

7. $\int_{2}^{7} \frac{\sqrt{x}}{\sqrt{x}+\sqrt{9-x}} dx = \frac{9}{2}.$

8. $\int_{4}^{7} \frac{(11-x)^2}{(11-x)^2+x^2} dx = \frac{3}{2}.$

### Answers

| | | | |
|---|---|---|---|
| 1. True | 2. True | 3. False | 4. False |
| 5. True | 6. True | 7. False | 8. True. |

## (IV) Solve the following :

1. $\int_{2}^{3} \frac{x}{(x+2)(x+3)} dx$

**Solution :** *Refer to the solution Q. 5 of Exercise 6.1.*

**Ans.** $\log\left(\frac{3456}{3125}\right).$

[Note : *Answer in the textbook is incorrect.*]

2. $\int_{1}^{2} \frac{x+3}{x(x+2)} dx$

**Solution :** Let $I = \int \frac{x+3}{x(x+2)} dx$

Let $\frac{x+3}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2}$

$\therefore \ x+3 = A(x+2) + Bx$

Put $x=0$, we get

$3 = A(2) + B(0) \qquad \therefore \ A = \frac{3}{2}$

Put $x+2=0$, i.e. $x=-2$, we get

$-2+3 = A(0) + B(-2)$

$\therefore \ 1 = -2B \qquad \therefore \ B = -\frac{1}{2}$

$\therefore \dfrac{x+3}{x(x+2)} = \dfrac{\left(\frac{3}{2}\right)}{x} + \dfrac{\left(-\frac{1}{2}\right)}{x+2}$

$\therefore I = \displaystyle\int_1^2 \left[\dfrac{\left(\frac{3}{2}\right)}{x} + \dfrac{\left(-\frac{1}{2}\right)}{x+2}\right] dx$

$= \dfrac{3}{2}\displaystyle\int_1^2 \dfrac{1}{x}\,dx - \dfrac{1}{2}\int_1^2 \dfrac{1}{x+2}\,dx$

$= \dfrac{3}{2}[\log|x|]_1^2 - \dfrac{1}{2}[\log|x+2|]_1^2$

$= \dfrac{3}{2}(\log 2 - \log 1) - \dfrac{1}{2}(\log 4 - \log 3)$

$= \dfrac{3}{2}\log 2 - \dfrac{1}{2}\log 4 + \dfrac{1}{2}\log 3$ ... $[\because \log 1 = 0]$

$= \dfrac{1}{2}(3\log 2 - \log 4 + \log 3)$

$= \dfrac{1}{2}(\log 8 - \log 4 + \log 3)$

$= \dfrac{1}{2}\log\left(\dfrac{8\times 3}{4}\right) = \dfrac{1}{2}\log 6.$

**3.** $\displaystyle\int_1^3 x^2 \log x\, dx$

**Solution :** $\displaystyle\int_1^3 x^2 \log x\, dx = \int_1^3 (\log x)\cdot x^2\, dx$

$= [(\log x)\int x^2 dx]_1^3 - \displaystyle\int_1^3 \left[\dfrac{d}{dx}(\log x)\int x^2 dx\right] dx$

$= \left[(\log x)\left(\dfrac{x^3}{3}\right)\right]_1^3 - \displaystyle\int_1^3 \dfrac{1}{x}\times\dfrac{x^3}{3}\,dx$

$= \dfrac{1}{3}[x^3\log x]_1^3 - \dfrac{1}{3}\displaystyle\int_1^3 x^2 dx$

$= \dfrac{1}{3}[27\log 3 - 0] - \dfrac{1}{3}\left[\dfrac{x^3}{3}\right]_1^3$ ... $[\because \log 1 = 0]$

$= 9\log 3 - \dfrac{1}{9}(27-1)$

$= 9\log 3 - \dfrac{26}{9}.$

**4.** $\displaystyle\int_0^1 e^{x^2}\cdot x^3\, dx$

**Solution :** Let $I = \displaystyle\int_0^1 e^{x^2}\cdot x^3\, dx = \int_0^1 e^{x^2}\cdot x^2\cdot x\, dx$

Put $x^2 = t$ $\therefore 2x\,dx = dt$

$\therefore x\,dx = \dfrac{dt}{2}$

When $x = 0,\ t = 0$

When $x = 1,\ t = 1$

$\therefore I = \displaystyle\int_0^1 e^t\cdot t\cdot\dfrac{dt}{2} = \dfrac{1}{2}\int_0^1 te^t\, dt$

$= \dfrac{1}{2}\left\{[t\int e^t\, dt]_0^1 - \displaystyle\int_0^1 \left[\dfrac{d}{dt}(t)\int e^t\, dt\right] dt\right\}$

$= \dfrac{1}{2}[te^t]_0^1 - \dfrac{1}{2}\displaystyle\int_0^1 1\cdot e^t\, dt$

$= \dfrac{1}{2}(e-0) - \dfrac{1}{2}[e^t]_0^1$

$= \dfrac{e}{2} - \dfrac{1}{2}(e-1)$

$= \dfrac{e}{2} - \dfrac{e}{2} + \dfrac{1}{2} = \dfrac{1}{2}.$

**5.** $\displaystyle\int_1^2 e^{2x}\left(\dfrac{1}{x} - \dfrac{1}{2x^2}\right) dx$

**Solution :** Let $I = \displaystyle\int_1^2 e^{2x}\left(\dfrac{1}{x} - \dfrac{1}{2x^2}\right) dx$

Put $2x = t$ $\therefore 2dx = dt$

$\therefore dx = \dfrac{dt}{2}$ and $x = \dfrac{t}{2}$

When $x = 1,\ t = 2$

When $x = 2,\ t = 4$

$\therefore I = \displaystyle\int_2^4 e^t\left(\dfrac{2}{t} - \dfrac{2}{t^2}\right)\dfrac{dt}{2} = \dfrac{1}{2}\int_2^4 e^t\left(\dfrac{2}{t} - \dfrac{2}{t^2}\right) dt$

Let $f(t) = \dfrac{2}{t}$

Then $f'(t) = 2\left(-\dfrac{1}{t^2}\right) = \dfrac{-2}{t^2}$

$\therefore I = \dfrac{1}{2}\displaystyle\int_2^4 e^t[f(t) + f'(t)]\, dt$

$= \dfrac{1}{2}[e^t\cdot f(t)]_2^4 = \dfrac{1}{2}\left[e^t\cdot\dfrac{2}{t}\right]_2^4$

$= \dfrac{1}{2}\left[e^4\times\dfrac{2}{4} - e^2\times\dfrac{2}{2}\right]$

$= \dfrac{e^4}{4} - \dfrac{e^2}{2}.$

**6.** $\displaystyle\int_4^9 \dfrac{1}{\sqrt{x}}\, dx$

**Solution :** *Refer to the solution of Q. 1 of Exercise 6.1.*

**Ans.** 2.

7. $\int_{-2}^{3} \frac{1}{x+5} dx$

**Solution :** *Refer to the solution of Q. 2 of Exercise 6.1.*

**Ans.** $\log\left(\frac{8}{3}\right)$.

8. $\int_{2}^{3} \frac{x}{x^2-1} dx$

**Solution :** *Refer to the solution of Q. 3 of Exercise 6.1.*

**Ans.** $\frac{1}{2}\log\left(\frac{8}{3}\right)$.

9. $\int_{0}^{1} \frac{x^2+3x+2}{\sqrt{x}} dx$

**Solution :** *Refer to the solution of Q. 4 of Exercise 6.1.*

**Ans.** $\frac{32}{5}$.

10. $\int_{3}^{5} \frac{dx}{\sqrt{x+4}+\sqrt{x-2}}$

**Solution :** $\int_{3}^{5} \frac{dx}{\sqrt{x+4}+\sqrt{x-2}}$

$= \int_{3}^{5} \frac{1}{\sqrt{x+4}+\sqrt{x-2}} \times \frac{\sqrt{x+4}-\sqrt{x-2}}{\sqrt{x+4}-\sqrt{x-2}} dx$

$= \int_{3}^{5} \frac{\sqrt{x+4}-\sqrt{x-2}}{x+4-x+2} dx$

$= \frac{1}{6}\int_{3}^{5} [(x+4)^{\frac{1}{2}}-(x-2)^{\frac{1}{2}}] dx$

$= \frac{1}{6}\left[\frac{(x+4)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} - \frac{(x-2)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right]_3^5$

$= \frac{1}{9}\left[(x+4)^{\frac{3}{2}}-(x-2)^{\frac{3}{2}}\right]_3^5$

$= \frac{1}{9}\left[(9^{\frac{3}{2}}-3^{\frac{3}{2}})-(7^{\frac{3}{2}}-1)\right]$

$= \frac{1}{9}(27-3\sqrt{3}-7\sqrt{7}+1)$

$= \frac{1}{9}(28-3\sqrt{3}-7\sqrt{7})$.

11. $\int_{2}^{3} \frac{x}{x^2+1} dx$

**Solution :** Let $I = \int_{2}^{3} \frac{x}{x^2+1} dx$

Put $x^2+1=t$ $\quad \therefore 2x\,dx = dt$

$\therefore x\,dx = \frac{dt}{2}$

When $x=2$, $t=4+1=5$

When $x=3$, $t=9+1=10$

$\therefore I = \int_{5}^{10} \frac{1}{t}\cdot\frac{dt}{2} = \frac{1}{2}\int_{5}^{10}\frac{1}{t} dt$

$= \frac{1}{2}[\log|t|]_5^{10}$

$= \frac{1}{2}(\log 10 - \log 5) = \frac{1}{2}\log\left(\frac{10}{5}\right)$

$= \frac{1}{2}\log 2 = \log\sqrt{2}$.

12. $\int_{1}^{2} x^2 dx$

**Solution :** $\int_{1}^{2} x^2 dx = \left[\frac{x^3}{3}\right]_1^2$

$= \frac{1}{3}(8-1) = \frac{7}{3}$.

13. $\int_{-4}^{-1} \frac{1}{x} dx$

**Solution :** $\int_{-4}^{-1} \frac{1}{x} dx = [\log|x|]_{-4}^{-1}$

$= \log|-1| - \log|-4|$

$= \log 1 - \log 4$

$= -\log 4.$ $\quad \ldots [\because \log 1 = 0]$

14. $\int_{0}^{1} \frac{1}{\sqrt{1+x}+\sqrt{x}} dx$

**Solution :** *Refer to the solution of Q. 9 of Exercise 6.1.*

**Ans.** $\frac{4}{3}(\sqrt{2}-1)$.

15. $\int_{0}^{4} \frac{1}{\sqrt{x^2+2x+3}} dx$

**Solution :** $\int_{0}^{4} \frac{1}{\sqrt{x^2+2x+3}} dx$

$\int_{0}^{4} \frac{1}{\sqrt{(x^2+2x+1)+2}} dx$

$\int_{0}^{4} \frac{1}{\sqrt{(x+1)^2+(\sqrt{2})^2}}$

$= [\log|(x+1)+\sqrt{(x+1)^2+\sqrt{2})^2}|]_0^4$

$= [\log|(x+1)+\sqrt{x^2+2x+3}|]_0^4$

$= \log(5+\sqrt{27}) - \log(1+\sqrt{3})$

$= \log\left(\frac{5+3\sqrt{3}}{1+\sqrt{3}}\right)$.

**16.** $\int_2^4 \frac{x}{x^2+1}\, dx$

**Solution :** *Refer to the solution of Q. 11.*

**Ans.** $\frac{1}{2}\log\left(\frac{17}{5}\right)$.

**17.** $\int_0^1 \frac{1}{2x-3}\, dx$

**Solution :** $\int_0^1 \frac{1}{2x-3}\, dx = \left[\frac{\log|2x-3|}{2}\right]_0^1$

$= \frac{1}{2}[\log|-1| - \log|-3|]$

$= \frac{1}{2}(\log 1 - \log 3)$

$= -\frac{1}{2}\log 3.$ ... [$\because \log 1 = 0$]

**18.** $\int_1^2 \frac{5x^2}{x^2+4x+3}\, dx$

**Solution :** Let $I = \int_1^2 \frac{5x^2}{x^2+4x+3}\, dx$

$= \int_1^2 \left[\frac{5(x^2+4x+3) - 5(4x+3)}{x^2+4x+3}\right] dx$

$= \int_1^2 \left[5 - \frac{20x+15}{x^2+4x+3}\right] dx$

$\therefore I = \int_1^2 5\, dx - \int_1^2 \frac{20x+15}{x^2+4x+3}\, dx$ ... (1)

Let $\frac{20x+15}{x^2+4x+3} = \frac{20x+15}{(x+1)(x+3)} = \frac{A}{x+1} + \frac{B}{x+3}$

$\therefore 20x + 15 = A(x+3) + B(x+1)$

Put $x+1=0$, i.e. $x=-1$, we get

$20(-1) + 15 = A(2) + B(0)$

$\therefore -5 = 2A \quad \therefore A = -\frac{5}{2}$

Put $x+3=0$, i.e. $x=-3$, we get

$20(-3)+15 = A(0) + B(-2)$

$\therefore -45 = -2B \quad \therefore B = \frac{45}{2}$

$\therefore \frac{20x+15}{x^2+4x+3} = \frac{\left(-\frac{5}{2}\right)}{x+1} + \frac{\left(\frac{45}{2}\right)}{x+3}$

$\therefore$ from (1),

$I = \int_1^2 5\, dx - \int_1^2 \left[\frac{\left(-\frac{5}{2}\right)}{x+1} + \frac{\left(\frac{45}{2}\right)}{x+3}\right] dx$

$= 5\int_1^2 1\, dx + \frac{5}{2}\int_1^2 \frac{1}{x+1}\, dx - \frac{45}{2}\int_1^2 \frac{1}{x+3}\, dx$

$= 5[x]_1^2 + \frac{5}{2}[\log|x+1|]_1^2 - \frac{45}{2}[\log|x+3|]_1^2$

$= 5(2-1) + \frac{5}{2}(\log 3 - \log 2) - \frac{45}{2}(\log 5 - \log 4)$

$= 5 + \frac{1}{2}(5\log 3 - 5\log 2 - 45\log 5 + 90\log 2)$

$= 5 + \frac{1}{2}(5\log 3 + 85\log 2 - 45\log 5).$

[**Note :** *Answer in the textbook is incorrect.*]

**19.** $\int_1^2 \frac{dx}{x(1+\log x)^2}$

**Solution :** Let $I = \int_1^2 \frac{dx}{x(1+\log x)^2}$

$= \int_1^2 \frac{1}{(1+\log x)^2} \cdot \frac{1}{x}\, dx$

Put $1 + \log x = t \quad \therefore \frac{1}{x}\, dx = dt$

When $x=1$, $t = 1 + \log 1 = 1 + 0 = 1$

When $x=2$, $t = 1 + \log 2$

$\therefore I = \int_1^{1+\log 2} \frac{1}{t^2}\, dt = \int_1^{1+\log 2} t^{-2}\, dt$

$= \left[\frac{t^{-1}}{-1}\right]_1^{1+\log 2} = -\left[\frac{1}{t}\right]_1^{1+\log 2}$

$= -\left[\frac{1}{1+\log 2} - 1\right]$

$= -\left[\frac{1-(1+\log 2)}{1+\log 2}\right] = \frac{\log 2}{1+\log 2}.$

**20.** $\int_0^9 \frac{1}{1+\sqrt{x}}\, dx$

**Solution :** Let $I = \int_0^9 \frac{1}{1+\sqrt{x}}\, dx$

Put $\sqrt{x} = t$, i.e. $x = t^2$

$\therefore dx = 2t\, dt$

When $x=0$, $t=0$

When $x=9$, $t=3$

$\therefore I = \int_0^3 \frac{1}{1+t} \cdot 2t\, dt$

$= 2\int_0^3 \left[\frac{(1+t)-1}{1+t}\right] dt$

$= 2\int_0^3 \left(1 - \frac{1}{1+t}\right) dt$

$= 2[t - \log|1+t|]_0^3$ ... [$\because$ log 1 = 0]

$= 2[(3 - \log 4) - 0]$

$= 6 - 2\log 4 = 6 - 4\log 2.$

**ACTIVITIES** | **Textbook pages 150 and 151**

**Complete the following activity :**

**1.** If $\int_a^b x^3\,dx = 0$, then

$\left[\frac{x^4}{\boxed{4}}\right]_a^b = 0$

$\therefore \frac{1}{4}(\boxed{b^4} - \boxed{a^4}) = 0$

$\therefore b^4 - \boxed{a^4} = 0$

$\therefore (b^2 - a^2)(\boxed{b^2} + \boxed{a^2}) = 0$

$\therefore b^2 - \boxed{a^2} = 0$ as $a^2 + b^2 \neq 0$

$\therefore b = \pm\boxed{a}.$

**2.** $\int_0^2 \frac{dx}{4 + x - x^2}$

$= \int_0^2 \frac{dx}{-x^2 + \boxed{x} + \boxed{4}}$

$= \int_0^2 \frac{dx}{-x^2 + x + \frac{1}{4} - \boxed{\frac{1}{4}} + 4}$

$= -\int_0^2 \frac{dx}{\left(x - \frac{1}{2}\right)^2 - \left(\boxed{\frac{\sqrt{17}}{2}}\right)^2}$

$= \frac{1}{\sqrt{17}} \log\left(\frac{20 + 4\sqrt{17}}{20 - 4\sqrt{17}}\right).$

**3.** $\int_0^1 \log\left(\frac{1}{x} - 1\right) dx$

$= \int_0^1 \log\left(\frac{1-x}{\boxed{x}}\right) dx$ ... (1)

$= \int_0^1 \log\left(\frac{1-(1-x)}{\boxed{1-x}}\right) dx$

$= \int_0^1 \log\left(\frac{\boxed{x}}{1-x}\right) dx$ ... (2)

Adding (1) and (2), we get

$2I = \int_0^1 \log\left[\frac{1-x}{x} \times \frac{x}{\boxed{1-x}}\right] dx$

$= \int_0^1 \log\boxed{1}\,dx = \int_0^1 0\,dx = \boxed{0}.$

**4.** $\int_{-8}^{8} \frac{x^5}{1-x^2}\,dx$

$f(x) = \frac{x^5}{1-x^2}$

$f(-x) = \frac{(-x)^5}{1-x^2} = \frac{\boxed{-x^5}}{1-x^2}$

Hence $f$ is $\boxed{\text{odd}}$ function.

$\int_{-8}^{8} \frac{x^5}{1-x^2}\,dx = \boxed{0}.$

## ACTIVITIES FOR PRACTICE

**1. Evaluate :** $\int_1^2 \frac{dx}{x^2 + 6x + 5}$.

**Solution :** $\int_1^2 \frac{dx}{x^2 + 6x + 5} = \int_1^2 \frac{dx}{(x^2 + 6x + 9) - \boxed{\phantom{0}}}$

$= \int_1^2 \frac{dx}{(x+3)^2 - \boxed{\phantom{0}}}$

$= \frac{1}{2\boxed{\phantom{0}}}\left[\log\left|\frac{x+3-\boxed{\phantom{0}}}{x+3+\boxed{\phantom{0}}}\right|\right]_1^2$

$= \frac{1}{\boxed{\phantom{0}}}\left[\log\left(\frac{3}{7}\right) - \log\boxed{\phantom{0}}\right]$

$= \frac{1}{\boxed{\phantom{0}}}\log\boxed{\phantom{0}}.$

**2. Evaluate :** $\int_0^1 \frac{x^3}{\sqrt{1-x^2}}\,dx.$

**Solution :** Let $I = \int_0^1 \frac{x^3}{\sqrt{1-x^2}}\,dx$

Put $1 - x^2 = t$ $\therefore -2x\,dx = dt$ $\therefore x\,dx = \frac{dt}{\boxed{\phantom{0}}}$

When $x = 0$, $t = 1$

When $x = 1$, $t = 0$

$$\therefore\ I = \int_1^0 \frac{1-\square}{\sqrt{t}}\,\frac{dt}{\square}$$

$$= \frac{1}{\square}\int_1^0 \left(t^{-\frac{1}{2}} - t^{\square}\right)dt$$

$$= \frac{1}{\square}\int_1^0 t^{-\frac{1}{2}}\,dt - \frac{1}{\square}\int_0^1 t^{\square}\,dt$$

$$= \frac{1}{\square}\left[\frac{t^{\frac{1}{2}}}{\left(\frac{1}{2}\right)}\right]_0^1 - \frac{1}{\square}\cdot\left[\frac{t^{\square}}{\square}\right]_0^1$$

$$= (1-0) - \frac{1}{\square}(1-0)$$

$$= 1 - \frac{1}{\square} = \square.$$

**3. Evaluate : $\int_1^e \log x\,dx$.**

**Solution :** $\int_1^e \log x\,dx = \int_1^e (\log x)\cdot 1\,dx$

$$= [(\log x)\int 1dx]_1^e - \int_1^e \left[\frac{d}{dx}(\log x)\cdot\int 1\,dx\right]dx$$

$$= \left[(\log x)\square\right]_1^e - \int_1^e \frac{1}{x}\times\square\,dx$$

$$= (e - \square) - \int_1^e \square\,dx$$

$$= e - [\square]_1^e$$

$$= e - \square = \square.$$

**4. Evaluate : $\int_1^3 \frac{2}{x(1+x^2)}\,dx$.**

**Solution :** Let $I = \int_1^3 \frac{2}{x(1+x^2)}\,dx$

$$= \int_1^3 \frac{2\square}{x^2(1+x^2)}\,dx$$

Put $1+x^2 = t$ $\quad\therefore\ 2x\,dx = dt$

When $x=1$, $t = 1+1 = 2$

When $x=3$, $t = 1+9 = 10$

$$\therefore\ I = \int_2^{10} \frac{1}{t(1+t)}\,dt$$

$$= \int_2^{10} \frac{(1+t)-\square}{t(1+t)}\,dt$$

$$= \int_2^{10}\left(\frac{1}{t} - \frac{1}{\square}\right)dt$$

$$= [\log t - \log \square]_2^{10}$$

$$= (\log 10 - \log \square) - (\log 2 - \log \square)$$

$$= \log \square - \log \square$$

$$= \log \square.$$

**5. Evaluate : $\int_{-5}^{5} \log\left(\frac{2-x}{2+x}\right)dx$**

**Solution :** Let $I = \int_{-5}^{5} \log\left(\frac{2-x}{2+x}\right)dx$

Let $f(x) = \log\left(\frac{2-x}{2+x}\right)$

Then $f(-x) = \log\left[\frac{2-(-x)}{2+(-x)}\right] = \log\left(\frac{2+x}{2-x}\right)$

$= \square$

$\therefore$ $f$ is an $\square$ function.

$\therefore\ I = \int_{-5}^{5} \log\left(\frac{2-x}{2+x}\right)dx = \square$.

**6. Evaluate : $\int_{-1}^{1} f(x)dx$, if**

**$f(x) = 4-3x$, for $x \leqslant 0$**

**$= 3x+4$, for $x > 0$.**

**Solution :** $\int_{-1}^{1} f(x)\,dx = \int_{-1}^{\square} f(x)\,dx + \int_{\square}^{1} f(x)\,dx$

$$= \int_{-1}^{\square} (4-3x)\,dx + \int_{\square}^{1} \square\,dx$$

$$= [4x - 3\cdot\square]_{-1}^{\square} + [3\square + 4\square]_{\square}^{1}$$

$$= [0 - (\square)] + [\square - 0]$$

$$= \square + \square = \square.$$

## OBJECTIVE SECTION

### MULTIPLE CHOICE QUESTIONS

**Select and write the correct answer from the given alternatives in each of the following questions :**

1. If $\int_0^{\alpha} (3x^2+2x+1)dx = 14$, then $\alpha = \ldots\ldots\ldots$

(a) 1 (b) 2 (c) $-1$ (d) $-2$

2. If $\int_0^{\alpha} 3x^2 dx = 8$, then the value of $\alpha$ is

(a) 0 (b) $-2$ (c) 2 (d) $\pm 2$

3. $\int_4^9 \sqrt{x}\, dx = \ldots\ldots\ldots$

(a) $\frac{10}{3}$ (b) $\frac{19}{3}$ (c) $\frac{38}{3}$ (d) $\frac{38}{9}$

4. $\int_0^3 \frac{1}{2x+1} dx = \ldots\ldots\ldots$

(a) $\frac{1}{2}\log 7$ (b) $\log 7$ (c) $2\log 7$ (d) $\log 7 + 1$

5. $\int_3^5 \frac{x}{x^2+5} dx = \ldots\ldots\ldots$

(a) $\log\left(\frac{15}{7}\right)$ (b) $\frac{1}{2}\log\left(\frac{15}{7}\right)$

(c) $2\log\left(\frac{15}{7}\right)$ (d) $\log\left(\frac{7}{15}\right)$

6. $\int_{-1}^2 |x|\, dx = \ldots\ldots\ldots$

(a) $\frac{3}{2}$ (b) 0 (c) 1 (d) $\frac{5}{2}$

7. $\int_0^1 \frac{dx}{x + x^{-4}} = \ldots\ldots\ldots$

(a) $\log 2$ (b) $5\log 2$ (c) $\frac{1}{5}\log 2$ (d) $-\frac{1}{5}\log 2$

8. $\int_1^2 x \log x\, dx = \ldots\ldots\ldots$

(a) $\log 2 - \frac{3}{4}$ (b) $2\log 2 + \frac{3}{4}$

(c) $2\log 2 - \frac{1}{4}$ (d) $2\log 2 - \frac{3}{4}$

9. $\int_0^{\infty} (a^{-x} - b^{-x})\, dx = \ldots\ldots\ldots$

(a) $\frac{1}{\log a} - \frac{1}{\log b}$ (b) $\log(ab)$

(c) $\log\left(\frac{a}{b}\right)$ (d) $\frac{1}{\log a} + \frac{1}{\log b}$

10. If $\int_0^1 \frac{dx}{\sqrt{1+x} - \sqrt{x}} = \frac{k}{3}$, then $k$ is equal to

(a) $\sqrt{2}(2\sqrt{2}-2)$ (b) $\frac{\sqrt{2}}{3}(2-2\sqrt{2})$

(c) $\frac{2\sqrt{2}-2}{3}$ (d) $4\sqrt{2}$

11. $\int_1^e \frac{e^x}{x}(1 + x\log x)\, dx = \ldots\ldots\ldots$

(a) $e^e$ (b) $e^e - e$ (c) $e^e + e$ (d) $1 - e^e$

12. $\int_2^e \left[\frac{1}{\log x} - \frac{1}{(\log x)^2}\right] dx = \ldots\ldots\ldots$

(a) $e - 2$ (b) $e + 2\log_2 e$

(c) $e - 2\log_2 e$ (d) $e + 2\log_e 2$

13. $\int_1^2 \frac{\sqrt{x} - \sqrt{3-x}}{1 + x\sqrt{3-x}} dx = \ldots\ldots\ldots$

(a) $-1$ (b) 0 (c) 1 (d) $\frac{3}{4}$

14. $\int_0^{2a} \frac{f(x)}{f(x) + f(2a-x)} dx = \ldots\ldots\ldots$

(a) 0 (b) $a$ (c) $2a$ (d) $\frac{a}{2}$

15. $\int_3^8 \frac{\sqrt{x}}{\sqrt{11-x} + \sqrt{x}} dx = \ldots\ldots\ldots$

(a) $\frac{3}{2}$ (b) $\frac{5}{2}$ (c) $\frac{7}{2}$ (d) $\frac{11}{2}$

16. $\int_{-4}^4 \log\left(\frac{5-x^3}{5+x^3}\right) dx = \ldots\ldots\ldots$

(a) $-1$ (b) 0 (c) 1 (d) 2

17. $\int_0^1 \frac{x^7}{\sqrt{1-x^4}} dx$ is equal to

(a) 1 (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $\frac{3}{2}$

18. $\int_a^b \frac{\log x}{x} dx$ is equal to

(a) $\frac{1}{2}\log(ab)\log\left(\frac{b}{a}\right)$ (b) $\log(ab)\log\left(\frac{b}{a}\right)$

(c) $\log(ab)\log\left(\frac{a}{b}\right)$ (d) $\frac{1}{2}\log(ab)\log\left(\frac{a}{b}\right)$

19. $\int_0^a [f(a+x) + f(a-x)]\, dx = \ldots\ldots\ldots$

(a) $\int_0^a f(x)dx$ (b) $2\int_0^a f(x)dx$

(c) $\int_0^{2a} f(x)dx$ (d) $\int_a^{2a} f(x)dx$

20. $\int_0^1 x^2\sqrt{1-x}\, dx = \ldots\ldots\ldots$

(a) $\frac{16}{105}$ (b) $-\frac{16}{105}$ (c) $\frac{15}{105}$ (d) $\frac{8}{105}$

### Answers

1. (b) 2 2. (c) 2 3. (c) $\frac{38}{3}$

4. (a) $\frac{1}{2}\log 7$ 5. (b) $\frac{1}{2}\log\left(\frac{15}{7}\right)$ 6. (d) $\frac{5}{2}$

7. (c) $\frac{1}{5}\log 2$ 8. (d) $2\log 2-\frac{3}{4}$ 9. (a) $\frac{1}{\log a}-\frac{1}{\log b}$

10. (d) $4\sqrt{2}$ 11. (a) $e^e$ 12. (c) $e-2\log_2 e$

13. (b) 0 14. (b) $a$ 15. (b) $\frac{5}{2}$

16. (b) 0 17. (b) $\frac{1}{3}$ 18. (a) $\frac{1}{2}\log(ab)\log\left(\frac{b}{a}\right)$

19. (c) $\int_0^{2a} f(x)dx$ 20. (a) $\frac{16}{105}$.

### TRUE OR FALSE

**State whether the following statements are *True* or *False* :**

1. If $f(a-x)=-f(x)$, then $\int_0^a f(x)\,dx=0$.
2. If $f(2a-x)=f(x)$, then $\int_0^{2a} f(x)\,dx=0$.
3. $\int_0^1 x(1-x)^5dx=\frac{1}{42}$.
4. $\int_a^a f(x)dx=1$.
5. $\int_a^b f(x)dx=\int_a^b f(a-b+x)\,dx$.
6. $\int_a^b f(x)dx=-\int_b^a f(t)\,dt$.
7. If $\int_a^b x^3dx=0$ and $\int_a^b x^2dx=\frac{2}{3}$, then $a=1$, $b=-1$.
8. If $f(x)=f(a-x)$, then $\int_0^a xf(x)dx=a\int_0^a f(x)\,dx$.
9. $\int_0^2 \frac{1}{\sqrt{4+x^2}}dx=\log(\sqrt{2}+1)$.
10. $\int_0^1 \frac{1-x}{1+x}dx=\log 4+1$.

### Answers

1. True 2. False 3. True 4. False 5. False
6. True 7. False 8. False 9. True 10. False.

### FILL IN THE BLANKS

**Fill in the following blanks with an appropriate words/numbers :**

1. If $\int_0^1 (3x^2+2x+a)dx=0$, then $a=$ .........
2. $\int_0^4 e^{2x}dx=$ .........
3. $\int_1^e \frac{\log x}{x}dx=$ .........
4. $\int_{-2}^{2} \frac{x^3}{9-x^2}dx=$ .........
5. If $f$ is an even function, then $\int_{-a}^{a} f(x)dx=$ .........
6. $\int_1^3 f(x)dx+\int_3^5 f(x)dx=$ .........
7. If $\int_{-a}^{a} f(x)dx=0$, then $f(x)$ is an ......... function.
8. $\int_1^2 \frac{1}{x^2}\cdot e^{-\frac{1}{x}}dx=$ .........
9. $\int_0^{4014} \frac{2^x}{2^x+2^{4014-x}}dx=$ .........
10. $\int_{-5}^{5} x^2[f(x)-f(-x)]dx=$ .........

### Answers

1. $-2$ 2. $\frac{1}{2}(e^8-1)$ 3. $\frac{1}{2}$

4. 0 5. $2\int_0^a f(x)dx$ 6. $\int_1^5 f(x)dx$

7. odd 8. $\frac{\sqrt{e}-1}{e}$ 9. 2007

10. 0.

***

# 7 APPLICATIONS OF DEFINITE INTEGRATION

CHAPTER OUTLINE



## IMPORTANT FORMULAE

1. The area bounded by the curve $y = f(x)$, the X-axis and the ordinates $x = a$ and $x = b$ is given by
$$A = \left|\int_a^b y\,dx\right| = \left|\int_a^b f(x)\,dx\right|.$$

2. If the two curves $y = f(x)$ and $y = g(x)$ intersect each other at $x = a$ and $x = b$, then the area between the curves is
$$\left|\int_a^b f(x)\,dx - \int_a^b g(x)\,dx\right|.$$

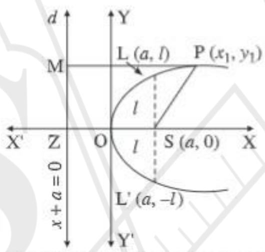
3. The area bounded by the curve $x = g(y)$, the Y-axis and the abscissas $y = c$ and $y = d$ is given by
$$A = \left|\int_c^d x\,dy\right| = \left|\int_c^d g(y)\,dy\right|.$$

### INTRODUCTION

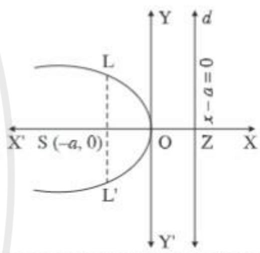
We have studied definite integral and the properties of definite integral in the previous chapter. The definite integral has many applications in Science and Mathematics. In this chapter, we shall study the geometrical application of definite integral, particularly, finding area under the curve.

## 7.1 : STANDARD FORMS OF PARABOLA AND THEIR SHAPES

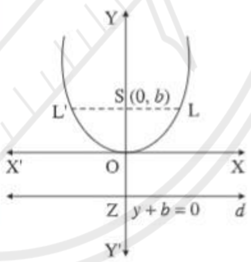
1. $y^2 = 4ax$



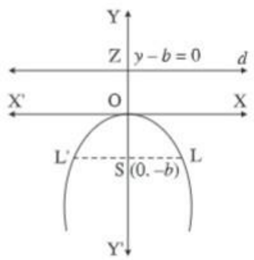

2. $y^2 = -4ax$




3. $x^2 = 4by$

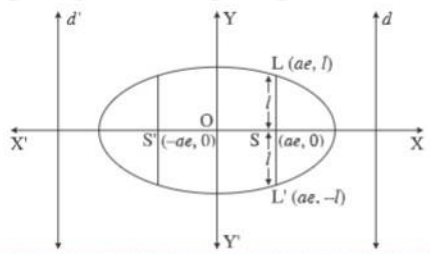



4. $x^2 = -4by$

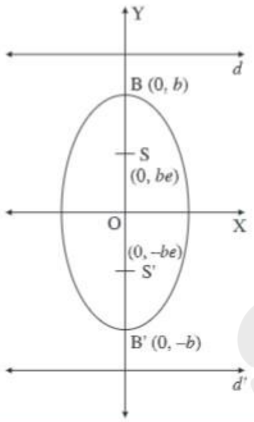



## 7.2 : STANDARD FORMS OF ELLIPSE

1. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \ (a > b)$



**2.** $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1\ (a<b)$

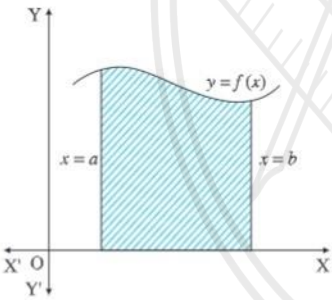



## 7.3 : AREA UNDER THE CURVE

**Area under the curve :** We state only formulae (without proof) to find the area under the curve.

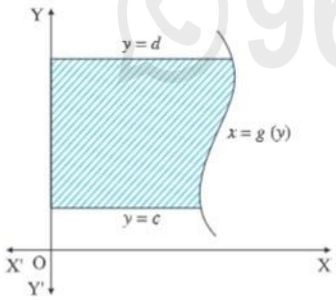
1. The area bounded by the curve $y=f(x)$, the X-axis and the lines $x=a$ and $x=b$ is given by

$$\int_a^b y\,dx=\int_a^b f(x)\,dx.$$



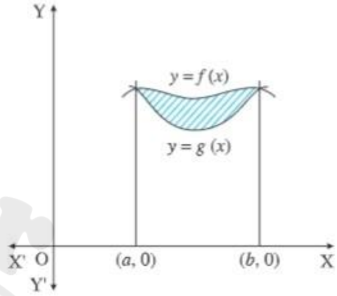

2. The area bounded by the curve $x=g(y)$, the Y-axis and the lines $y=c$ and $y=d$ is given by

$$\int_c^d x\,dy=\int_c^d g(y)\,dy.$$




3. Consider the two curves $y=f(x)$ and $y=g(x)$ which intersect each other at $x=a$ and $x=b$ as shown in the figure. Then the area between the curves is

$$\int_a^b f(x)\,dx-\int_a^b g(x)\,dx \text{ where } f(x)>g(x) \text{ for } a<x<b.$$



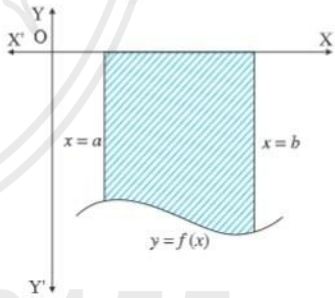

If the relative position of the curves is not known, then the area between the curves is

$$\left|\int_a^b f(x)\,dx-\int_a^b g(x)\,dx\right|.$$

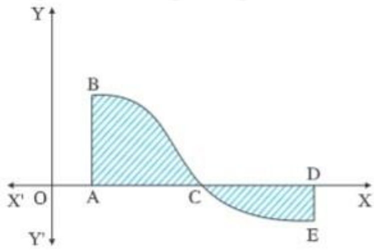
**Remarks :**

1. The area of the portion of the curve lying above the X-axis is positive.
2. If the curve, under consideration, is below the X-axis, then the area bounded by the curve, X-axis and the lines $x=a$, $x=b$ is negative. So we consider the absolute value in this case.

   Thus, required area $=\left|\int_a^b f(x)\,dx\right|$.




3. If the curve crosses the X-axis, i.e. some part of the curve lies above and some part lies below the X-axis



(see the figure), then we find the area of the regions ABC and CDE separately by definite integrals. The definite integral corresponding to the area CDE is negative. We make it positive and then add the areas of the two regions.

**EXERCISE 7.1** | **Textbook page 157**

**1. Find the area of the region bounded by the following curves, the X-axis and the given lines :**

**(i)** $y = x^4,\ x = 1,\ x = 5$

**(ii)** $y = \sqrt{6x+4},\ x = 0,\ x = 2$

**(iii)** $y = \sqrt{16 - x^2},\ x = 0,\ x = 4$

**[Given :** $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$ **and** $\sin^{-1}(1) = \frac{\pi}{2}$, $\sin^{-1}(0) = 0.$**]**

**(iv)** $2y = 5x + 7,\ x = 2,\ x = 8$

**(v)** $2y + x = 8,\ x = 2,\ x = 4$

**(vi)** $y = x^2 + 1,\ x = 0,\ x = 3$

**(vii)** $y = 2 - x^2,\ x = -1,\ x = 1.$

**Solution :**

**(i)** Required area $= \int_1^5 y\,dx$, where $y = x^4$

$$= \int_1^5 x^4\,dx = \left[\frac{x^5}{5}\right]_1^5$$

$$= \frac{1}{5}[3125 - 1] = \frac{3124}{5} \text{ sq units.}$$

**(ii)** Required area $= \int_0^2 y\,dx$, where $y = \sqrt{6x+4}$

$$= \int_0^2 \sqrt{6x+4}\,dx = \int_0^2 (6x+4)^{\frac{1}{2}}\,dx$$

$$= \left[\frac{(6x+4)^{\frac{3}{2}}}{3/2} \times \frac{1}{6}\right]_0^2$$

$$= \frac{1}{9}\left[(6x+4)^{\frac{3}{2}}\right]_0^2$$

$$= \frac{1}{9}[64 - 8]$$

$$= \frac{56}{9} \text{ sq units.}$$

[*Note : Answer in the textbook is incorrect.*]

**(iii)** Required area $= \int_0^4 y\,dx$, where $y = \sqrt{16 - x^2}$

$$= \int_0^4 \sqrt{16 - x^2}\,dx$$

$$= \left[\frac{x}{2}\sqrt{16 - x^2} + \frac{16}{2}\sin^{-1}\left(\frac{x}{4}\right)\right]_0^4$$

$$\ldots \left[\because \int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]$$

$$= 0 + 8\sin^{-1}(1) - 0 - 0 \qquad \ldots [\because \sin^{-1}(0) = 0]$$

$$= 8 \times \frac{\pi}{2} = 4\pi \text{ sq units.} \qquad \ldots \left[\because \sin^{-1}(1) = \frac{\pi}{2}\right]$$

**(iv)** Required area $= \int_2^8 y\,dx$, where $2y = 5x + 7$

i.e. $y = \frac{5x+7}{2}$

$$= \int_2^8 \left(\frac{5x+7}{2}\right)dx = \frac{1}{2}\int_2^8 (5x+7)\,dx$$

$$= \frac{1}{2}\left[5 \cdot \frac{x^2}{2} + 7x\right]_2^8$$

$$= \frac{1}{2}[160 + 56 - 10 - 14]$$

$$= \frac{1}{2}(192) = 96 \text{ sq units.}$$

**(v)** Required area $= \int_2^4 y\,dx$, where $2y + x = 8$

i.e. $y = \frac{8-x}{2}$

$$= \int_2^4 \left(\frac{8-x}{2}\right)dx = \frac{1}{2}\int_2^4 (8-x)\,dx$$

$$= \frac{1}{2}\left[8x - \frac{x^2}{2}\right]_2^4$$

$$= \frac{1}{2}[(32-8) - (16-2)]$$

$$= \frac{1}{2}(24 - 14) = 5 \text{ sq units.}$$

**(vi)** Required area $= \int_0^3 y\,dx$, where $y = x^2 + 1$

$$= \int_0^3 (x^2 + 1)\,dx$$

$$= \left[\frac{x^3}{3} + x\right]_0^3$$

$$= 9 + 3 - 0 = 12 \text{ sq units.}$$

**(vii)** Required area $= \int_{-1}^{1} y\,dx$, where $y = 2 - x^2$

$= \int_{-1}^{1} (2 - x^2)\,dx$

$= 2\int_{0}^{1} (2 - x^2)\,dx$

... [$\because f(x) = 2 - x^2$ is an even function]

$= 2\left[2x - \frac{x^3}{3}\right]_0^1$

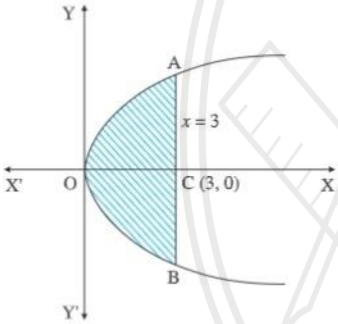
$= 2\left[2 - \frac{1}{3} - 0\right]$

$= 2\left(\frac{5}{3}\right)$

$= \frac{10}{3}$ sq units.

**2. Find the area of the region bounded by the parabola $y^2 = 4x$ and the line $x = 3$.**

**Solution :**




Required area = area of the region OABO

$= 2$ (area of the region OACO)

$= 2\int_{0}^{3} y\,dx$, where $y^2 = 4x$, i.e. $y = 2\sqrt{x}$

$= 2\int_{0}^{3} 2\sqrt{x}\,dx$

$= 4\int_{0}^{3} x^{\frac{1}{2}}\,dx$

$= 4\cdot\left[\frac{x^{\frac{3}{2}}}{3/2}\right]_0^3$

$= \frac{8}{3}[x^{\frac{3}{2}}]_0^3$

$= \frac{8}{3}(3\sqrt{3} - 0)$

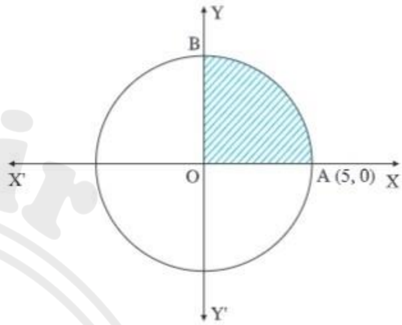
$= 8\sqrt{3}$ sq units.

**3. Find the area of the circle $x^2 + y^2 = 25$.**

**[Given : $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$**

**and $\sin^{-1}(1) = \frac{\pi}{2}$, $\sin^{-1}(0) = 0$.]**

**Solution :**




By the symmetry of the circle, its area is equal to 4 times the area of the region OABO. Clearly for this region the limits of integration are 0 and 5.

From the equation of the circle, $y^2 = 25 - x^2$.

In the first quadrant $y > 0$

$\therefore\ y = \sqrt{25 - x^2}$

$\therefore$ area of the circle $= 4$(area of region OABO)

$= 4\int_{0}^{5} y\,dx = 4\int_{0}^{5} \sqrt{25 - x^2}\,dx$

$= 4\left[\frac{x}{2}\sqrt{25 - x^2} + \frac{25}{2}\sin^{-1}\left(\frac{x}{5}\right)\right]_0^5$

... $\left[\because \int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]$

$= 4\left[\left\{\frac{5}{2}\sqrt{25 - 25} + \frac{25}{2}\sin^{-1}(1)\right\} - \left\{\frac{0}{2}\sqrt{25 - 0} + \frac{25}{2}\sin^{-1}(0)\right\}\right]$

$= 4\cdot\frac{25}{2}\cdot\frac{\pi}{2} = 25\pi$ sq units.

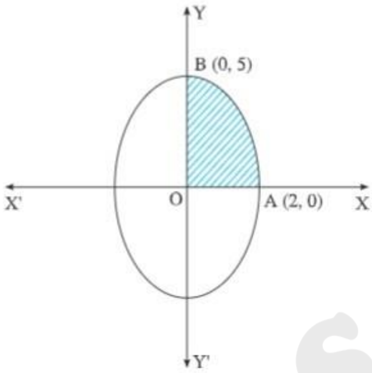
... $\left[\because \sin^{-1}(1) = \frac{\pi}{2}, \sin^{-1}(0) = 0.\right]$

**4. Find the area of the ellipse $\frac{x^2}{4} + \frac{y^2}{25} = 1$.**

**[Given : $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$**

**and $\sin^{-1}(1) = \frac{\pi}{2}$, $\sin^{-1}(0) = 0$.]**

**Solution :**




By the symmetry of the ellipse, its area is equal to 4 times the area of the region OABO. Clearly for this region, the limits of integration are 0 and 2.

From the equation of the ellipse,

$$\frac{y^2}{25} = 1 - \frac{x^2}{4} = \frac{4-x^2}{4}$$

$$\therefore\ y^2 = \frac{25}{4}(4-x^2)$$

In the first quadrant, $y > 0$

$$\therefore\ y = \frac{5}{2}\sqrt{4-x^2}$$

$\therefore$ area of ellipse $= 4$(area of the region OABO)

$$= 4\int_0^2 y\,dx$$

$$= 4\int_0^2 \frac{5}{2}\sqrt{4-x^2}\,dx$$

$$= 10\int_0^2 \sqrt{4-x^2}\,dx$$

$$= 10\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

$$\ldots\left[\because \int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]$$

$$= 10\left[\left\{\frac{2}{2}\sqrt{4-4} + 2\sin^{-1}(1)\right\} - \left\{\frac{0}{2}\sqrt{4-0} + 2\sin^{-1}(0)\right\}\right]$$

$$= 10 \times 2 \times \frac{\pi}{2} = 10\pi \text{ sq units.}$$

$$\ldots\left[\because \sin^{-1}(1) = \frac{\pi}{2},\ \sin^{-1}(0) = 0.\right]$$

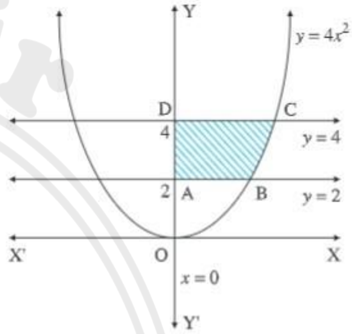
## ADDITIONAL SOLVED PROBLEMS – 7

**1. Find the area of the region lying in the first quadrant and bounded by $y = 4x^2$, $x = 0$, $y = 2$ and $y = 4$.**

**Solution :**

The equation of the curve is $y = 4x^2$, i.e. $x^2 = \frac{y}{4}$

$$\therefore\ x = \frac{\sqrt{y}}{2}$$




Required area = area of the region ABCDA

$$= \int_2^4 x\,dy, \text{ where } x = \frac{\sqrt{y}}{2}$$

$$= \int_2^4 \frac{\sqrt{y}}{2}\,dy = \frac{1}{2}\int_2^4 y^{\frac{1}{2}}\,dy$$

$$= \frac{1}{2}\cdot\left[\frac{y^{\frac{3}{2}}}{3/2}\right]_2^4 = \frac{1}{3}\left[y^{\frac{3}{2}}\right]_2^4$$

$$= \frac{1}{3}\left[4^{\frac{3}{2}} - 2^{\frac{3}{2}}\right]$$

$$= \frac{1}{3}(8 - 2\sqrt{2}) \text{ sq units.}$$

**2. Find the area under the curve $y = 3 - 2x - x^2$, up to the X-axis, bounded by the ordinates at $x = -1$ and $x = 3$.**

**Solution :**

When $x = -1$, $y = 3 + 2 - 1 = 4$

When $x = 3$, $y = 3 - 6 - 9 = -12$

This shows that some portion of the curve lies below the X-axis.

The curve intersects the X-axis, where $y = 0$

$\therefore$ putting $y = 0$, we get $3 - 2x - x^2 = 0$

$\therefore$ $x^2 + 2x - 3 = 0$ $\quad\therefore$ $(x+3)(x-1) = 0$

$\therefore$ $x = -3$, $x = 1$

Since, $x=1$ lies between given ordinates $x=-1$ and $x=3$.

$\therefore$ the portion of the curve between $x=-1$ and $x=1$ lies above the X-axis and the portion between $x=1$ and $x=3$ lies below the X-axis.

Area under the curve between $x=-1$ and $x=1$ is given by

$$\int_{-1}^{1} y\,dx=\int_{-1}^{1}(3-2x-x^2)\,dx$$

$$=\left[3x-2\cdot\frac{x^2}{2}-\frac{x^3}{3}\right]_{-1}^{1}$$

$$=\left(3-1-\frac{1}{3}\right)-\left(-3-1+\frac{1}{3}\right)=\frac{16}{3}$$

Area under the curve between $x=1$ and $x=3$ is given by

$$\int_{1}^{3} y\,dx=\int_{1}^{3}(3-2x-x^2)\,dx$$

$$=\left[3x-2\cdot\frac{x^2}{2}-\frac{x^3}{3}\right]_{1}^{3}$$

$$=\left[3(3)-(3)^2-\frac{(3)^3}{3}\right]-\left[3(1)-(1)^2-\frac{(1)^3}{3}\right]$$

$$=(9-9-9)-\left(3-1-\frac{1}{3}\right)=\frac{-32}{3}$$

The magnitude of this area $=\frac{32}{3}$

$\therefore$ required area $=\frac{16}{3}+\frac{32}{3}=\frac{48}{3}=16$ sq units.

## EXAMPLES FOR PRACTICE 7.1

1. Find the area of the region bounded by the curve $y=x^2$, X-axis and the lines $x=1$, $x=3$.
2. Find the area of the region bounded by the parabola $y^2=9x$, X-axis and the lines $x=4$, $x=9$.
3. Find the area of the region bounded by the curve $9x^2-y^2=36$, X-axis and the lines $x=2$, $x=4$ in the first quadrant.
4. Find the area under the curve $y=2x+3x^2$ between $y=0$, $x=0$ and $x=2$.
5. Find the area under the curve $y=(x^2+2)^2+2x$ between the lines $x=0$, $x=2$ and the X-axis.
6. Find the area under the curve $y=4\sqrt{x-1}$, $1\leqslant x\leqslant 3$ and the X-axis.
7. Find the area of the region bounded by the curve $y=x^2-2x+2$, the X-axis and the lines $x=1$ and $x=2$.
8. Find the area of the region bounded by the parabola $y^2=16x$ and the line $x=4$.
9. Find the area under the curve $y=5-4x-x^2$, up to the X-axis, bounded by the ordinates at $x=-1$ and $x=2$.
10. Find the area of the region bounded by $x^2=16y$, $y=1$, $y=4$ and the Y-axis, lying in the first quadrant.
    [Hint : Required area $=\int_1^4 x\,dy$.]
11. Find the area of the region bounded by the curve $y=x^2$ and the line $y=4$.
    [Hint : Required area $=2\int_0^4 x\,dy$.]
12. Find the area of the circle $x^2+y^2=r^2$.
    $\ldots\left[\text{Given : }\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\sin^{-1}\left(\frac{x}{a}\right)\right.$ and $\left.\sin^{-1}(1)=\frac{\pi}{2},\ \sin^{-1}(0)=0.\right]$
13. Find the area of the ellipse $\frac{x^2}{144}+\frac{y^2}{81}=1$.
    $\ldots\left[\text{Given : }\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\sin^{-1}\left(\frac{x}{a}\right)\right.$ and $\left.\sin^{-1}(1)=\frac{\pi}{2},\ \sin^{-1}(0)=0.\right]$
14. Find the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.
    $\ldots\left[\text{Given : }\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\sin^{-1}\left(\frac{x}{a}\right)\right.$ and $\left.\sin^{-1}(1)=\frac{\pi}{2},\ \sin^{-1}(0)=0.\right]$

### Answers

1. $\frac{26}{3}$ sq units
2. 38 sq units
3. $12\sqrt{3}-6\log(2+\sqrt{3})$ sq units
4. 12 sq units
5. $\frac{436}{15}$ sq units
6. 72 sq units
7. $\frac{1}{2}\left[\sqrt{2}+\log(1+\sqrt{2})\right]$ sq units
8. $\frac{128}{3}$ sq units
9. $\frac{38}{3}$ sq units
10. $\frac{56}{3}$ sq units
11. $\frac{32}{3}$ sq units
12. $\pi r^2$ sq units
13. $108\pi$ sq units
14. $\pi ab$ sq units.

## MISCELLANEOUS EXERCISE – 7

**(Textbook pages 157 and 158)**

**(I) Choose the correct alternatives :**

1. Area of the region bounded by the curve $x^2 = y$, the X-axis and the lines $x = 1$ and $x = 3$ is

   (a) $\frac{26}{3}$ sq units (b) $\frac{3}{26}$ sq unit

   (c) 26 sq units (d) 3 sq units

2. The area of the region bounded by $y^2 = 4x$, the X-axis and the lines $x = 1$ and $x = 4$ is

   (a) 28 sq units (b) 3 sq unit

   (c) $\frac{28}{3}$ sq units (d) $\frac{3}{28}$ sq unit

3. Area of the region bounded by $x^2 = 16y$, $y = 1$ and $y = 4$ and the Y-axis, lying in the first quadrant is

   (a) 63 sq units (b) $\frac{3}{56}$ sq unit

   (c) $\frac{56}{3}$ sq units (d) $\frac{63}{7}$ sq unit

4. Area of the region bounded by $y = x^4$, $x = 1$, $x = 5$ and the X-axis is

   (a) $\frac{3142}{5}$ sq units (b) $\frac{3124}{5}$ sq units

   (c) $\frac{3142}{3}$ sq units (d) $\frac{3124}{3}$ sq units

5. Using definite integration area of circle $x^2 + y^2 = 25$ is

   (a) $5\pi$ sq units (b) $4\pi$ sq units

   (c) $25\pi$ sq units (d) 25 sq units

### Answers

1. (a) $\frac{26}{3}$ sq units 2. (c) $\frac{28}{3}$ sq units

3. (c) $\frac{56}{3}$ sq units 4. (b) $\frac{3124}{5}$ sq units

5. (c) $25\pi$ sq units.

**(II) Fill in the blanks :**

1. Area of the region bounded by $y = x^4$, $x = 1$, $x = 5$ and the X-axis is .........
2. Using definite integration area of the circle $x^2 + y^2 = 49$ is .........
3. Area of the region bounded by $x^2 = 16y$, $y = 1$, $y = 4$ and the Y-axis lying in the first quadrant is .........
4. The area of the region bounded by the curve $x^2 = y$, the X-axis and the lines $x = 3$ and $x = 9$ is .........
5. The area of the region bounded by $y^2 = 4x$, the X-axis and the lines $x = 1$ and $x = 4$ is .........

### Answers

1. $\frac{3124}{5}$ sq units 2. $49\pi$ sq units 3. $\frac{56}{3}$ sq units

4. 234 sq units

   [*Note : Answer in the textbook is incorrect.*]

5. $\frac{28}{3}$ sq units.

**(III) State whether each of the following is *True* or *False*.**

1. The area bounded by the curve $x = g(y)$, Y-axis and bounded between the lines $y = c$ and $y = d$ is given by $\int_c^d x\,dy = \int_{y=c}^{y=d} g(y)\,dy$.
2. The area bounded by two curves $y = f(x)$, $y = g(x)$ and X-axis is $\left|\int_a^b f(x)\,dx - \int_b^a g(x)\,dx\right|$.
3. The area bounded by the curve $y = f(x)$, X-axis and lines $x = a$ and $x = b$ is $\left|\int_a^b f(x)\,dx\right|$.
4. If the curve, under consideration, is below the X-axis, then the area bounded by curve, X-axis and lines $x = a$, $x = b$ is positive.
5. The area of the portion lying above the X-axis is positive.

### Answers

1. True 2. False 3. True 4. False 5. True.

**(IV) Solve the following :**

**1. Find the area of the region bounded by the curve $xy = c^2$, the X-axis, and the lines $x = c$, $x = 2c$.**

**Solution :**

Required area $= \int_c^{2c} y\,dx$, where $xy = c^2$, i.e. $y = \frac{c^2}{x}$

$$= \int_c^{2c} \frac{c^2}{x}\,dx = c^2 \int_c^{2c} \frac{1}{x}\,dx$$

$$= c^2 [\log x]_c^{2c}$$

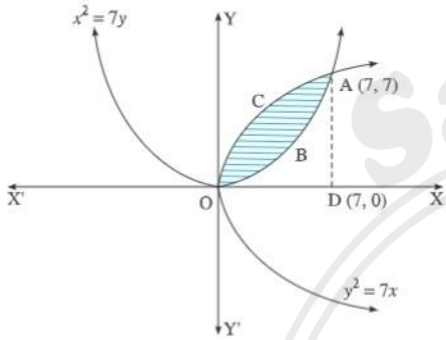
$$= c^2 (\log 2c - \log c)$$

$= c^2 \log\left(\frac{2c}{c}\right)$

$= c^2 \cdot \log 2$ sq units.

**2. Find the area between the parabolas $y^2 = 7x$ and $x^2 = 7y$.**

**Solution :**




For finding the points of intersection of the two parabolas, we equate the values of $y^2$ from their equations.

From the equation $x^2 = 7y$, $y^2 = \frac{x^4}{49}$

$\therefore \frac{x^4}{49} = 7x \qquad \therefore x^4 = 343x$

$\therefore x^4 - 343x = 0 \qquad \therefore x(x^3 - 343) = 0$

$\therefore x = 0$ or $x^3 = 343$, i.e. $x = 7$

When $x = 0$, $y = 0$

When $x = 7$, $7y = 49 \qquad \therefore y = 7$

$\therefore$ the points of intersection are O(0, 0) and A(7, 7)

Required area = area of the region OBACO

= (area of the region ODACO) – (area of the region ODABO)

Now, area of the region ODACO

= area under the parabola $y^2 = 7x$, i.e. $y = \sqrt{7}\sqrt{x}$

$= \int_0^7 \sqrt{7}\sqrt{x}\,dx = \sqrt{7}\left[\frac{x^{\frac{3}{2}}}{3/2}\right]_0^7$

$= \sqrt{7} \times \frac{2}{3}\left[7^{\frac{3}{2}} - 0\right] = \frac{2\sqrt{7}}{3}\,[7\sqrt{7} - 0]$

$= \frac{98}{3}$

Area of the region ODABO = Area under the parabola $x^2 = 7y$ i.e. $y = \frac{x^2}{7}$

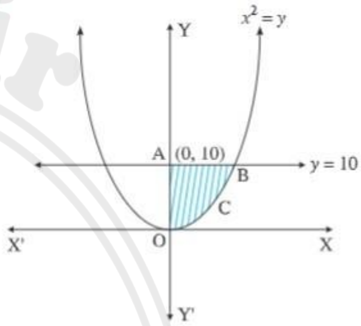
$= \int_0^7 \frac{x^2}{7}\,dx = \frac{1}{7}\left[\frac{x^3}{3}\right]_0^7 = \frac{1}{7}\left[\frac{7^3}{3} - 0\right]$

$= \frac{7^2}{3} = \frac{49}{3}$

$\therefore$ required area $= \frac{98}{3} - \frac{49}{3} = \frac{49}{3}$ sq units.

**3. Find the area of the region bounded by the curve $y = x^2$ and the line $y = 10$.**

**Solution :**




By the symmetry of the parabola, the required area is twice the area of the region OABCO

Now, area of the region OABCO

$= \int_0^{10} x\,dy$, where $x^2 = y$ i.e. $x = \sqrt{y}$

$= \int_0^{10} \sqrt{y}\,dy = \left[\frac{y^{\frac{3}{2}}}{3/2}\right]_0^{10} = \frac{10^{\frac{3}{2}}}{3/2} - 0 = \frac{2 \times 10\sqrt{10}}{3} = \frac{20\sqrt{10}}{3}$

$\therefore$ required area $= 2\left[\frac{20\sqrt{10}}{3}\right]$

$= \frac{40\sqrt{10}}{3}$ sq units.

**4. Find the area of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.**

**Solution :** *Refer to the solution of Q. 4. of Exercise 7.1.*

**Ans.** $12\pi$ sq units.

**5. Find the area of the region bounded by $y = x^2$, the X-axis and $x = 1$, $x = 4$.**

**Solution :** Required area $= \int_1^4 y\,dx$, where $y = x^2$

$= \int_1^4 x^2\,dx$

$= \left[\frac{x^3}{3}\right]_1^4 = \frac{4^3}{3} - \frac{1}{3} = \frac{64 - 1}{3} = 21$ sq units.

**6. Find the area of the region bounded by the curve $x^2 = 25y$, $y = 1$, $y = 4$ and the Y-axis.**

**Solution :**

Required area $= \int_1^4 x\,dy$, where $x^2 = 25y$, i.e. $x = 5\sqrt{y}$

$$= \int_1^4 5\sqrt{y}\,dy = 5\left[\frac{y^{\frac{3}{2}}}{3/2}\right]_1^4$$

$$= 5 \times \frac{2}{3}[4^{\frac{3}{2}} - 1] = \frac{10}{3}[(2^2)^{\frac{3}{2}} - 1]$$

$$= \frac{10}{3}[8 - 1] = \frac{70}{3} \text{ sq units.}$$

**7. Find the area of the region bounded by the parabola $y^2 = 25x$ and the line $x = 5$.**

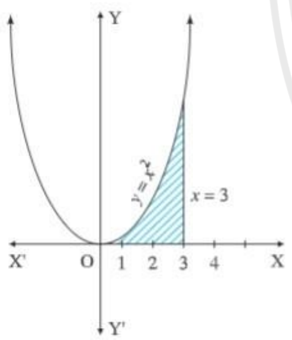
**Solution :** *Refer to the solution of Q. 2. of Exercise 7.1.*

**Ans.** $\frac{100\sqrt{5}}{3}$ sq units.

| **ACTIVITIES** | **Textbook pages 158 and 159** |
|---|---|

**From the following information find the area of the shaded regions :**

**1.**




**Solution :**

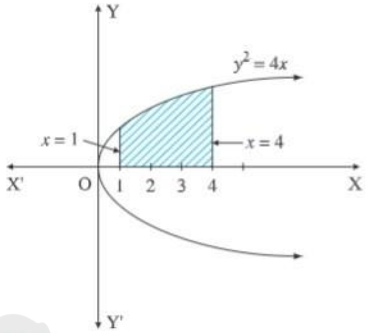
The shaded region is bounded by the curve $y = x^2$, the X-axis and the lines $x = 1$, $x = 3$.

$\therefore$ required area $= \int_1^3 y\,dx$, where $y = x^2$

$$= \int_1^3 x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_1^3 = \frac{3^3}{3} - \frac{1}{3} = \frac{26}{3} \text{ sq units.}$$

**2.**




**Solution :**

The shaded region is bounded by the curve $y^2 = 4x$, the X-axis and the lines $x = 1$, $x = 4$.

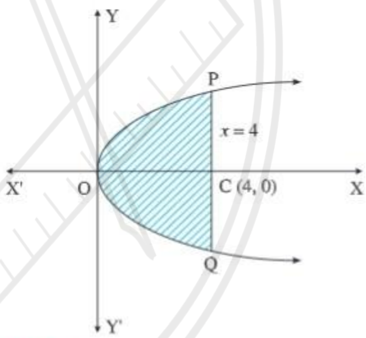
$\therefore$ required area $= \int_1^4 y\,dx$, where $y^2 = 4x$, i.e. $y = 2\sqrt{x}$

$$= 2\int_1^4 \sqrt{x}\,dx = 2\left[\frac{x^{\frac{3}{2}}}{3/2}\right]_1^4$$

$$= 2 \times \frac{2}{3}[4^{\frac{3}{2}} - 1] = \frac{8}{3}[(2^2)^{\frac{3}{2}} - 1]$$

$$= \frac{8}{3}[8 - 1] = \frac{56}{3} \text{ sq units.}$$

**3.**




**Solution :**

We take the equation of the parabola as $y^2 = 4ax$.

The shaded region is bounded by the curve $y^2 = 4ax$ and the line $x = 4$.

By the symmetry of the parabola,

required area $= 2$[area of the region above X-axis]

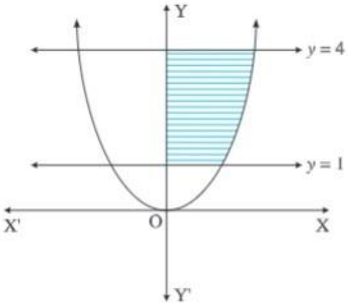
$$= 2\int_0^4 y\,dx, \text{ where } y^2 = 4ax, \text{ i.e. } y = 2\sqrt{a}\sqrt{x}$$

$$= 2(2\sqrt{a})\int_0^4 \sqrt{x}\,dx$$

$$= 4\sqrt{a}\left[\frac{x^{\frac{3}{2}}}{3/2}\right]_0^4 = 4\sqrt{a}\left[\frac{4^{\frac{3}{2}}}{3/2} - 0\right]$$

$$= 4\sqrt{a} \times \frac{2}{3}[(2^2)^{\frac{3}{2}}] = \frac{64\sqrt{a}}{3} \text{ sq units.}$$

4.




**Solution :**

We take the equation of the parabola as $x^2 = 4by$

The shaded region is bounded by the curve $x^2 = 4by$, the Y-axis and the lines $y = 1$, $y = 4$.

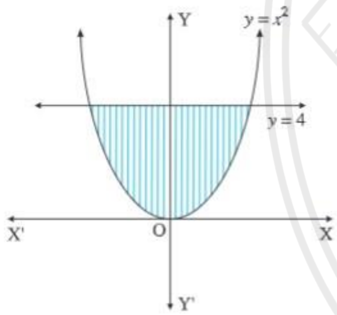
$\therefore$ the required area

$$= \int_1^4 x\,dy, \text{ where } x^2 = 4by, \text{ i.e. } x = 2\sqrt{b}\sqrt{y}$$

$$= 2\sqrt{b}\int_1^4 \sqrt{y}\,dy = 2\sqrt{b}\left[\frac{y^{\frac{3}{2}}}{3/2}\right]_1^4$$

$$= 2\sqrt{b} \times \frac{2}{3}[(2^2)^{\frac{3}{2}} - 1] = \frac{4\sqrt{b}}{3}[8 - 1] = \frac{28\sqrt{b}}{3} \text{ sq units.}$$

5.




**Solution :**

The shaded region is bounded by the curve $y = x^2$ and the line $y = 4$.

By the symmetry of the parabola,

required area = 2 [area of the region in the first quadrant]

$$= 2\int_0^4 x\,dy, \text{ where } x^2 = y, \text{ i.e. } x = \sqrt{y}$$

$$= 2\int_0^4 \sqrt{y}\,dy = 2\left[\frac{y^{\frac{3}{2}}}{3/2}\right]_0^4$$

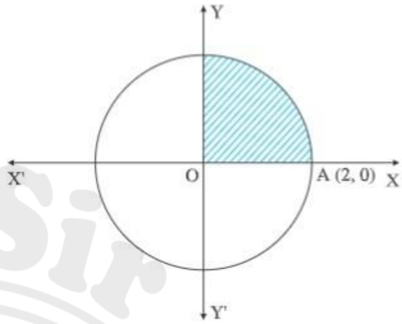
$$= 2 \times \frac{2}{3}[(2^2)^{\frac{3}{2}} - 0]$$

$$= \frac{4}{3}[8] = \frac{32}{3} \text{ sq units.}$$

## ACTIVITIES FOR PRACTICE

**From the following information, find the area of the shaded regions :**

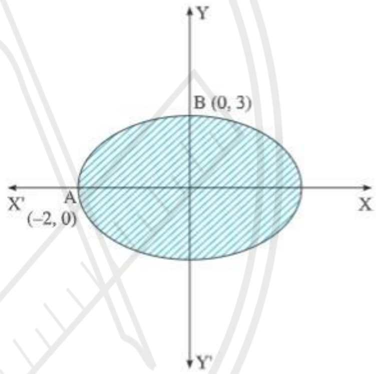
**1.**




$$\left[\text{Given : } \int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right.$$

$$\left.\text{and } \sin^{-1}(1) = \frac{\pi}{2}, \sin^{-1}(0) = 0.\right]$$

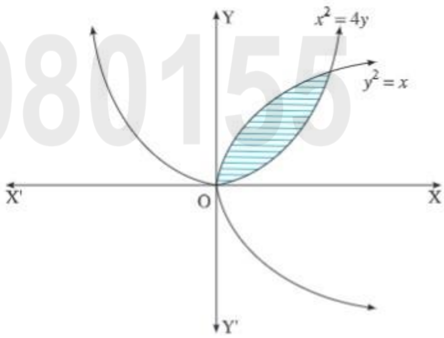
**2.**



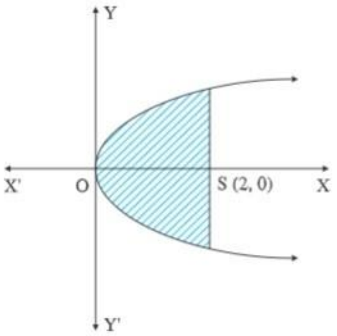

$$\left[\text{Given : } \int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right.$$

$$\left.\text{and } \sin^{-1}(1) = \frac{\pi}{2}, \sin^{-1}(0) = 0.\right]$$

**3.**



4.




## OBJECTIVE SECTION

### MULTIPLE CHOICE QUESTIONS

**Select and write the correct answer from the given alternatives in each of the following questions :**

1. The area of the region bounded by the curve $xy = 1$, X-axis and the lines $x = a$, $x = 4a$ is
   (a) $2\log 2$ sq units (b) $\log 2$ sq units
   (c) $(\log 2)^2$ sq units (d) $\frac{2}{\log 2}$ sq units

2. The area bounded by $y = 4\sqrt{x-1}$, X-axis and the lines $x = 1$, $x = 3$ is
   (a) $\frac{8\sqrt{2}}{3}$ sq units (b) $\frac{4\sqrt{2}}{3}$ sq units
   (c) $\frac{16\sqrt{2}}{3}$ sq units (d) $\frac{32\sqrt{2}}{3}$ sq units

3. The area of the region bounded by the curve $y^2 = 4x$, X-axis and the lines $x = 1$, $x = 4$ is
   (a) $\frac{16}{3}$ sq units (b) $\frac{28}{3}$ sq units
   (c) $\frac{8}{3}$ sq units (d) 9 sq units

4. The area of the region bounded by the curve $y = 2x^2 + 1$, X-axis and the lines $x = 0$, $x = 2$ is $\frac{a}{b}$ sq units. Then $a - b$ is equal to
   (a) 22 (b) 25 (c) $\frac{22}{3}$ (d) 19

. The area of the ellipse $2x^2 + y^2 = 2$ is
   (a) $2\pi$ sq units (b) $\sqrt{2}\pi$ sq units
   (c) $\frac{\pi}{\sqrt{2}}$ sq units (d) $2\sqrt{2}\pi$ sq units

5. The area enclosed by the parabolas $x^2 = 4ay$ and $y^2 = 4bx$ is
   (a) $\frac{16ab}{3}$ sq units (b) $\frac{32ab}{3}$ sq units
   (c) $\frac{8ab}{3}$ sq units (d) $\frac{64ab}{3}$ sq units

6. If the area enclosed by the parabolas $x^2 = 4y$ and $y^2 = 4by$ is $\frac{1}{3}$ sq unit, then $b$ is equal to
   (a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{1}{8}$ (d) $\frac{1}{16}$

7. The area bounded by the parabola $y^2 = 16x$ and the line $x = 3$ is $4^m(\sqrt{3})$ sq units. Then $m$ is equal to
   (a) 1 (b) 2 (c) $-1$ (d) 3

8. The area bounded by the parabola $y^2 = 4ax$, X-axis and the lines $x = 0$, $x = a$ is
   (a) $\frac{8a^2}{3}$ sq units (b) $\frac{4a^2}{3}$ sq units
   (c) $\frac{2a^2}{3}$ sq units (d) $\frac{16a^2}{3}$ sq units

9. The area of the region bounded by the parabola $y^2 = 16x$ and its latus rectum is
   (a) $\frac{32}{3}$ sq units (b) $\frac{64}{3}$ sq units
   (c) $\frac{128}{3}$ sq units (d) $\frac{256}{3}$ sq units

10. Area of the region bounded by the curve $y^2 = 4x$, Y-axis and the line $y = 3$ is
   (a) 2 sq units (b) $\frac{9}{4}$ sq units
   (c) $\frac{9}{3}$ sq units (d) $\frac{9}{2}$ sq units

11. The area of the region bounded by the curve $y = x^2$ and the line $y = 4$ is
   (a) $\frac{16}{3}$ sq units (b) $\frac{32}{3}$ sq units
   (c) $\frac{64}{3}$ sq units (d) $\frac{8}{3}$ sq units

12. The area enclosed by the parabola $x^2 = 4y$ and its latus rectum is $\frac{8}{6m}$ sq units. Then the value of $m$ is
   (a) 2 (b) 1 (c) $\frac{1}{2}$ (d) $\frac{1}{3}$

13. The area enclosed by $y = 3x + 2$, X-axis, $x = 1$, $x = 2$ is

(a) $\frac{11}{2}$ sq units (b) $\frac{15}{2}$ sq units

(c) $\frac{21}{2}$ sq units (d) $\frac{13}{2}$ sq units

14. The area of the triangle enclosed by the line $x - \sqrt{3}y = 0$, X-axis, $x = 0$, $x = 1$ is

(a) $\frac{\sqrt{3}}{2}$ sq units (b) $2\sqrt{3}$ sq units

(c) $\frac{1}{2\sqrt{3}}$ sq units (d) $\frac{2}{\sqrt{3}}$ sq units

### Answers

1. (a) $2 \log 2$ sq units
2. (c) $\frac{16\sqrt{2}}{3}$ sq units
3. (b) $\frac{28}{3}$ sq units
4. (d) 19
5. (b) $\sqrt{2}\pi$ sq units
6. (a) $\frac{16ab}{3}$ sq units
7. (d) $\frac{1}{16}$
8. (b) 2
9. (a) $\frac{8a^2}{3}$ sq units
10. (c) $\frac{128}{3}$ sq units
11. (b) $\frac{9}{4}$ sq units
12. (b) $\frac{32}{3}$ sq units
13. (c) $\frac{1}{2}$
14. (d) $\frac{13}{2}$ sq units
15. (c) $\frac{1}{2\sqrt{3}}$ sq units.

## TRUE OR FALSE

**State whether the following statements are *True* or *False* :**

1. The area enclosed by an unbounded curve and a straight line cannot be found using integration.
2. If the area enclosed by $y = f(x)$, X-axis, $x = a$, $x = b$ and $y = g(x)$, X-axis, $x = a$, $x = b$ are equal, then $f(x) = g(x)$.
3. $A_1$ is the area enclosed by $y = f(x)$, $x = a$, $x = b$ and X-axis, $A_2$ is the area enclosed by $y = f(x)$, $x = c$, $x = d$ and X-axis. If $A_1 = A_2$, then $a = c$, $b = d$.
4. The area of an ellipse found using integration is equal to the area found using formula.
5. Using integration, area of a region can be found even if part of the region lies above the X-axis and part of the region lies below the X-axis.

### Answers

1. False 2. False 3. False 4. True 5. True.

## FILL IN THE BLANKS

**Fill in the following blanks with an appropriate words/numbers :**

1. The area enclosed by $y = x$, X-axis and the lines $x = 1$, $x = 3$ is .........
2. The area enclosed by $y = x^3$, X-axis, $x = 0$ and $x = 4$ is .........
3. The area enclosed by $y = x^2$ and its latus rectum is ......
4. The area of the triangle formed by O(0, 0), A(4, 0), B = (4, 2) using integration is ......... .
5. The area enclosed by $y = \sqrt{x - 2}$, X-axis and the lines $x = 6$, $x = 11$ is .........

### Answers

1. 4 sq units 2. 64 sq units
3. $\frac{1}{6}$ sq unit 4. 4 sq units
5. $\frac{38}{3}$ sq units.

❄❄❄

# 8 DIFFERENTIAL EQUATIONS AND APPLICATIONS

CHAPTER OUTLINE



## IMPORTANT FORMULAE

1. An equation which contains one independent and one or more dependent variables and their derivatives $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots$ or differentials $dx$, $dy$ is called a **Differential Equation (D.E.).**
2. A differential equation in which the dependent variable $y$ depends only on one independent variable $x$ is called an **ordinary differential equation.**
3. The **order** of a differential equation is the order of the highest order differential coefficient appearing in it.
4. The **degree** of a differential equation is the degree of the highest order derivative occurring in it, when the differential equation is so written that the derivatives are free from negative or fractional indices.
5. Order and degree of a differential equation are always positive integers.
6. The general differential equation of first order and first degree is of the form $M + N\frac{dy}{dx} = 0$, where $M$ and $N$ are the functions of $x$ and $y$.
7. If the differential equation can be put in the form $f_1(x)\,dx + f_2(y)\,dy = 0$, then such differential equation are said to be in the **variable separable form.**

   The solution of this equation is $\int f_1(x)\,dx + \int f_2(y)\,dy = c$.
8. A differential equation of first order and first degree is said to be **homogeneous**, if it is of the form $\frac{dy}{dx} = \frac{f_1(x, y)}{f_2(x, y)}$, where $f_1$ and $f_2$ are functions of same degree in $x$ and $y$.

   Such an equation can be reduced to the variable separable form by substituting $y = vx$.

**9.** The general form of a linear differential equation of the first order is

$$\frac{dy}{dx}+P\cdot y=Q$$

where $P$ and $Q$ are the functions of $x$ only or constants. The solution of the linear differential equation is given by

$y\,(\text{I.F.})=\int Q\,(\text{I.F.})\,dx+c$, where I.F. $=e^{\int P\,dx}$.

**10.** If the linear differential equation is of the form $\frac{dx}{dy}+P\cdot x=Q$, where $P$ and $Q$ are functions of $y$ or constants, then its solution is

$x\cdot(\text{I.F.})=\int Q\cdot(\text{I.F.})\,dy+c$, where I.F. $=e^{\int P\,dy}$.

**11.** The solution of the differential equation in which the number of arbitrary constants is equal to the order of the differential equation, is called the **general solution** of the differential equation.

**12.** The solution obtained from the general solution by giving particular values to the arbitrary constants is called the **particular solution** of the differential equation.

**13.** If $x$ denotes the amount or quantity (which grows or decay) at time $t$, then rate of change of $x$ w.r.t. time $t$ is $\frac{dx}{dt}$.

In case of growth, $x$ increases as $t$ increases.

$\therefore$ $\frac{dx}{dt}$ is positive. Hence, $\frac{dx}{dt}=kx$, where $k>0$.

In case of decay, $x$ decreases as $t$ increases.

$\therefore$ $\frac{dx}{dt}$ is negative. Hence, $\frac{dx}{dt}=-kx$, where $k>0$.

**14.** The solution of the differential equation $\frac{dx}{dt}=kx$ is in the form $x=ae^{kt}$, where $a$ is the initial value of $x$.

## INTRODUCTION

We represent the rate of change of $y$ with respect to $x$ by $\frac{dy}{dx}$.

Let $x$ be the quantity of radioactive disintegrating material at time $t$. It is found that the rate of decay of the material is proportional to the quantity of disintegrating substance.

i.e. $\frac{dx}{dt}\propto x$

i.e. $\frac{dx}{dt}=-kx$, where $k>0$ and is called constant of proportionality (the negative sign indicates rate of decrease).

Here we have a relation between $\frac{dx}{dt}$ and $x$.

This relation is called differential equation.

## 8.1 : DIFFERENTIAL EQUATION

An equation which contains one independent and one or more dependent variables and their derivatives $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, ... or the differentials $dx$, $dy$ is called a **Differential Equation (D.E.)**.

The following are some differential equations :

(1) $\left(\frac{dy}{dx}\right)^2-e^x=0$

(2) $y^2\,dx+x^2\,dy=0$

(3) $x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}-y=0$

(4) $\frac{d^3y}{dx^3}+5\frac{dy}{dx}-y=0.$

### 8.1.1 : Ordinary Differential Equation

A differential equation in which the dependent variable $y$ depends only on one independent variable $x$ is called an **ordinary differential equation.**

In this chapter, we will study only ordinary differential equations.

**Notes :**

1. The following notations are used for derivatives :

   $\frac{dy}{dx}=y'$, $\frac{d^2y}{dx^2}=y''$, $\frac{d^3y}{dx^3}=y'''$, ..., etc.

2. If $y=f(x)$, then $f'(x)$, $f''(x)$, ..., etc. are used for derivatives.

### 8.1.2 : Order of a Differential Equation

The **order** of a differential equation is the order of the highest order derivative occurring in it.

### 8.1.3 : Degree of a Differential Equation

The **degree** of differential equation is the degree of the highest order derivative occurring in it, when the D.E. is so written that the derivatives are free from negative or fractional indices. **OR**

The **degree** of a differential equation is the power of the highest order derivative occurring in a differential equation, when it is written as a polynomial in differential coefficients.

**Remark :** The order and degree of a D.E. are positive integers. Hence, in order to find the order and degree of a given D.E. we have to remove the negative or fractional indices, if they occur in the D.E.

### 8.1.4 : Solution of a Differential Equation

A relation between independent and dependent variable which does not involve differentials is called a **solution** of a differential equation, if this relation and the drivative obtained from it satisfy the differential equation. e.g. consider the differential equation

$$\frac{dy}{dx} = e^x$$

Then $y = e^x + c$, where $c$ is a constant, is the solution of this differential equation.

**Note :** There are two types of solutions for a differential equation :

1. **General solution :** A solution of a differential equation in which the number of arbitrary constants is equal to the order of the differential equation, is called the general solution of the differential equation.
2. **Particular solution :** A solution of a differential equation which can be obtained from the general solution by giving particular values to the arbitrary constants is called the particular solution of the differential equation.

   **For example :** $y = e^x + c$ is the general solution of the differential equation $\frac{dy}{dx} = e^x$ whereas $y = e^x + 5$ is its particular solution.

**EXERCISE 8.1** | **Textbook page 162**

**1. Determine the order and degree of each of the following differential equations :**

(i) $\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + 8 = 0$

(ii) $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = a^x$

(iii) $\frac{d^4y}{dx^4} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3$

(iv) $(y''')^2 + 2(y'')^2 + 6y' + 7y = 0$

(v) $\sqrt{1 + \frac{1}{\left(\frac{dy}{dx}\right)^2}} = \left(\frac{dy}{dx}\right)^{\frac{3}{2}}$

(vi) $\frac{dy}{dx} = 7\frac{d^2y}{dx^2}$

(vii) $\left(\frac{d^3y}{dx^3}\right)^{\frac{1}{6}} = 9.$

**Solution :**

(i) The given D.E. is $\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + 8 = 0$

This D.E. has highest order derivative $\frac{d^2x}{dt^2}$ with power 1.

$\therefore$ the given D.E. is of order **2** and degree **1.**

(ii) The given D.E. is $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = a^x$

This D.E. has highest order derivative $\frac{d^2y}{dx^2}$ with power 2.

$\therefore$ the given D.E. is of order **2** and degree **2.**

(iii) The given D.E. is $\frac{d^4y}{dx^4} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3$.

This D.E. has highest order derivative $\frac{d^4y}{dx^4}$ with power 1.

$\therefore$ the given D.E. is of order **4** and degree **1.**

(iv) The given D.E. is

$(y''')^2 + 2(y'')^2 + 6y' + 7y = 0$

This can be written as :

$\left(\frac{d^3y}{dx^3}\right)^2 + 2\left(\frac{d^2y}{dx^2}\right)^2 + 6\frac{dy}{dx} + 7y = 0$

This D.E. has highest order derivative $\frac{d^3y}{dx^3}$ with power 2.

$\therefore$ the given D.E. is of order **3** and degree **2.**

(v) The given D.E. is

$$\sqrt{1 + \frac{1}{\left(\frac{dy}{dx}\right)^2}} = \left(\frac{dy}{dx}\right)^{\frac{3}{2}}$$

On squaring both sides, we get

$$1+\frac{1}{\left(\frac{dy}{dx}\right)^2}=\left(\frac{dy}{dx}\right)^3$$

$$\therefore \left(\frac{dy}{dx}\right)^2+1=\left(\frac{dy}{dx}\right)^5$$

This D.E. has highest order derivative $\frac{dy}{dx}$ with power 5.

$\therefore$ the given D.E. is of order **1** and degree **5.**

**(vi)** The given D.E. is $\frac{dy}{dx}=7\frac{d^2y}{dx^2}$.

This D.E. has highest order derivative $\frac{d^2y}{dx^2}$ with power 1.

$\therefore$ the given D.E. is of order **2** and degree **1.**

**(vii)** The given D.E. is

$$\left(\frac{d^3y}{dx^3}\right)^{\frac{1}{6}}=9, \text{ i.e. } \frac{d^3y}{dx^3}=9^6$$

This D.E. has highest order derivative $\frac{d^3y}{dx^3}$ with power 1.

$\therefore$ the given D.E. is of order **3** and degree **1.**

**2. In each of the following examples, verify that the given function is a solution of the corresponding differential equation :**

| | Solution | D.E. |
|---|---|---|
| (i) | $xy=\log y+k$ | $y'(1-xy)=y^2$ |
| (ii) | $y=x^n$ | $x^2\frac{d^2y}{dx^2}-nx\frac{dy}{dx}+ny=0$ |
| (iii) | $y=e^x$ | $\frac{dy}{dx}=y$ |
| (iv) | $y=1-\log x$ | $x^2\frac{d^2y}{dx^2}=1$ |
| (v) | $y=ae^x+be^{-x}$ | $\frac{d^2y}{dx^2}=y$ |
| (vi) | $ax^2+by^2=5$ | $xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2=y\cdot\frac{dy}{dx}$ |

**Solution :**

**(i)** $xy=\log y+k$

Differentiating w.r.t. $x$, we get

$$x\cdot\frac{dy}{dx}+y\times 1=\frac{1}{y}\cdot\frac{dy}{dx}+0$$

$$\therefore\ x\frac{dy}{dx}+y=\frac{1}{y}\frac{dy}{dx}$$

$$\left(x-\frac{1}{y}\right)\frac{dy}{dx}=-y$$

$$\therefore\ \left(\frac{xy-1}{y}\right)\frac{dy}{dx}=-y$$

$$\therefore\ \frac{dy}{dx}=\frac{-y^2}{xy-1}=\frac{y^2}{1-xy},\text{ if } xy\neq 1$$

$$\therefore\ y'=\frac{y^2}{1-xy},\text{ if } xy\neq 1.$$

$$\therefore\ y'(1-xy)=y^2$$

Hence, $xy=\log y+k$ is a solution of the D.E. $y'(1-xy)=y^2$.

**(ii)** $y=x^n$

Differentiating twice w.r.t. $x$, we get

$$\frac{dy}{dx}=\frac{d}{dx}(x^n)=nx^{n-1}$$

and $\frac{d^2y}{dx^2}=\frac{d}{dx}(nx^{n-1})=n\frac{d}{dx}(x^{n-1})=n(n-1)x^{n-2}$

$$\therefore\ x^2\frac{d^2y}{dx^2}-nx\frac{dy}{dx}+ny$$

$$=x^2\cdot n(n-1)x^{n-2}-nx\cdot nx^{n-1}+n\cdot x^n$$

$$=n(n-1)x^n-n^2x^n+nx^n$$

$$=(n^2-n-n^2+n)x^n=0$$

This shows that $y=x^n$ is a solution of the D.E. $x^2\frac{d^2y}{dx^2}-nx\frac{dy}{dx}+ny=0$.

**(iii)** $y=e^x$

Differentiating w.r.t. $x$, we get

$$\frac{dy}{dx}=e^x=y$$

Hence, $y=e^x$ is a solution of the D.E. $\frac{dy}{dx}=y$.

**(iv)** $y=1-\log x$

Differentiating w.r.t. $x$, we get

$$\frac{dy}{dx}=\frac{d}{dx}(1)-\frac{d}{dx}(\log x)$$
$$=0-\frac{1}{x}=-\frac{1}{x}$$

Differentiating again w.r.t. $x$, we get

$$\frac{d^2y}{dx^2}=-\frac{d}{dx}(x^{-1})=-(-1)x^{-2}=\frac{1}{x^2}$$

$$\therefore\ x^2\frac{d^2y}{dx^2}=1$$

Hence, $y=1-\log x$ is a solution of the D.E. $x^2\frac{d^2y}{dx^2}=1$.

**(v)** $y=ae^x+be^{-x}$

Differentiating w.r.t. $x$, we get

$$\frac{dy}{dx}=a(e^x)+b(-e^{-x})=ae^x-be^{-x}$$

Differentiating again w.r.t. $x$, we get

$$\frac{d^2y}{dx^2}=a(e^x)-b(-e^{-x})=ae^x+be^{-x}=y$$

Hence, $y=ae^x+be^{-x}$ is a solution of the D.E. $\frac{d^2y}{dx^2}=y$.

**(vi)** $ax^2+by^2=5$

Differentiating w.r.t. $x$, we get

$$a(2x)+b\left(2y\frac{dy}{dx}\right)=0$$

$$\therefore\ ax+by\frac{dy}{dx}=0$$

$$\therefore\ ax=-by\frac{dy}{dx} \qquad \ldots(1)$$

Differentiating again w.r.t. $x$, we get

$$a\cdot 1=-b\left[y\frac{d}{dx}\left(\frac{dy}{dx}\right)+\frac{dy}{dx}\cdot\frac{dy}{dx}\right]$$

$$\therefore\ a=-b\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right] \qquad \ldots(2)$$

Dividing (1) by (2), we get

$$x=\frac{y\frac{dy}{dx}}{y\left(\frac{d^2y}{dx^2}\right)+\left(\frac{dy}{dx}\right)^2}$$

$$\therefore\ xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2=y\frac{dy}{dx}$$

Hence, $ax^2+by^2=5$ is a solution of the D.E.

$$xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2=y\left(\frac{dy}{dx}\right).$$

## EXAMPLES FOR PRACTICE 8.1

**1. Find the order and degree of the following differential equations :**

(1) $\left(\frac{d^2y}{dx^2}\right)^2+\left(\frac{dy}{dx}\right)^3=e^x$

(2) $\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$

(3) $\left(\frac{d^2s}{dt^2}\right)^3+3\left(\frac{ds}{dt}\right)^2+5=0$

(4) $\frac{d^3y}{dx^3}+\left(\frac{dy}{dx}\right)^3+y=e^x$

(5) $\frac{d^3y}{dx^3}+6\frac{d^2y}{dx^2}+y=0$

(6) $x\frac{d^3y}{dx^3}-\left(\frac{d^2y}{dx^2}\right)^2=0$

(7) $\left(y+\frac{dy}{dx}\right)^2+x\frac{dy}{dx}=x^2$

(8) $\frac{d^2y}{dx^2}+\frac{1}{(dy/dx)^2}=y$

(9) $\frac{d^2y}{dx^2}=\sqrt[3]{1-\left(\frac{dy}{dx}\right)^4}$

(10) $y=x\frac{dy}{dx}+5\sqrt{1+\left(\frac{dy}{dx}\right)^4}$

(11) $\frac{d^2y}{dx^2}+\frac{dy}{dx}+x=\sqrt{1+\frac{d^3y}{dx^3}}$

(12) $\frac{dy}{dx}=\frac{2x+3}{\left(\frac{dy}{dx}\right)}$

(13) $\sqrt{y'''+y'}=2$

(14) $\frac{d^2y}{dx^2}=\left(1+\frac{dy}{dx}\right)^{3/2}$

(15) $\frac{d^4y}{dx^4}=\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}$

(16) $y=\frac{dy}{dx}+\left(1+\frac{dy}{dx}\right)^{1/2}$

(17) $\left(\frac{d^3y}{dx^3}+x\right)^{5/2}=\frac{d^2y}{dx^2}$

(18) $\begin{vmatrix}1 & y\\ 1 & x\frac{dy}{dx}+\sqrt{1+\left(\frac{dy}{dx}\right)^2}\end{vmatrix}-y=0.$

2. Show that $y = 2(x^2 - 1) + ce^{-x^2}$ is a solution of the D.E. $\frac{dy}{dx} + 2xy = 4x^3$.

3. Verify that $y = a + \frac{b}{x}$ is a solution of $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$.

4. Verify that $y = ae^{-bx}$ is a solution of $\frac{d^2y}{dx^2} = \frac{1}{y}\left(\frac{dy}{dx}\right)^2$.

5. Show that $x^2 + y^2 = r^2$ is a solution of the D.E. $y = x\frac{dy}{dx} + r.\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$.

6. Verify that $y = e^{-x} + Ax + B$ is a solution of the D.E. $e^x\left(\frac{d^2y}{dx^2}\right) = 1$.

7. Verify that $y = ae^x + be^{-2x}$, $a, b \in R$, is a solution of the D.E. $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 2y$.

8. If $y = e^{ax}$, show that $x\frac{dy}{dx} = y \log y$.

### Answers

1. (1) order = 2, degree = 2 (2) order = 2, degree = 1
(3) order = 2, degree = 3 (4) order = 3, degree = 1
(5) order = 3, degree = 1 (6) order = 3, degree = 1
(7) order = 1, degree = 2 (8) order = 2, degree = 1
(9) order = 2, degree = 3 (10) order = 1, degree = 4
(11) order = 3, degree = 1 (12) order = 1, degree = 2
(13) order = 3, degree = 1 (14) order = 2, degree = 2
(15) order = 4, degree = 2 (16) order = 1, degree = 2
(17) order = 3, degree = 5 (18) order = 1, degree = 2.

### 8.1.5 : Formation of a Differential Equation

If a relation between variables $x$ and $y$ containing a number of arbitrary constants is given, then we can form a differential equation from it by differentiating the given relation enough number of times so as to eliminate all the arbitrary constants.

If an equation contains only one arbitrary constant, we differentiate it once. Thus we get two equations. From these two equations, we eliminate the arbitrary constant.

If an equation contains two arbitrary constants, we differentiate it twice. Thus we get three equations. From these three equations, we eliminate the two arbitrary constants.

Hence to eliminate $n$ arbitrary constants, we have to differentiate $n$ times successively and the final equation will be of the $n^{\text{th}}$ order.

## EXERCISE 8.2 | Textbook page 163

**1. Obtain the differential equation by eliminating arbitrary constants from the following equations :**

**(i)** $y = Ae^{3x} + Be^{-3x}$

**(ii)** $y = c_2 + \frac{c_1}{x}$

**(iii)** $y = (c_1 + c_2x)e^x$

**(iv)** $y = c_1e^{3x} + c_2e^{2x}$

**(v)** $y^2 = (x + c)^3$.

**Solution :**

(i) $y = Ae^{3x} + Be^{-3x}$ ... (1)

Differentiating twice w.r.t. $x$, we get

$\frac{dy}{dx} = Ae^{3x} \times 3 + Be^{-3x} \times (-3)$

$\therefore \frac{dy}{dx} = 3Ae^{3x} - 3Be^{-3x}$

and $\frac{d^2y}{dx^2} = 3Ae^{3x} \times 3 - 3Be^{-3x} \times (-3)$

$= 9Ae^{3x} + 9Be^{-3x}$

$= 9(Ae^{3x} + Be^{-3x}) = 9y$ ... [By (1)]

$\therefore \frac{d^2y}{dx^2} = 9y$

This is the required D.E.

(ii) $y = c_2 + \frac{c_1}{x}$

$\therefore xy = c_2x + c_1$

Differentiating w.r.t. $x$, we get

$x\frac{dy}{dx} + y \cdot 1 = c_2 \cdot 1 + 0 = c_2$

$\therefore x\frac{dy}{dx} + y = c_2$

Differentiating again w.r.t. $x$, we get

$x \cdot \frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx} \cdot \frac{d}{dx}(x) + \frac{dy}{dx} = 0$

$\therefore x\frac{d^2y}{dx^2} + \frac{dy}{dx} \times 1 + \frac{dy}{dx} = 0$

$\therefore x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$ is the required D.E.

(iii) $y = (c_1 + c_2x)e^x$

$\therefore e^{-x}y = c_1 + c_2x$

Differentiating w.r.t. $x$, we get

$e^{-x} \cdot \frac{dy}{dx} + y \cdot e^{-x}(-1) = 0 + c_2 \times 1$

$\therefore\ e^{-x}\left(\frac{dy}{dx}-y\right)=c_2$

Differentiating again w.r.t. $x$, we get

$$e^{-x}\left(\frac{d^2y}{dx^2}-\frac{dy}{dx}\right)+\left(\frac{dy}{dx}-y\right)\cdot e^{-x}(-1)=0$$

$$\therefore\ e^{-x}\left(\frac{d^2y}{dx^2}-\frac{dy}{dx}-\frac{dy}{dx}+y\right)=0$$

$$\therefore\ \frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=0$$

This is the required D.E.

**(iv)** $y=c_1e^{3x}+c_2e^{2x}$ ... (1)

Differentiating twice w.r.t. $x$, we get

$$\frac{dy}{dx}=3c_1e^{3x}+2c_2e^{2x} \quad \text{... (2)}$$

$$\frac{d^2y}{dx^2}=9c_1e^{3x}+4c_2e^{2x} \quad \text{... (3)}$$

These three equations in $c_1e^{3x}$ and $c_2e^{2x}$ are consistent.

$\therefore$ determinant of their consistency condition is zero.

$$\therefore\ \begin{vmatrix} y & 1 & 1 \\ \frac{dy}{dx} & 3 & 2 \\ \frac{d^2y}{dx^2} & 9 & 4 \end{vmatrix}=0$$

$$\therefore\ y(12-18)-1\left(4\frac{dy}{dx}-2\frac{d^2y}{dx^2}\right)+1\left(9\frac{dy}{dx}-3\frac{d^2y}{dx^2}\right)=0$$

$$\therefore\ -6y-4\frac{dy}{dx}+2\frac{d^2y}{dx^2}+9\frac{dy}{dx}-3\frac{d^2y}{dx^2}=0$$

$$\therefore\ \frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0$$

This is the required D.E.

*Alternative Method :*

$y=c_1e^{3x}+c_2e^{2x}$

Dividing both sides by $e^{2x}$, we get

$e^{-2x}y=c_1e^{x}+c_2$

Differentiating w.r.t. $x$, we get

$$e^{-2x}\frac{dy}{dx}+y\cdot e^{-2x}(-2)=c_1e^{x}+0$$

$$\therefore\ e^{-2x}\left(\frac{dy}{dx}-2y\right)=c_1e^{x}$$

Dividing both sides by $e^{x}$, we get

$$e^{-3x}\left(\frac{dy}{dx}-2y\right)=c_1$$

Differentiating w.r.t. $x$, we get

$$e^{-3x}\left(\frac{d^2y}{dx^2}-2\frac{dy}{dx}\right)+\left(\frac{dy}{dx}-2y\right)\cdot e^{-3x}(-3)=0$$

$$\therefore\ e^{-3x}\left(\frac{d^2y}{dx^2}-2\frac{dy}{dx}-3\frac{dy}{dx}+6y\right)=0$$

$$\therefore\ \frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0$$

This is the required D.E.

**(v)** $y^2=(x+c)^3$ ... (1)

Differentiating w.r.t. $x$, we get

$$2y\frac{dy}{dx}=3(x+c)^2\cdot(1)=3(x+c)^2$$

$$\therefore\ (x+c)^2=\frac{2y}{3}\cdot\frac{dy}{dx}$$

$$\therefore\ (x+c)^6=\left(\frac{2y}{3}\cdot\frac{dy}{dx}\right)^3$$

$$\therefore\ (y^2)^2=\frac{8y^3}{27}\cdot\left(\frac{dy}{dx}\right)^3 \quad \text{... [By (1)]}$$

$$\therefore\ 27y^4=8y^3\left(\frac{dy}{dx}\right)^3$$

$$\therefore\ 27y=8\left(\frac{dy}{dx}\right)^3$$

$$\therefore\ \left(\frac{dy}{dx}\right)^3=\frac{27y}{8}$$

$$\therefore\ \frac{dy}{dx}=\frac{3}{2}(\sqrt[3]{y})$$

This is the required D.E.

**2. Find the differential equation by eliminating arbitrary constant from the relation $x^2+y^2=2ax$.**

**Solution :** $x^2+y^2=2ax$ ... (1)

Differentiating both sides w.r.t. $x$, we get

$$2x+2y\frac{dy}{dx}=2a$$

Substituting value of $2a$ in equation (1), we get

$$x^2+y^2=\left[2x+2y\frac{dy}{dx}\right]x$$

$$=2x^2+2xy\frac{dy}{dx}$$

$\therefore\ 2xy\frac{dy}{dx}=y^2-x^2$ is the required D.E.

**3. Form the differential equation by eliminating arbitrary constants from the relation $bx+ay=ab$.**

**Solution :** $bx+ay=ab$

$\therefore\ ay=-bx+ab$

$$\therefore\ y=-\frac{b}{a}x+b$$

Differentiating w.r.t. $x$, we get

$\frac{dy}{dx} = -\frac{b}{a} \times 1 + 0 = -\frac{b}{a}$

Differentiating again w.r.t. $x$, we get

$\frac{d^2y}{dx^2} = 0$ is the required D.E.

**4. Find the differential equation whose general solution is $x^3 + y^3 = 35\,ax$.**

**Solution :** *Refer to the solution of Q. 2.*

**Ans.** $2x^3 - y^3 + 3xy^2 \frac{dy}{dx} = 0$.

**5. Form the differential equation from the relation $x^2 + 4y^2 = 4b^2$.**

**Solution :** $x^2 + 4y^2 = 4b^2$

Differentiating w.r.t. $x$, we get

$2x + 4\left(2y\frac{dy}{dx}\right) = 0$

i.e. $x + 4y\frac{dy}{dx} = 0$ is the required D.E.

## EXAMPLES FOR PRACTICE 8.2

**1. Form the differential equations by eliminating the arbitrary constants from each of the following relations :**

(1) $y^2 = 4ax$ (2) $y = a + \frac{a}{x}$

(3) $y = x + 2ax^2$ (4) $x^3 + y^3 = 4ax$

(5) $x^2 + cy^2 = 4$ (6) $(x - a)^2 + y^2 = 1$

(7) $y = cx^2 + c^3$ (8) $y = Ax^2 + Bx$

(9) $y = Ae^{5x} + Be^{-5x}$ (10) $y = c_1e^{2x} + c_2e^{-2x}$

(11) $xy = ae^{3x} + be^{-3x}$ (12) $xy = Ae^x + Be^{-x} + x^2$

(13) $y = Ae^{3x} + Be^{-2x}$ (14) $y = Ae^{2x} + Be^x$

(15) $Ax^3 + By^2 = 5$ (16) $y = ax + \frac{b}{x}$

(17) $ax^2 + by^2 = 7$ (18) $(y - a)^2 = 4(x - b)$.

2. Form the differential equation of the family of lines having slope $m$ and $y$-intercept 5, where $m$ is an arbitrary constant.

3. Form the differential equation of the family of lines having $x$-intercept $a$ and $y$-intercept $b$.

### Answers

**1.** (1) $y = 2x\frac{dy}{dx}$ (2) $x(x+1)\frac{dy}{dx} + y = 0$

(3) $2y = x\left(1 + \frac{dy}{dx}\right)$ (4) $2x^3 - y^3 + 3xy^2\frac{dy}{dx} = 0$

(5) $(x^2 - 4)\frac{dy}{dx} = xy$ (6) $y^2\left[\left(\frac{dy}{dx}\right)^2 + 1\right] = 1$

(7) $\left(\frac{dy}{dx}\right)^3 + 4x^4\frac{dy}{dx} = 8x^3y$

(8) $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$

(9) $\frac{d^2y}{dx^2} - 25y = 0$ (10) $\frac{d^2y}{dx^2} - 4y = 0$

(11) $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 9xy$

(12) $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy + x^2 - 2 = 0$

(13) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$

(14) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$

(15) $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - 2y\frac{dy}{dx} = 0$

(16) $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$

(17) $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$

(18) $2\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$.

**2.** $y = x\frac{dy}{dx} + 5$ [**Hint :** Equation of the line is $y = mx + 5$]

**3.** $\frac{d^2y}{dx^2} = 0$.

## 8.2 : SOLUTION OF A DIFFERENTIAL EQUATION

In this section we shall study the method of solving the differential equation of first order and first degree.

### 8.2.1 : Variable Separable Method

A D.E. of the first order and first degree is of the type $f_1(x, y) + f_2(x, y)\frac{dy}{dx} = 0$, where $f_1(x, y)$ and $f_2(x, y)$ are functions of $x$ and $y$. For convenience, this equation is also written as $f_1(x, y)\,dx + f_2(x, y)\,dy = 0$.

We are going to deal with differential equations of the type $f_1(x)\,dx + f_2(y)\,dy = 0$ or which can be put in this form. Such differential equations are said to be in the *variable separable form*. The general solution of such a D.E. is obtained by directly integrating it. Thus its general solution is $\int f_1(x)\,dx + \int f_2(y)\,dy = c$.

**EXERCISE 8.3** | **Textbook page 165**

**1. Solve the following differential equations :**

**(i)** $\frac{dy}{dx} = x^2y + y$

**Solution :** $\frac{dy}{dx} = x^2y + y$

$\therefore \frac{dy}{dx} = y(x^2+1) \qquad \therefore \frac{1}{y}dy = (x^2+1)dx$

Integrating, we get

$$\int \frac{1}{y}dy = \int (x^2+1)dx$$

$\therefore \log|y| = \frac{x^3}{3} + x + c$

This is the general solution.

**(ii)** $\frac{d\theta}{dt} = -k(\theta - \theta_0)$

**Solution :** $\frac{d\theta}{dt} = -k(\theta - \theta_0)$

$\therefore \frac{1}{\theta - \theta_0}d\theta = -k\,dt$

Integrating, we get

$$\int \frac{1}{\theta - \theta_0}d\theta = -k\int dt$$

$\therefore \log|\theta - \theta_0| = -kt + \log c_1$

$\therefore \log|\theta - \theta_0| - \log c_1 = -kt$

$\therefore \log\left|\frac{\theta - \theta_0}{c_1}\right| = -kt$

$\therefore \frac{\theta - \theta_0}{c_1} = e^{-kt}$

$\therefore \theta - \theta_0 = c_1 e^{-kt}$

$\therefore \theta - \theta_0 = e^c \cdot e^{-kt}$, where $c_1 = e^c$

$\therefore \theta - \theta_0 = e^{-kt+c}$

This is the general solution.

**(iii)** $(x^2 - yx^2)dy + (y^2 + xy^2)dx = 0$

**Solution :** $(x^2 - yx^2)dy + (y^2 + xy^2)dx = 0$

$\therefore x^2(1-y)dy + y^2(1+x)dx = 0$

$\therefore \frac{1-y}{y^2}dy + \frac{1+x}{x^2}dx = 0$

Integrating, we get

$$\int \frac{1-y}{y^2}dy + \int \frac{1+x}{x^2}dx = c$$

$\therefore \int \left(\frac{1}{y^2} - \frac{1}{y}\right)dy + \int \left(\frac{1}{x^2} + \frac{1}{x}\right)dx = c$

$\therefore \int y^{-2}dy - \int \frac{1}{y}dy + \int x^{-2}dx + \int \frac{1}{x}dx = c$

$\therefore \frac{y^{-1}}{-1} - \log|y| + \frac{x^{-1}}{-1} + \log|x| = c$

$\therefore -\frac{1}{y} - \log|y| - \frac{1}{x} + \log|x| = c$

$\therefore \log|x| - \log|y| = \frac{1}{x} + \frac{1}{y} + c$

This is the general solution.

**(iv)** $y^3 - \frac{dy}{dx} = x\frac{dy}{dx}$

**Solution :** $y^3 - \frac{dy}{dx} = x\frac{dy}{dx}$

$\therefore y^3 = x\frac{dy}{dx} + \frac{dy}{dx} = (x+1)\frac{dy}{dx}$

$\therefore \frac{dx}{x+1} = \frac{dy}{y^3}$

Integrating, we get

$$\int \frac{dx}{x+1} = \int y^{-3}dy$$

$\therefore \log|x+1| = \frac{y^{-2}}{(-2)} + c = \frac{-1}{2y^2} + c$

$\therefore 2y^2\log|x+1| = 2cy^2 - 1$ is the required solution.

**2. For each of the following differential equations find the particular solution :**

**(i)** $(x - y^2x)dx - (y + x^2y)dy = 0$**, when** $x = 2$, $y = 0$.

**Solution :** $(x - y^2x)dx - (y + x^2y)dy = 0$

$\therefore x(1-y^2)dx - y(1+x^2)dy = 0$

$\therefore \frac{x}{1+x^2}dx - \frac{y}{1-y^2}dy = 0$

$\therefore \frac{2x}{1+x^2} - \frac{2y}{1-y^2}dy = 0$

Integrating, we get

$$\int \frac{2x}{1+x^2}dx + \int \frac{-2y}{1-y^2}dy = c_1$$

Each of these integrals is of the type

$$\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + c$$

$\therefore$ the general solution is

$\log|1+x^2| + \log|1-y^2| = \log c$, where $c_1 = \log c$

$\therefore \log|(1+x^2)(1-y^2)| = \log c$

$\therefore (1+x^2)(1-y^2) = c$

When $x=2$, $y=0$, we have

$(1+4)(1-0)=c \qquad \therefore\ c=5$

$\therefore$ the particular solution is $(1+x^2)(1-y^2)=5$.

**(ii)** $(x+1)\dfrac{dy}{dx}-1=2e^{-y}$**, when** $y=0$, $x=1$.

**Solution :** $(x+1)\dfrac{dy}{dx}-1=2e^{-y}$

$\therefore\ (x+1)\dfrac{dy}{dx}=\dfrac{2}{e^y}+1=\dfrac{2+e^y}{e^y}$

$\therefore\ \dfrac{e^y}{2+e^y}dy=\dfrac{1}{x+1}dx$

Integrating, we get

$\displaystyle\int\frac{e^y}{2+e^y}dy=\int\frac{1}{x+1}dx$

$\therefore\ \log|2+e^y|=\log|x+1|+\log c$

$\quad \ldots \left[\because\ \dfrac{d}{dy}(2+e^y)=e^y \text{ and } \displaystyle\int\frac{f'(y)}{f(y)}dy=\log|f(y)|+c\right]$

$\therefore\ \log|2+e^y|=\log|c(x+1)|$

$\therefore\ 2+e^y=c(x+1)$

This is the general solution.

Now, $y=0$, when $x=1$

$\therefore\ 2+e^0=c(1+1)$

$\therefore\ 3=2c \qquad \therefore\ c=3/2$

$\therefore$ the particular solution is

$2+e^y=\dfrac{3}{2}(x+1)$

$\therefore\ 4+2e^y=3x+3$

$\therefore\ 3x-2e^y-1=0$

**(iii)** $y(1+\log x)\dfrac{dx}{dy}-x\log x=0$**, when** $x=e$, $y=e^2$.

**Solution :** $y(1+\log x)\dfrac{dx}{dy}-x\log x=0$

$\therefore\ \dfrac{1+\log x}{x\log x}dx-\dfrac{dy}{y}=0$

$\therefore\ \displaystyle\int\frac{1+\log x}{x\log x}dx-\int\frac{dy}{y}=c_1 \qquad \ldots (1)$

Put $x\log x=t$.

Then $\left[x\cdot\dfrac{d}{dx}(\log x)+(\log x)\cdot\dfrac{d}{dx}(x)\right]dx=dt$

$\therefore\ \left[\dfrac{x}{x}+(\log x)(1)\right]dx=dt \quad \therefore\ (1+\log x)\,dx=dt$

$\therefore\ \displaystyle\int\frac{1+\log x}{x\log x}dx=\int\frac{dt}{t}=\log|t|=\log|x\log x|$

$\therefore$ from (1), the general solution is

$\log|x\log x|-\log|y|=\log c$, where $c_1=\log c$

$\therefore\ \log\left|\dfrac{x\log x}{y}\right|=\log c \quad \therefore\ \dfrac{x\log x}{y}=c$

$\therefore\ x\log x=cy$

This is the general solution.

Now, $y=e^2$, when $x=e$

$\therefore\ e\log e=c\cdot e^2 \qquad 1=c\cdot e \qquad \ldots [\because\ \log e=1]$

$\therefore\ c=\dfrac{1}{e}$

$\therefore$ the particular solution is $x\log x=\left(\dfrac{1}{e}\right)y$

$\therefore\ y=ex\log x$.

**METHOD OF SUBSTITUTION**

Some differential equations, which are not in the variable separable form, can be reduced to this form by appropriate substitutions. In the question papers, substitutions are expected to be given. The following example will make the procedure clear.

**(iv)** $\dfrac{dy}{dx}=4x+y+1$**, when** $y=1$, $x=0$.

**Solution :** $\dfrac{dy}{dx}=4x+y+1$

Put $4x+y+1=v$

$\therefore\ 4+\dfrac{dy}{dx}=\dfrac{dv}{dx}$

$\therefore\ \dfrac{dy}{dx}=\dfrac{dv}{dx}-4$

$\therefore$ the given D.E. becomes

$\dfrac{dv}{dx}-4=v \qquad \therefore\ \dfrac{dv}{dx}=4+v$

$\therefore\ \dfrac{dv}{v+4}=dx$

Integrating we get

$\displaystyle\int\frac{dv}{v+4}=\int dx$

$\therefore\ \log|v+4|=x+c$

$\therefore\ \log|4x+y+1+4|=x+c$

i.e. $\log|4x+y+5|=x+c$

This is the general solution.

Now, $y=1$ when $x=0$

$\therefore\ \log|0+1+5|=0+c$, i.e. $c=\log 6$

$\therefore$ the particular solution is

$\log|4x+y+5|=x+\log 6$

$\therefore \log\left|\frac{4x+y+5}{6}\right| = x$

[*Note : Answer in the textbook is incorrect.*]

## ADDITIONAL SOLVED PROBLEMS–8(A)

**Solve the following differential equations :**

**1. (i)** $\log\left(\frac{dy}{dx}\right) = 2x + 3y$

**(ii)** $\frac{dy}{dx} + x^2 = x^2 e^{3y}$

**(iii)** $\frac{dy}{dx} = e^{x+y} + x^2 e^y$.

**Solution :** (i) $\log\left(\frac{dy}{dx}\right) = 2x + 3y \quad \therefore \frac{dy}{dx} = e^{2x+3y}$

$\therefore \frac{dy}{dx} = e^{2x} \cdot e^{3y}$

$\therefore e^{-3y} dy = e^{2x} dx$

$\therefore \int e^{2x}\, dx - \int e^{-3y}\, dy = c_1$

$\therefore \frac{e^{2x}}{2} + \frac{e^{-3y}}{3} = c_1$

$\therefore 3e^{2x} + 2e^{-3y} = 6c_1$

$\therefore 3e^{2x} + 2e^{-3y} = c$, where $c = 6c_1$

This is the general solution.

(ii) $\frac{dy}{dx} + x^2 = x^2 e^{3y}$

$\therefore \frac{dy}{dx} = x^2(e^{3y} - 1) \quad \therefore \frac{dy}{e^{3y} - 1} = x^2 dx$

$\therefore \frac{e^{-3y}}{1 - e^{-3y}}\, dy = x^2\, dx$

$\therefore \int x^2 dx = \int \frac{e^{-3y}}{1 - e^{-3y}}\, dy \quad \ldots (1)$

Put $1 - e^{-3y} = t$. Then $e^{-3y}\, dy = \frac{dt}{3}$

$\therefore$ from (1), the general solution is

$\frac{x^3}{3} = \frac{1}{3}\int \frac{dt}{t} = \frac{1}{3}\log|t| + c_1$

$\therefore \frac{x^3}{3} = \frac{1}{3}\log|1 - e^{-3y}| + c_1$

$\therefore x^3 = \log|1 - e^{-3y}| + 3c_1$

$\therefore x^3 = \log|1 - e^{-3y}| + c$, where $3c_1 = c$

(iii) $\frac{dy}{dx} = e^{x+y} + x^2 e^y = e^x \cdot e^y + x^2 e^y$

$\therefore \frac{dy}{dx} = e^y (e^x + x^2)$

$\therefore \frac{1}{e^y}\, dy = (e^x + x^2)\, dx$

Integrating, we get

$\int e^{-y}\, dy = \int (e^x + x^2)\, dx$

$\therefore \frac{e^{-y}}{-1} = e^x + \frac{x^3}{3} + c$

$\therefore \frac{-1}{e^y} = e^x + \frac{x^3}{3} + c$

$\therefore \frac{1}{e^y} + e^x + \frac{x^3}{3} + c = 0$

This is the general solution.

**2.** $\frac{dy}{dx} = x\sqrt{25 - x^2}$.

**Solution :** $\frac{dy}{dx} = x\sqrt{25 - x^2}$

$\therefore dy = x\sqrt{25 - x^2}\, dx$

Integrating, we get

$\int dy = \int x\sqrt{25 - x^2}\, dx + c_1$

In RHS, put $25 - x^2 = t \quad \therefore -2x\, dx = dt$

$\therefore x\, dx = \frac{-dt}{2}$

$\therefore \int dy = \int \sqrt{t} \cdot \frac{(-dt)}{2} + c_1$

$\therefore \int dy = -\frac{1}{2}\int t^{\frac{1}{2}}\, dt + c_1$

$\therefore y = -\frac{1}{2} \cdot \frac{t^{\frac{3}{2}}}{3/2} + c_1$

$\therefore y = -\frac{1}{3}(25 - x^2)^{\frac{3}{2}} + c_1$

$\therefore 3y = -(25 - x^2)^{\frac{3}{2}} + 3c_1$

$\therefore 3y + (25 - x^2)^{\frac{3}{2}} = c$, where $c = 3c_1$

This is the general solution.

**3.** $2e^{x+2y}\, dx - 3dy = 0$.

**Solution :** $2e^{x+2y}\, dx - 3dy = 0$

$\therefore 2e^x \cdot e^{2y}\, dx - 3dy = 0$

$\therefore 2e^x dx - \frac{3}{e^{2y}}\, dy = 0$

Integrating, we get

$2\int e^x dx - 3\int e^{-2y} dy = c_1$

$\therefore\ 2e^x - 3\cdot\frac{e^{-2y}}{(-2)} = c_1$

$\therefore\ 4e^x + 3e^{-2y} = 2c_1$

$\therefore\ 4e^x + 3e^{-2y} = c$, where $c = 2c_1$

This is the general solution.

**4.** $\frac{dx}{dt} = \frac{x\log x}{t}$.

**Solution :** $\frac{dx}{dt} = \frac{x\log x}{t}$

$\therefore\ \frac{1}{x\log x}dx = \frac{1}{t}dt$

Integrating, we get

$\int \frac{(1/x)}{\log x}dx = \int \frac{1}{t}dt$

$\therefore\ \log|\log x| = \log|t| + \log c$

$\ldots \left[\because\ \frac{d}{dx}(\log x) = \frac{1}{x} \text{ and } \int\frac{f'(x)}{f(x)}dx = \log|f(x)| + c\right]$

$\therefore\ \log|\log x| = \log|ct|$

$\therefore\ \log x = ct$

$\therefore\ x = e^{ct}$

This is the general solution.

**5.** $(x-y)^2\frac{dy}{dx} = a^2$.

**Solution :** $(x-y)^2\frac{dy}{dx} = a^2$ ... (1)

Put $x - y = u$ $\therefore\ x - u = y$ $\therefore\ 1 - \frac{du}{dx} = \frac{dy}{dx}$

$\therefore$ (1) becomes, $u^2\left(1 - \frac{du}{dx}\right) = a^2$

$\therefore\ u^2 - u^2\frac{du}{dx} = a^2$

$\therefore\ u^2 - a^2 = u^2\frac{du}{dx}$ $\therefore\ dx = \frac{u^2}{u^2 - a^2}du$

Integrating, we get

$\int dx = \int \frac{(u^2 - a^2) + a^2}{u^2 - a^2}du$

$\therefore\ x = \int 1\,du + a^2\int\frac{du}{u^2 - a^2}$

$= u + a^2\cdot\frac{1}{2a}\log\left|\frac{u-a}{u+a}\right| + c_1$

$\therefore\ x = x - y + \frac{a}{2}\log\left|\frac{x-y-a}{x-y+a}\right| + c_1$

$\therefore\ -c_1 + y = \frac{a}{2}\log\left|\frac{x-y-a}{x-y+a}\right|$

$\therefore\ -2c_1 + 2y = a\log\left|\frac{x-y-a}{x-y+a}\right|$

$\therefore\ c + 2y = a\log\left|\frac{x-y-a}{x-y+a}\right|$, where $c = -2c_1$

This is the general solution.

**6. Solve** $y\left(\frac{dy}{dx} - y\right) = x(x-1)$ **by substituting** $x^2 + y^2 = u$.

**Solution :** $y\left(\frac{dy}{dx} - y\right) = x(x-1)$

$\therefore\ y\frac{dy}{dx} - y^2 = x^2 - x$

$x + y\frac{dy}{dx} = x^2 + y^2$ ... (1)

Put $x^2 + y^2 = u$ $\therefore\ 2x + 2y\frac{dy}{dx} = \frac{du}{dx}$

$\therefore\ x + y\frac{dy}{dx} = \frac{1}{2}\cdot\frac{du}{dx}$ ... (2)

From (1) and (2), we have

$\frac{1}{2}\cdot\frac{du}{dx} = u$ $\therefore\ \frac{du}{u} = 2dx$

On integrating, we get

$\int\frac{1}{u}du = 2\int dx$

$\therefore\ \log u = 2x + c$ $\therefore\ \log(x^2 + y^2) = 2x + c$

This is the general solution.

**7.** $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$.

**Solution :** $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$

$\therefore\ y - x\frac{dy}{dx} = ay^2 + a\frac{dy}{dx}$ $\therefore\ y - ay^2 = x\frac{dy}{dx} + a\frac{dy}{dx}$

$\therefore\ y(1 - ay) = (x + a)\frac{dy}{dx}$ $\therefore\ \frac{dx}{x+a} = \frac{dy}{y(1-ay)}$

$\therefore\ \int\frac{dx}{x+a} - \int\frac{dy}{y(1-ay)} = c_1$ ... (1)

Now, $\int\frac{dy}{y(1-ay)} = \int\frac{(1-ay) + ay}{y(1-ay)}dy = \int\left(\frac{1}{y} + \frac{a}{1-ay}\right)dy$

$= \log|y| + a\frac{\log|1-ay|}{-a}$

$= \log|y| - \log|1-ay| = \log\left|\frac{y}{1-ay}\right|$

$\therefore$ (1) gives, $\log|x+a| - \log\left|\frac{y}{1-ay}\right| = \log c$,

where $c_1 = \log c$

$\therefore \log\left|\frac{(x+a)(1-ay)}{y}\right| = \log c$

$\therefore \frac{(x+a)(1-ay)}{y} = c \quad \therefore (x+a)(1-ay) = cy$

This is the general solution.

**8. Find the particular solution of the differential equation $\log\frac{dy}{dx} = 3x+4y$, given that $y=0$, when $x=0$.**

**Solution :** $\log\frac{dy}{dx} = 3x+4y$

$\therefore \frac{dy}{dx} = e^{3x+4y} = e^{3x} \cdot e^{4y}$

$\therefore \frac{1}{e^{4y}} dy = e^{3x} dx$

On integrating, we get

$\int e^{-4y} dy = \int e^{3x} dx$

$\therefore \frac{e^{-4y}}{-4} = \frac{e^{3x}}{3} + c_1$

$\therefore -3e^{-4y} = 4e^{3x} + 12c_1$

$\therefore 4e^{3x} + 3e^{-4y} = c$, where $c = -12c_1$

This is the general solution.

Now, $y=0$, when $x=0$.

$\therefore 4e^0 + 3e^0 = c \quad \therefore c = 7$

$\therefore$ the particular solution is $4e^{3x} + 3e^{-4y} = 7$

$\therefore 4e^{3x} + 3e^{-4y} - 7 = 0$.

## EXAMPLES FOR PRACTICE 8.3

**1. Solve the following differential equations :**

(1) $xe^{-y} dx + y\,dy = 0$

(2) $y - x\frac{dy}{dx} = 3\left(1 + x^2\frac{dy}{dx}\right)$

(3) $\frac{dy}{dx} = 4^{x+y}$

(4) $\frac{dy}{dx} = e^{x-y} + x^2e^{-y}$

(5) $x(1+y^2)\,dx + y(1+x^2)\,dy = 0$

(6) $y\sqrt{1-x^2}\,dy + x\sqrt{1-y^2}\,dx = 0$

(7) $3e^x\,dx + (1+e^x)\,dy = 0$

(8) $y\frac{dy}{dx} = \frac{x}{e^y}$.

**2. Solve the following differential equations and find its particular solution :**

(1) $x\,dx + y\,dy = 0$, when $x=3$, $y=4$

(2) $\frac{dy}{dx} + xy = xy^2$, when $y=4$, $x=1$

(3) $\frac{dy}{dx} = 3^{x+y}$, if $x=y=0$

(4) $\frac{dx}{x+2} + \frac{dy}{y+2} = 0$, when $x=1$, $y=2$

(5) $x\,dy = y(1-y)\,dx$, if $y=2$, when $x=-4$.

**3. Solve the following differential equations using the substitution shown against them :**

(1) $(x+y)\frac{dy}{dx} + y = 0$, $x+y=u$

(2) $(x-y)\left(1-\frac{dy}{dx}\right) = e^x$, $x-y=u$

(3) $x + y\frac{dy}{dx} = x^2+y^2$, $x^2+y^2=u$.

**4. Solve the following differential equations using the substitution shown against them :**

(1) $(2x-2y+5)\frac{dy}{dx} = x-y+3$, $x-y=u$

(2) $(x+2y+1)\,dx - (2x+4y+3)\,dy = 0$, $x+2y=u$

(3) $\frac{dy}{dx} = \frac{6x-4y+3}{3x-2y+1}$, $3x-2y=u$

5. Solve : $(2x-2y+3)\,dx - (x-y+1)\,dy = 0$.

Also, find its particular solution when $x=0$ and $y=1$.

### Answers

**1.** (1) $\frac{x^2}{2} + (y-1)e^y = c$ (2) $(y-3)(3x+1) = cx$

(3) $4^x + 4^{-y} = c$ (4) $e^y = e^x + \frac{x^3}{3} + c$

(5) $(1+x^2)(1+y^2) = c$ (6) $\sqrt{1-x^2} + \sqrt{1-y^2} = c$

(7) $3\log|1+e^x| + y = c$ (8) $e^y(y-1) = \frac{x^2}{2} + c$.

**2.** (1) $x^2+y^2 = 25$

(2) $\frac{x^2}{2} + \log\left|\frac{y}{y-1}\right| = \frac{1}{2} + \log\left(\frac{4}{3}\right)$

(3) $3^x + 3^{-y} = 2$

(4) $xy + 2(x+y) = 8$

(5) $2y = x(1-y)$.

**3.** (1) $y(2x+y) = c$

(2) $(x-y)^2 = 2e^x + c$

(3) $\log(x^2+y^2) = 2x + c$.

4. (1) $x-2y+\log|x-y+2|=c$

(2) $\log|4x+8y+5|=4x-8y+c$

(3) $4x-2y-2\log|3x-2y+3|=c$

5. $(2x-y)-\log|(x-y+2)|=c$,

$(2x-y)-\log|(x-y+2)|+1=0$.

## 8.3 : HOMOGENEOUS DIFFERENTIAL EQUATION

### Homogeneous Function :

A function $f(x, y)$ is said to be homogeneous function of degree $n$ in $x$ and $y$, if $f(\lambda x, \lambda y)=\lambda^n f(x, y)$.

**For example :**

1. If $f(x, y)=x^4+3x^3y+2x^2y^2+y^4$, then

$$f(\lambda x, \lambda y)=\lambda^4x^4+3\lambda^3x^3\cdot\lambda y+2\lambda^2x^2\cdot\lambda^2y^2+\lambda^4y^4$$
$$=\lambda^4(x^4+3x^3y+2x^2y^2+y^4)$$
$$=\lambda^4 f(x, y).$$

$\therefore$ $f(x, y)$ is a homogeneous function of degree 4.

2. If $h(x, y)=x^3+3xy$, then

$h(\lambda x, \lambda y)=\lambda^3x^3+3\lambda x\cdot\lambda y=\lambda^3x^3+3\lambda^2xy\neq\lambda^n h(x, y)$ for any $n$

$\therefore$ $h(x, y)$ is not a homogeneous function.

### Solution of a Homogeneous Differential Equation :

A first order, differential equation of the form $f_1(x, y)\,dx+f_2(x, y)\,dy=0$ is said to be a *homogeneous differential equation,* if $f_1(x, y)$ and $f_2(x, y)$ are homogeneous functions of the same degree in $x$ and $y$.

This homogeneous differential equation can be put in the form

$$f_1(x, y)+f_2(x, y)\frac{dy}{dx}=0 \quad\text{or}\quad \frac{dy}{dx}=-\frac{f_1(x, y)}{f_2(x, y)}.$$

Such an equation can be reduced to the variables separable form by the substitution $y=vx$.

**Note :** If the differential equation is of the form $\frac{dx}{dy}=F(x, y)$, where $F(x, y)$ is a homogeneous function of degree zero, then we use the substitution $x=vy$.

**EXERCISE 8.4** | **Textbook page 167**

**Solve the following differential equations :**

**1. $x\,dx+2y\,dy=0$**

**Solution :** $x\,dx+2y\,dy=0$

Integrating, we get

$$\int x\,dx+2\int y\,dy=c_1$$

$$\therefore \frac{x^2}{2}+2\left(\frac{y^2}{2}\right)=c_1$$

$\therefore$ $x^2+2y^2=c$, where $c=2c_1$

This is the general solution.

[*Note : Question is modified.*]

**2. $y^2\,dx+(xy+x^2)\,dy=0$**

**Solution :** $y^2\,dx+(xy+x^2)\,dy=0$

$$\therefore (xy+x^2)\,dy=-y^2dx$$

$$\therefore \frac{dy}{dx}=\frac{-y^2}{xy+x^2} \quad \dots (1)$$

Put $y=vx$ $\quad\therefore \frac{dy}{dx}=v+x\frac{dv}{dx}$

Substituting these values in (1), we get

$$v+x\frac{dv}{dx}=\frac{-v^2x^2}{x\cdot vx+x^2}=\frac{-v^2}{v+1}$$

$$\therefore x\frac{dv}{dx}=\frac{-v^2}{v+1}-v=\frac{-v^2-v^2-v}{v+1}$$

$$\therefore x\frac{dv}{dx}=\frac{-2v^2-v}{v+1}=-\left(\frac{2v^2+v}{v+1}\right)$$

$$\therefore \frac{v+1}{2v^2+v}dv=-\frac{1}{x}dx$$

Integrating, we get

$$\int\frac{v+1}{2v^2+v}dv=-\int\frac{1}{x}dx$$

$$\therefore \int\frac{v+1}{v(2v+1)}dv=-\int\frac{1}{x}dx$$

$$\therefore \int\frac{(2v+1)-v}{v(2v+1)}dv=-\int\frac{1}{x}dx$$

$$\therefore \int\left(\frac{1}{v}-\frac{1}{2v+1}\right)dv=-\int\frac{1}{x}dx$$

$$\therefore \int\frac{1}{v}dv-\int\frac{1}{2v+1}dv=-\int\frac{1}{x}dx$$

$$\therefore \log|v|-\frac{1}{2}\log|2v+1|=-\log|x|+\log c$$

$$\therefore 2\log|v|-\log|2v+1|=-2\log|x|+2\log c$$

$$\therefore \log|v^2|-\log|2v+1|=-\log|x^2|+\log c^2$$

$$\therefore \log\left|\frac{v^2}{2v+1}\right|=\log\left|\frac{c^2}{x^2}\right|$$

$$\therefore \frac{v^2}{2v+1}=\frac{c^2}{x^2}$$

$$\therefore \frac{\left(\frac{y^2}{x^2}\right)}{2\left(\frac{y}{x}\right)+1}=\frac{c^2}{x^2} \qquad \therefore \frac{y^2}{x(2y+x)}=\frac{c^2}{x^2}$$

$$\therefore xy^2=c^2(x+2y)$$

This is the general solution.

**Remark :**

The answer can also be given as follows :

$$\int \frac{v+1}{2v^2+v}dv=-\int \frac{1}{x}dx$$

$$\therefore \int \frac{\frac{1}{4}(4v+1)+\frac{3}{4}}{2v^2+v}dv=-\int \frac{1}{x}dx$$

$$\therefore \frac{1}{4}\int \frac{4v+1}{2v^2+v}dv+\frac{3}{4}\int \frac{1}{2v^2+v}dv=-\int \frac{1}{x}dx$$

$$\therefore \frac{1}{4}\int \frac{4v+1}{2v^2+v}dv+\frac{3}{4}\int \frac{(2v+1)-2v}{v(2v+1)}dv=-\int \frac{1}{x}dx$$

$$\therefore \frac{1}{4}\int \frac{4v+1}{2v^2+v}dv+\frac{3}{4}\int \left(\frac{1}{v}-\frac{2}{2v+1}\right)dv=-\int \frac{1}{x}dx$$

$$\therefore \frac{1}{4}\log|2v^2+v|+\frac{3}{4}[\log|v|-\log|2v+1|]=-\log x+c$$

$$\ldots \left[\because \frac{d}{dv}(2v^2+v)=4v+1 \text{ and } \int \frac{f'(v)}{f(v)}dv=\log|f(v)|+c\right]$$

$$\therefore \frac{1}{4}\log|2v^2+v|+\frac{3}{4}\log\left|\frac{v}{2v+1}\right|=-\log x+c$$

$$\therefore \frac{1}{4}\log\left|\frac{2y^2}{x^2}+\frac{y}{x}\right|+\frac{3}{4}\log\left|\frac{\left(\frac{y}{x}\right)}{2\left(\frac{y}{x}\right)+1}\right|=-\log x+c$$

$$\therefore \log x+\frac{1}{4}\log\left|\frac{2y^2+xy}{x^2}\right|+\frac{3}{4}\log\left[\frac{y}{x+2y}\right]=c$$

This is the general solution.

[*Note : Answer in the textbook is incorrect.*]

**3.** $x^2y\,dx-(x^3+y^3)\,dy=0$ ***(4 marks)***

**Solution :** $x^2y\,dx-(x^3+y^3)\,dy=0$

$$\therefore (x^3+y^3)\,dy=x^2y\,dx$$

$$\therefore \frac{dy}{dx}=\frac{x^2y}{x^3+y^3} \qquad \ldots (1)$$

Put $y=vx$ $\quad \therefore \frac{dy}{dx}=v+x\frac{dv}{dx}$

$$\therefore \text{(1) becomes, } v+x\frac{dv}{dx}=\frac{x^2\cdot vx}{x^3+v^3x^3}=\frac{v}{1+v^3}$$

$$\therefore x\frac{dv}{dx}=\frac{v}{1+v^3}-v=\frac{v-v-v^4}{1+v^3}$$

$$\therefore x\frac{dv}{dx}=\frac{-v^4}{1+v^3}$$

$$\therefore \frac{1+v^3}{v^4}dv=-\frac{1}{x}dx$$

Integrating, we get

$$\int \frac{1+v^3}{v^4}dv=-\int \frac{1}{x}dx$$

$$\therefore \int \left(\frac{1}{v^4}+\frac{1}{v}\right)dv=-\int \frac{1}{x}dx$$

$$\therefore \int v^{-4}dv+\int \frac{1}{v}dv=-\int \frac{1}{x}dx$$

$$\therefore \frac{v^{-3}}{-3}+\log|v|=-\log|x|+c_1$$

$$\therefore -\frac{1}{3v^3}+\log|v|=-\log|x|+c_1$$

$$\therefore -\frac{1}{3}\cdot\frac{1}{\left(\frac{y}{x}\right)^3}+\log\left|\frac{y}{x}\right|=-\log|x|+c_1$$

$$\therefore -\frac{x^3}{3y^3}+\log|y|-\log|x|=-\log|x|-\log c,$$

where $c_1=-\log c$

$$\therefore \frac{x^3}{3y^3}=\log c+\log y$$

$$\therefore \frac{x^3}{3y^3}=\log|cy|$$

This is the general solution.

**4.** $\dfrac{dy}{dx}+\dfrac{x-2y}{2x-y}=0$

**Solution :** $\dfrac{dy}{dx}+\dfrac{x-2y}{2x-y}=0$

$$\therefore \frac{dy}{dx}=-\left(\frac{x-2y}{2x-y}\right) \qquad \ldots (1)$$

Put $y=vx$. $\quad \therefore \frac{dy}{dx}=v+x\frac{dv}{dx}$

$$\therefore \text{(1) becomes, } v+x\frac{dv}{dx}=-\left(\frac{x-2vx}{2x-vx}\right)$$

$$\therefore v+x\frac{dv}{dx}=-\left(\frac{1-2v}{2-v}\right)$$

$$\therefore x\frac{dv}{dx}=-\left(\frac{1-2v}{2-v}\right)-v$$

$\therefore\ x\dfrac{dv}{dx}=\dfrac{-1+2v-2v+v^2}{2-v}$

$\therefore\ x\dfrac{dv}{dx}=\dfrac{v^2-1}{2-v}$

$\therefore\ \dfrac{2-v}{v^2-1}dv=\dfrac{1}{x}dx$

Integrating, we get

$\int\dfrac{2-v}{v^2-1}dv=\int\dfrac{1}{x}dx$

$\therefore\ 2\int\dfrac{1}{v^2-1}dv-\dfrac{1}{2}\int\dfrac{2v}{v^2-1}dv=\int\dfrac{1}{x}dx$

$\therefore\ 2\times\dfrac{1}{2}\log\left|\dfrac{v-1}{v+1}\right|-\dfrac{1}{2}\log|v^2-1|=\log|x|+\log c_1$

$\ldots\left[\because\ \dfrac{d}{dv}(v^2-1)=2v \text{ and } \int\dfrac{f'(x)}{f(x)}dx=\log|f(x)|+c\right]$

$\therefore\ \log\left|\dfrac{v-1}{v+1}\right|-\log|(v^2-1)^{\frac{1}{2}}|=\log|c_1x|$

$\therefore\ \log\left|\dfrac{v-1}{v+1}\cdot\dfrac{1}{\sqrt{v^2-1}}\right|=\log|c_1x|$

$\therefore\ \dfrac{v-1}{v+1}\cdot\dfrac{1}{\sqrt{v^2-1}}=c_1x$

$\therefore\ \dfrac{\frac{y}{x}-1}{\frac{y}{x}+1}\cdot\dfrac{1}{\sqrt{\frac{y^2}{x^2}-1}}=c_1x$

$\therefore\ \dfrac{y-x}{y+x}\cdot\dfrac{x}{\sqrt{y^2-x^2}}=c_1x$

$\therefore\ \dfrac{y-x}{y+x}=c_1\sqrt{y^2-x^2}$

$\therefore\ \dfrac{y-x}{y+x}=c_1\sqrt{y-x}\cdot\sqrt{y+x}$

$\therefore\ \sqrt{y-x}=c_1(y+x)^{\frac{3}{2}}$

$\therefore\ y-x=c_1{}^2(x+y)^3$

$\therefore\ y-x=c(x+y)^3$, where $c=c_1{}^2$

This is the general solution.

**Remark :**

The answer can also be given as follows :

$\therefore\ \log\left|\dfrac{v-1}{v+1}\right|-\dfrac{1}{2}\log|v^2-1|=\log|x|+\log c_1$

$\therefore\ \log\left|\dfrac{\frac{y}{x}-1}{\frac{y}{x}+1}\right|-\dfrac{1}{2}\log\left|\dfrac{y^2}{x^2}-1\right|=\log|x|+\log c_1$

$\therefore\ \log\left|\dfrac{y-x}{y+x}\right|-\dfrac{1}{2}\log\left|\dfrac{y^2-x^2}{x^2}\right|=\log|x|+\log c_1$

$\therefore\ \log\left|\dfrac{y-x}{y+x}\right|-\dfrac{1}{2}\log|y^2-x^2|+\dfrac{1}{2}\log x^2=\log|x|+\log c_1$

$\therefore\ \log\left|\dfrac{y-x}{y+x}\right|-\dfrac{1}{2}\log|x^2-y^2|+\log|x|=\log|x|+\log c_1$

$\therefore\ -\log\left|\dfrac{x+y}{x-y}\right|-\dfrac{1}{2}\log|x^2-y^2|=\log c_1$

$\therefore\ \log\left|\dfrac{x+y}{x-y}\right|+\dfrac{1}{2}\log|x^2-y^2|=\log c,$

where $\log c=-\log c_1$

[*Note : Answer in the textbook is incorrect.*]

**5. $(x^2-y^2)dx+2xy\,dy=0$**

**Solution :** $(x^2-y^2)dx+2xy\,dy=0$

$\therefore\ 2xy\,dy=-(x^2-y^2)dx=(y^2-x^2)dx$

$\therefore\ \dfrac{dy}{dx}=\dfrac{y^2-x^2}{2xy}$ … (1)

Put $y=vx$ $\quad\therefore\ \dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$

$\therefore$ (1) becomes, $v+x\dfrac{dv}{dx}=\dfrac{v^2x^2-x^2}{2x\cdot vx}$

$\therefore\ v+x\dfrac{dv}{dx}=\dfrac{v^2-1}{2v}$

$\therefore\ x\dfrac{dv}{dx}=\dfrac{v^2-1}{2v}-v=\dfrac{v^2-1-2v^2}{2v}$

$\therefore\ x\dfrac{dv}{dx}=\dfrac{-1-v^2}{2v}=-\left(\dfrac{1+v^2}{2v}\right)$

$\therefore\ \dfrac{2v}{1+v^2}dv=-\dfrac{1}{x}dx$

Integrating, we get

$\int\dfrac{2v}{1+v^2}dv=-\int\dfrac{1}{x}dx$

$\therefore\ \log|1+v^2|=-\log x+\log c$

$\ldots\left[\because\ \dfrac{d}{dv}(1+v^2)=2v \text{ and } \int\dfrac{f'(x)}{f(x)}dx=\log|f(x)|+c\right]$

$\therefore\ \log\left|1+\dfrac{y^2}{x^2}\right|=-\log x+\log c$

$\therefore\ \log\left|\dfrac{x^2+y^2}{x^2}\right|=\log\left|\dfrac{c}{x}\right|$

$\therefore\ \dfrac{x^2+y^2}{x^2}=\dfrac{c}{x}$

$\therefore\ x^2+y^2=cx$

This is the general solution.

**6.** $xy\dfrac{dy}{dx}=x^2+2y^2$

**Solution :** $xy\dfrac{dy}{dx}=x^2+2y^2$

$\therefore\ \dfrac{dy}{dx}=\dfrac{x^2+2y^2}{xy}$ ... (1)

Put $y=vx$. Then $\dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$

$\therefore$ (1) becomes, $v+x\dfrac{dv}{dx}=\dfrac{x^2+2v^2x^2}{x\cdot vx}=\dfrac{1+2v^2}{v}$

$\therefore\ x\dfrac{dv}{dx}=\dfrac{1+2v^2}{v}-v=\dfrac{1+2v^2-v^2}{v}$

$\therefore\ x\dfrac{dv}{dx}=\dfrac{1+v^2}{v}$

$\therefore\ \dfrac{v}{1+v^2}dv=\dfrac{1}{x}dx$

Integrating, we get

$$\int\frac{v}{1+v^2}dv=\int\frac{1}{x}dx$$

$\therefore\ \dfrac{1}{2}\displaystyle\int\frac{2v}{1+v^2}dv=\int\frac{1}{x}dx+\log c_1$

$\therefore\ \dfrac{1}{2}\log|1+v^2|=\log|x|+\log c_1$

$\therefore\ \log|1+v^2|=2\log|x|+2\log c_1$

$\therefore\ \log|1+v^2|=\log|x^2|+\log {c_1}^2$

$\therefore\ \log|1+v^2|=\log|cx^2|$, where $c={c_1}^2$

$\therefore\ 1+v^2=cx^2$

$\therefore\ 1+\dfrac{y^2}{x^2}=cx^2$

$\therefore\ \dfrac{x^2+y^2}{x^2}=cx^2$

$\therefore\ x^2+y^2=cx^4$

This is the general solution.

**7.** $x^2\dfrac{dy}{dx}=x^2+xy-y^2$

**Solution :**

$x^2\dfrac{dy}{dx}=x^2+xy-y^2$

$\therefore\ \dfrac{dy}{dx}=1+\dfrac{y}{x}-\dfrac{y^2}{x^2}$ ... (1)

Put $y=vx$ i.e. $\dfrac{y}{x}=v$

$\therefore\ \dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$

$\therefore$ (1) becomes

$v+x\dfrac{dv}{dx}=1+v-v^2$

$\therefore\ x\dfrac{dv}{dx}=1-v^2$

$\therefore\ \dfrac{dv}{1-v^2}=\dfrac{dx}{x}$

Integrating, we get

$$\int\frac{dv}{1-v^2}=\int\frac{dx}{x}$$

$\therefore\ \dfrac{1}{2}\log\left|\dfrac{1+v}{1-v}\right|=\log x+\log c_1$

$\therefore\ \dfrac{1}{2}\log\left|\dfrac{1+\frac{y}{x}}{1-\frac{y}{x}}\right|=\log(xc_1)$

$\therefore\ \log\left|\dfrac{x+y}{x-y}\right|=2\log(xc_1)$

$=\log(x^2{c_1}^2)$

$\therefore\ \dfrac{x+y}{x-y}=cx^2$, $c={c_1}^2$, is the required solution.

## ADDITIONAL SOLVED PROBLEMS–8(B)

**Solve the following differential equations :**

**1.** $(1+e^{x/y})\,dx+e^{x/y}\left(1-\dfrac{x}{y}\right)dy=0$

**Solution :** $(1+e^{x/y})\,dx+e^{x/y}\left(1-\dfrac{x}{y}\right)dy=0$

$\therefore\ (1+e^{x/y})\dfrac{dx}{dy}+e^{x/y}\left(1-\dfrac{x}{y}\right)=0$ ... (1)

Put $\dfrac{x}{y}=u$ $\therefore\ x=uy$ $\therefore\ \dfrac{dx}{dy}=u+y\dfrac{du}{dy}$

$\therefore$ (1) becomes, $(1+e^u)\left(u+y\dfrac{du}{dy}\right)+e^u(1-u)=0$

$\therefore\ u+ue^u+y(1+e^u)\dfrac{du}{dy}+e^u-ue^u=0$

$\therefore\ (u+e^u)+y(1+e^u)\dfrac{du}{dy}=0$

$\therefore\ \dfrac{dy}{y}+\dfrac{1+e^u}{u+e^u}du=0$

$\therefore\ \displaystyle\int\frac{dy}{y}+\int\frac{1+e^u}{u+e^u}du=c_1$ ... (2)

$\dfrac{d}{du}(u+e^u)=1+e^u$ and $\displaystyle\int\frac{f'(u)}{f(u)}du=\log|f(u)|+c$

$\therefore$ from (2), the general solution is

$\log|y| + \log|u+e^u| = \log c$, where $c_1 = \log c$

$\therefore \log|y(u+e^u)| = \log c \quad \therefore y(u+e^u) = c$

$\therefore y\left(\frac{x}{y} + e^{x/y}\right) = c \quad \therefore x + ye^{x/y} = c$

This is the general solution.

**2.** $(x^2 + 3xy + y^2)dx - x^2dy = 0$

**Solution :** $(x^2 + 3xy + y^2)\,dx - x^2dy = 0$

$\therefore x^2dy = (x^2 + 3xy + y^2)\,dx$

$\therefore \frac{dy}{dx} = \frac{x^2 + 3xy + y^2}{x^2}$ ... (1)

Put $y = vx \quad \therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ (1) becomes, $v + x\frac{dv}{dx} = \frac{x^2 + 3x\cdot vx + v^2x^2}{x^2}$

$\therefore v + x\frac{dv}{dx} = 1 + 3v + v^2$

$x\frac{dv}{dx} = v^2 + 2v + 1 = (v+1)^2$

$\therefore \frac{1}{(v+1)^2}dv = \frac{1}{x}dx$

Integrating, we get

$\int (v+1)^{-2}\,dv = \int \frac{1}{x}dx + c_1$

$\therefore \frac{(v+1)^{-1}}{-1} = \log|x| + c_1$

$\therefore -\frac{1}{v+1} = \log|x| + c_1$

$\therefore -\frac{1}{\frac{y}{x}+1} = \log|x| + c_1$

$\therefore -\frac{x}{y+x} = \log|x| + c_1$

$\therefore \log|x| + \frac{x}{x+y} = -c_1$

$\therefore \log|x| + \frac{x}{x+y} = c$, where $c = -c_1$

This is the general solution.

**3.** $y^2 - x^2\frac{dy}{dx} = xy\frac{dy}{dx}$

**Solution :** $y^2 - x^2\frac{dy}{dx} = xy\frac{dy}{dx}$

$\therefore x^2\frac{dy}{dx} + xy\frac{dy}{dx} = y^2$

$\therefore (x^2 + xy)\frac{dy}{dx} = y^2$

$\therefore \frac{dy}{dx} = \frac{y^2}{x^2 + xy}$ ... (1)

Put $y = vx \quad \therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ (1) becomes, $v + x\frac{dv}{dx} = \frac{v^2x^2}{x^2 + x\cdot vx} = \frac{v^2}{1+v}$

$\therefore x\frac{dv}{dx} = \frac{v^2}{1+v} - v = \frac{v^2 - v - v^2}{1+v}$

$\therefore x\frac{dv}{dx} = \frac{-v}{1+v}$

$\therefore \frac{1+v}{v}dv = -\frac{1}{x}dx$

Integrating, we get

$\int\left(\frac{1+v}{v}\right)dv = -\int\frac{1}{x}dx$

$\int\left(\frac{1}{v} + 1\right)dv = -\int\frac{1}{x}dx$

$\therefore \int\frac{1}{v}dv + \int 1\,dv = -\int\frac{1}{x}dx$

$\therefore \log|v| + v = -\log|x| + c$

$\therefore \log\left|\frac{y}{x}\right| + \frac{y}{x} = -\log|x| + c$

$\therefore \log|y| - \log|x| + \frac{y}{x} = -\log|x| + c$

$\therefore \frac{y}{x} + \log|y| = c$

This is the general solution.

## EXAMPLES FOR PRACTICE 8.4

**Solve the following differential equations :**

**1.** $\frac{dy}{dx} = \frac{y + \sqrt{x^2+y^2}}{x}$

**2.** $(x+y)\frac{dy}{dx} = y$

**3.** $xy\frac{dy}{dx} = x^2 + y^2$

**4.** $x^2\frac{dy}{dx} = xy - y^2$

**5.** $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$

**6.** $(x^2+y^2)\,dx - 2xy\,dy = 0$

7. $x^2\frac{dy}{dx} - 3xy - 2y^2 = 0$

8. $x\frac{dy}{dx} + \frac{y^2}{x} = y$

9. $\frac{dy}{dx} = \frac{4x-3y}{3x-2y}$

10. $(1+e^{x/y})\,dx + 2e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$

11. $(x^2+y^2)dx = 2xy\,dy$, when $x=1,\ y=0$

12. $(x^2+y^2)dy = 2(x^2+xy-y^2)\,dx,\ y(0)=2$

**Answers**

1. $y+\sqrt{x^2+y^2} = cx^2$
2. $y\log|y| = x+cy$
3. $\frac{y^2}{x^2} = 2\log|x|+c$
4. $\frac{x}{y} = \log|x|+c$
5. $\frac{y}{x} - \log|y| = c$
6. $x^2-y^2 = cx$
7. $y = cx^2(x+y)$
8. $x = ce^{x/y}$
9. $y^2-3xy+2x^2 = c^2$
10. $x+2ye^{x/y} = c$
11. $x^2-y^2 = x$
12. $(2x+y)^5(x-y)+8(x+y)^3 = 0.$

## 8.4 : LINEAR DIFFERENTIAL EQUATION

The general form of a linear differential equation of the first degree is

$$\frac{dy}{dx} + Py = Q \qquad \ldots (1)$$

where $P$ and $Q$ are the functions of $x$ only or constants.

e.g. (i) $\frac{dy}{dx} + xy = x^2$

(ii) $x\frac{dy}{dx} + y = x^3$

(iii) $\frac{dy}{dx} + y = e^x$

are linear differential equations.

The equation (1) may not be in variable separable form.

To solve equation (1), we multiply both sides of (1) by $e^{\int P\,dx}$, we get

$$e^{\int P\,dx}\cdot\left[\frac{dy}{dx} + Py\right] = Q\cdot e^{\int P\,dx} \qquad \ldots (2)$$

Now, $\frac{d}{dx}\left(y\cdot e^{\int P\,dx}\right) = y\cdot\frac{d}{dx}\left(e^{\int P\,dx}\right) + e^{\int P\,dx}\cdot\frac{dy}{dx}$

$= y\cdot e^{\int P\,dx}\frac{d}{dx}(\int P\,dx) + e^{\int P\,dx}\cdot\frac{dy}{dx}$

$= y\cdot e^{\int P\,dx}\cdot P + e^{\int P\,dx}\cdot\frac{dy}{dx}$

$= e^{\int P\,dx}\left[\frac{dy}{dx} + Py\right]$

$\therefore$ (2) can be written as :

$$\frac{d}{dx}\left[y\cdot e^{\int P\,dx}\right] = Q\cdot e^{\int P\,dx}$$

Integrating both sides w.r.t. $x$, we get

$$y\cdot e^{\int P\,dx} = \int Q\cdot e^{\int P\,dx}\,dx + c \qquad \ldots (3)$$

This is the required general solution of the differential equation, where $c$ is the constant of integration.

**Note :**

(i) The function $e^{\int P\,dx}$ is called **integrating factor (I.F.)** of the given equation.

(ii) The solution (3) can be written as :

$y\cdot(\text{I.F.}) = \int Q\cdot(\text{I.F.})\,dx + c$

**Working rule to solve the linear equation of first order and first degree :**

**Step 1 :** Write the given equation in the form

$\frac{dy}{dx} + Py = Q$

where $P$ and $Q$ are functions of $x$ only or constants.

**Step 2 :** Find the integrating factor $(\text{I.F.}) = e^{\int P\,dx}$

**Step 3 :** Multiply both sides of the equation in step 1 by I.F.

**Step 4 :** Integrate both sids of the equation obtained in step 3 w.r.t. $x$ to obtain

$y\cdot(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + c$

This is the required general solution of the differential equation.

**Note :** If the equation is of the form $\frac{dx}{dy} + P\cdot x = Q$ where $P$ and $Q$ are the functions of $y$ only or constants, then $\text{I.F.} = e^{\int P\,dy}$

and the solution of this equation is given by

$x\cdot(\text{I.F.}) = \int Q\,(\text{I.F.})\,dy + c.$

| EXERCISE 8.5 | Textbook page 168 |
|---|---|

**Solve the following differential equations :**

1. $\frac{dy}{dx} + y = e^{-x}$

**Solution :** $\frac{dy}{dx} + y = e^{-x} \qquad \ldots (1)$

This is the linear differential equation of the form

$\frac{dy}{dx}+P\cdot y=Q$, where $P=1$ and $Q=e^{-x}$

$\therefore$ I.F. $=e^{\int P\,dx}=e^{\int 1\,dx}=e^{x}$

$\therefore$ the solution of (1) is given by

$y\cdot(\text{I.F.})=\int Q\cdot(\text{I.F.})\,dx+c$

$\therefore\ y\cdot e^{x}=\int e^{-x}\cdot e^{x}\,dx+c$

$\therefore\ e^{x}\cdot y=\int 1\,dx+c$

$\therefore\ e^{x}\cdot y=x+c \qquad \therefore\ ye^{x}=x+c$

This is the general solution.

**2.** $\frac{dy}{dx}+y=3.$

**Solution :** $\frac{dy}{dx}+y=3$

This is the linear differential equation of the form

$\frac{dy}{dx}+P\cdot y=Q$, where $P=1$, $Q=3$

$\therefore$ I.F. $=e^{\int P\,dx}=e^{\int 1 dx}=e^{x}$

$\therefore$ the solution of (1) is given by

$y\cdot(\text{I.F.})=\int Q(\text{I.F.})\,dx+c$

$\therefore\ ye^{x}=\int 3e^{x}dx+c=3e^{x}+c$

$\therefore\ ye^{x}=3e^{x}+c$

This is the general solution.

**3.** $x\frac{dy}{dx}+2y=x^2\cdot\log x.$

**Solution :** $x\frac{dy}{dx}+2y=x^2\cdot\log x$

$\therefore\ \frac{dy}{dx}+\left(\frac{2}{x}\right)\cdot y=x\cdot\log x \qquad \dots (1)$

This is the linear differential equation of the form

$\frac{dy}{dx}+P\cdot y=Q$, where $P=\frac{2}{x}$ and $Q=x\cdot\log x$

$\therefore$ I.F. $=e^{\int P\,dx}=e^{\int \frac{2}{x}dx}=e^{2\int\frac{1}{x}dx}$

$=e^{2\log x}=e^{\log x^2}=x^2$

$\therefore$ the solution of (1) is given by

$y\cdot(\text{I.F.})=\int Q\cdot(\text{I.F.})\,dx+c$

$\therefore\ y\cdot x^2=\int(x\log x)\cdot x^2\,dx+c$

$\therefore\ x^2\cdot y=\int x^3\cdot\log x\,dx+c$

$=(\log x)\int x^3\,dx-\int\left[\frac{d}{dx}(\log x)\int x^3\,dx\right]dx+c$

$=(\log x)\cdot\frac{x^4}{4}-\int\frac{1}{x}\cdot\frac{x^4}{4}dx+c$

$=\frac{1}{4}x^4\log x-\frac{1}{4}\int x^3dx+c$

$\therefore\ x^2\cdot y=\frac{1}{4}x^4\log x-\frac{1}{4}\cdot\frac{x^4}{4}+c$

$\therefore\ y\cdot x^2=\frac{x^4\log x}{4}-\frac{x^4}{16}+c$

This is the general solution.

**4.** $(x+y)\frac{dy}{dx}=1.$

**Solution :** $(x+y)\frac{dy}{dx}=1$

$\therefore\ \frac{dx}{dy}=x+y$

$\therefore\ \frac{dx}{dy}-x=y$

$\therefore\ \frac{dx}{dy}+(-1)x=y \qquad \dots (1)$

This is the linear differential equation of the form

$\frac{dx}{dy}+P\cdot x=Q$, where $P=-1$ and $Q=y$

$\therefore$ I.F. $=e^{\int P\,dy}=e^{\int -1\,dy}=e^{-y}$

$\therefore$ the solution of (1) is given by

$x\cdot(\text{I.F.})=\int Q\cdot(\text{I.F.})\,dy+c$

$\therefore\ x\cdot e^{-y}=\int y\cdot e^{-y}\,dy+c$

$\therefore\ e^{-y}\cdot x=y\int e^{-y}\,dy-\int\left[\frac{d}{dy}(y)\int e^{-y}\,dy\right]dy+c$

$=y\cdot\frac{e^{-y}}{-1}-\int 1\cdot\frac{e^{-y}}{-1}dy+c$

$=-ye^{-y}+\int e^{-y}\,dy+c$

$\therefore\ e^{-y}\cdot x=-ye^{-y}+\frac{e^{-y}}{-1}+c$

$\therefore\ e^{-y}\cdot x+ye^{-y}+e^{-y}=c$

$\therefore\ e^{-y}(x+y+1)=c$

$\therefore\ x+y+1=ce^{y}$

This is the general solution.

**5.** $y\,dx+(x-y^2)\,dy=0.$

**Solution :** $y\,dx+(x-y^2)\,dy=0$

$\therefore\ y\,dx=-(x-y^2)dy$

$\therefore \ \dfrac{dx}{dy} = -\dfrac{(x-y^2)}{y} = -\dfrac{x}{y} + y$

$\therefore \ \dfrac{dx}{dy} + \left(\dfrac{1}{y}\right) \cdot x = y$ ... (1)

This is the linear differential equation of the form

$\dfrac{dx}{dy} + P \cdot x = Q$, where $P = \dfrac{1}{y}$ and $Q = y$

$\therefore$ I.F. $= e^{\int P\,dy} = e^{\int \frac{1}{y} dy} = e^{\log y} = y_1$

$\therefore$ the solution of (1) is given by

$x \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.})\, dy + c_1$

$\therefore \ xy = \int y \cdot y\, dy + c_1$

$\therefore \ xy = \int y^2\, dy + c_1$

$\therefore \ xy = \dfrac{y^3}{3} + c_1$

$\therefore \ 3xy = y^3 + 3c_1$

$\therefore \ 3xy = y^2 + c$, where $c = 3c_1$

This is the general solution.

**6.** $\boldsymbol{\dfrac{dy}{dx} + 2xy = x.}$

**Solution :** $\dfrac{dy}{dx} + 2xy = x$ ... (1)

This is the linear differential equation of the form

$\dfrac{dy}{dx} + P \cdot y = Q$, where $P = 2x$, $Q = x$

$\therefore$ I.F. $= e^{\int P\,dx} = e^{\int 2x\,dx}$

$= e^{2\int x\,dx} = e^{2\left(\frac{x^2}{2}\right)} = e^{x^2}$

$\therefore$ the solution of (1) is given by

$y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.})\, dx + c$

$\therefore \ ye^{x^2} = \int xe^{x^2}\, dx + c$ ... (1)

Put $x^2 = t$ $\quad \therefore \ 2x\,dx = dt$

$\therefore \ x\,dx = \dfrac{1}{2} dt$

$\therefore$ (1) becomes

$ye^{x^2} = \dfrac{1}{2}\displaystyle\int e^t\, dt + c$

$\therefore \ ye^{x^2} = \dfrac{1}{2} e^{x^2} + c$

This is the general solution.

**7.** $\boldsymbol{(x+a)\dfrac{dy}{dx} = -y + a.}$

**Solution :** $(x+a)\dfrac{dy}{dx} + y = a$

$\therefore \ \dfrac{dy}{dx} + \left(\dfrac{1}{x+a}\right) y = \dfrac{a}{x+a}$ ... (1)

This is the linear differential equation of the form

$\dfrac{dy}{dx} + P \cdot y = Q$, where $P = \dfrac{1}{x+a}$, $Q = \dfrac{a}{x+a}$.

I.F. $= e^{\int P\,dx} = e^{\int \frac{1}{x+a} dx}$

$= e^{\log(x+a)} = x + a$

$\therefore$ the solution of (1) is given by

$y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.})\, dx + c$

$\therefore \ y(x+a) = \displaystyle\int \left(\frac{a}{x+a}\right)(x+a)\, dx + c$

$= a \int dx + c$

$\therefore \ y(x+a) = ax + c$

This is the general solution.

**8.** $\boldsymbol{dy + (2y)\,dx = 8\,dx.}$

**Solution :** $dy + (2y)\,dx = 8\,dx$

$\therefore \ \dfrac{dy}{dx} + 2y = 8$ ... (1)

This is the linear differential equation of the form

$\dfrac{dy}{dx} + Py = Q$, where $P = 2$, $Q = 8$

$\therefore$ I.F. $= e^{\int P\,dx} = e^{2\int dx} = e^{2x}$

$\therefore$ the solution $f(1)$ is given by

$y \cdot (\text{I.F.}) = \int Q\,(\text{I.F.})\, dx + c$

$\therefore \ ye^{2x} = \int 8e^{2x}\, dx + c$

$= 8\left(\dfrac{e^{2x}}{2}\right) + c$

$\therefore \ ye^{2x} = 4e^{2x} + c$

This is the general solution.

[*Note : Question is modified.*]

## ADDITIONAL SOLVED PROBLEMS–8(C)

**Solve the following differential equations :**

**1.** $\boldsymbol{(x+a)\dfrac{dy}{dx} - 3y = (x+a)^5.}$

**Solution :** $(x+a)\dfrac{dy}{dx} - 3y = (x+a)^5$

$\therefore \frac{dy}{dx}-\frac{3y}{x+a}=(x+a)^4$

$\therefore \frac{dy}{dx}+\left(\frac{-3}{x+a}\right)y=(x+a)^4$ ... (1)

This is the linear differential equation of the form

$\frac{dy}{dx}+P\cdot y=Q$, where $P=\frac{-3}{x+a}$ and $Q=(x+a)^4$

$\therefore$ I.F. $=e^{\int P\,dx}=e^{\int\frac{-3}{x+a}dx}=e^{-3\int\frac{1}{x+a}dx}$

$=e^{-3\log|x+a|}=e^{\log(x+a)^{-3}}$

$=(x+a)^{-3}=\frac{1}{(x+a)^3}$

$\therefore$ the solution of (1) is given by

$y\cdot(\text{I.F.})=\int Q\cdot(\text{I.F.})\,dx+c$

$\therefore y\cdot\frac{1}{(x+a)^3}=\int(x+a)^4\cdot\frac{1}{(x+a)^3}dx+c$

$\therefore \frac{y}{(x+a)^3}=\int(x+a)\,dx+c$

$\therefore \frac{y}{(x+a)^3}=\frac{(x+a)^2}{2}+c$

$\therefore 2y=(x+a)^5+2c(x+a)^3$

This is the general solution.

**2. $(1-x^2)\frac{dy}{dx}+2xy=x(1-x^2)^{\frac{1}{2}}$.**

**Solution :** $(1-x^2)\frac{dy}{dx}+2xy=x(1-x^2)^{\frac{1}{2}}$

$\therefore \frac{dy}{dx}+\left(\frac{2x}{1-x^2}\right)y=\frac{x}{(1-x^2)^{\frac{1}{2}}}$

This is the linear differential equation of the form

$\frac{dy}{dx}+P\cdot y=Q$, where $P=\frac{2x}{1-x^2}$ and $Q=\frac{x}{(1-x^2)^{\frac{1}{2}}}$

$\therefore$ I.F. $=e^{\int P\,dx}=e^{\int\frac{2x}{1-x^2}dx}$

$=e^{-\int\frac{-2x}{1-x^2}dx}=e^{-\log|1-x^2|}$

$=e^{\log\left|\frac{1}{1-x^2}\right|}=\frac{1}{1-x^2}$

$\therefore$ the solution of (1) is given by

$y\cdot(\text{I.F.})=\int Q\cdot(\text{I.F.})\,dx+c$

$\therefore y\cdot\frac{1}{(1-x^2)}=\int\frac{x}{(1-x^2)^{\frac{1}{2}}}\cdot\frac{1}{1-x^2}dx+c$

$\therefore \frac{y}{(1-x^2)}=\int\frac{x}{(1-x^2)^{\frac{3}{2}}}dx+c$

Put $1-x^2=t$ $\therefore -2x\,dx=dt$

$\therefore x\,dx=-\frac{dt}{2}$

$\therefore \frac{y}{1-x^2}=\int\frac{1}{t^{\frac{3}{2}}}\cdot\frac{(-dt)}{2}+c$

$\therefore \frac{y}{1-x^2}=-\frac{1}{2}\int t^{-\frac{3}{2}}dt+c$

$\therefore \frac{y}{1-x^2}=-\frac{1}{2}\cdot\frac{t^{-\frac{1}{2}}}{(-1/2)}+c$

$\therefore \frac{y}{1-x^2}=\frac{1}{(1-x^2)^{\frac{1}{2}}}+c$

$\therefore y=\sqrt{1-x^2}+c(1-x^2)$

This is the general solution.

## EXAMPLES FOR PRACTICE 8.5

**1. Solve the following differential equations :**

(1) $\frac{dy}{dx}-\frac{y}{x}=2x^2,\ x>0$ (2) $4\frac{dy}{dx}+8y=5e^{-3x}$

(3) $\frac{dy}{dx}+2y=4x$ (4) $\frac{dy}{dx}+\frac{y}{x}=x^3-3$

(5) $x\frac{dy}{dx}+y=xe^x,\ x>0$ (6) $x\frac{dy}{dx}-y=(x-1)e^x$

(7) $(1+x^3)\frac{dy}{dx}+6x^2y=1+x^2$

(8) $(x+2y^3)dy=y\,dx$

(9) $\frac{dy}{dx}=x+y$ (10) $\frac{dy}{dx}=y+2x$.

2. Solve the differential equation $\frac{dy}{dx}-y=e^x$.

Hence, find the particular solution for $x=0$ and $y=1$.

### Answers

1. (1) $y=x^3+cx$ (2) $y=-\frac{5}{4}e^{-3x}+ce^{-2x}$

(3) $y=(2x-1)+ce^{-2x}$ (4) $xy=\frac{x^5}{5}-\frac{3}{2}x^2+c$

(5) $y=\left(\frac{x-1}{x}\right)e^x+\frac{c}{x}$ (6) $y=e^x+cx$

(7) $y(1+x^3)^2 = x + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^6}{6} + c$

(8) $x = y^3 + cy$ (9) $x + y + 1 = ce^x$

(10) $y + 2(x+1) = ce^x$.

**2.** $y = (x+c)e^x$, $y = (x+1)e^x$.

## 8.5 : APPLICATIONS OF DIFFERENTIAL EQUATIONS

We studied about the differential equation, its formation and the solution of the differential equation of first order and first degree.

Differential equations have wide range of applications. In most of the physical problems, the relation between the variables can be expressed in terms of rate of change of one variable with respect to another. Hence, in all such problems, we have to deal with differential equations.

In this section, we shall study about some of the applications.

### 8.5.1 : Population Growth and Growth of Bacteria

Generally the population of a city or country, the number of bacteria in the culture increases, i.e. grow with time.

If the population $P$ increases at time $t$, then the rate of change of $P$ is generally proportional to the population present at that time.

$\therefore \frac{dP}{dt} \propto P$ $\therefore \frac{dP}{dt} = kP$, where $k$ is a constant.

This is the differential equation of first order and first degree.

$\frac{dP}{dt} = kP$ can be written as

$\frac{dP}{P} = k\,dt$

On integrating, we get

$\int \frac{1}{P} dP = k \int dt$

$\therefore \log P = kt + c$

$\therefore P = e^{kt+c} = e^{kt} \cdot e^c$

$\therefore P = a \cdot e^{kt}$, where $a = e^c$

This gives the population at any time $t$.

### 8.5.2 : Radioactive Decay

We know that the radioactive element like radium disintegrates and there is a decay in its mass. The rate of disintegration is always proportional to the amount present at that time.

If $x$ is the amount at time $t$, then

$\frac{dx}{dt} = -kx$, where $k$ is a constant and $k > 0$

The negative sign indicates that $x$ decreases as $t$ increases.

$\frac{dx}{dt} = -kx$ can be written as

$\frac{1}{x} dx = -k\,dt$

On integrating, we get

$\int \frac{1}{x} dx = -k \int dt$

$\therefore \log x = -kt + c$

$\therefore x = e^{-kt+c} = e^{-kt} . e^c$

$\therefore x = ae^{-kt}$, when $a = e^c$

If $x_0$ is the initial amount of radioactive element, i.e. $x = x_0$ when $t = 0$, then

$x_0 = a \cdot e^0$ $\therefore a = x_0$

$\therefore x = x_0 e^{-kt}$

This expression gives the amount of radioactive element at any time $t$.

**EXERCISE 8.6** | **Textbook page 170**

**1. In a certain culture of bacteria, the rate of increase is proportional to the number present. If it is found that the number doubles in 4 hours, find the number of times the bacteria are increased in 12 hours.**

**Solution :** Let $x$ be the number of bacteria in the culture at time $t$.

Then the rate of increase is $\frac{dx}{dt}$ which is proportional to $x$.

$\therefore \frac{dx}{dt} \propto x$

$\therefore \frac{dx}{dt} = kx$, where $k$ is a constant

$\therefore \frac{dx}{x} = k\,dt$

On integrating, we get

$\int \frac{dx}{x} = k \int dt$

$\therefore \log x = kt + c$

Initially, i.e. when $t = 0$, let $x = x_0$

$\therefore \log x_0 = k \times 0 + c$ $\therefore c = \log x_0$

$\therefore \log x = kt + \log x_0$

$\therefore \ \log x - \log x_0 = kt$

$\therefore \ \log\left(\frac{x}{x_0}\right) = kt$ ... (1)

Since the number doubles in 4 hours, i.e. when $t = 4$, $x = 2x_0$

$\therefore \ \log\left(\frac{2x_0}{x_0}\right) = 4k \quad \therefore \ k = \frac{1}{4}\log 2$

$\therefore$ (1) becomes, $\log\left(\frac{x}{x_0}\right) = \frac{t}{4}\log 2$

When $t = 12$, we get

$\log\left(\frac{x}{x_0}\right) = \frac{12}{4}\log 2 = 3\log 2$

$\therefore \ \log\left(\frac{x}{x_0}\right) = \log 8$

$\therefore \ \frac{x}{x_0} = 8 \quad \therefore \ x = 8x_0$

$\therefore$ number of bacteria will be 8 times the original number in 12 hours.

**2. If the population of a town increases at a rate proportional to the population at that time. If the population increases from 40 thousands to 60 thousands in 40 years, what will be the population in another 20 years?**

**(Given : $\sqrt{\frac{3}{2}} = 1.2247$)**

**Solution :** Let P be the population of the city at time $t$.

Then $\frac{dP}{dt}$, the rate of increase of population, is proportional to $P$.

$\therefore \ \frac{dP}{dt} \propto P$

$\therefore \ \frac{dP}{dt} = kP$, $k$ is a constant

$\therefore \ \frac{dP}{P} = k\,dt$

Integrating, we get

$\int \frac{dP}{P} = k\int dt$

$\therefore \ \log P = kt + c$

Initially, i.e. when $t = 0$, $P = 40000$

$\therefore \ \log 40000 = 0 + c \quad \therefore \ c = \log 40000$

$\therefore \ \log P = kt + \log 40000$

$\therefore \ \log P - \log 40000 = kt$

$\therefore \ \log\left(\frac{P}{40000}\right) = kt$ ... (1)

When $t = 40$, $P = 60000$

$\therefore \ \log\left(\frac{60000}{40000}\right) = 40k$

$\therefore \ k = \frac{1}{40}\log\left(\frac{3}{2}\right)$

$\therefore$ (1) becomes

$\log\left(\frac{P}{40000}\right) = \frac{t}{40}\log\left(\frac{3}{2}\right)$

$= \log\left(\frac{3}{2}\right)^{\frac{t}{40}}$

$\therefore \ \frac{P}{40000} = \left(\frac{3}{2}\right)^{\frac{t}{40}}$

We have to find $P$ in another 20 years

i.e. at $t = 40 + 20 = 60$

If $t = 60$, then

$\frac{P}{40000} = \left(\frac{3}{2}\right)^{\frac{60}{40}} = \left(\frac{3}{2}\right)^{\frac{3}{2}} = \frac{3}{2}\sqrt{\frac{3}{2}}$

$\therefore \ P = \frac{40000 \times 3}{2} \times 1.2247$ ... [By data]

$= 73482$

$\therefore$ population after 60 years will be 73482.

**3. The rate of growth of bacteria is proportional to the number present. If initially, there were 1000 bacteria and the number doubles in 1 hour, find the number of bacteria after $\frac{5}{2}$ hours. [Given : $\sqrt{2} = 1.414$]**

**Solution :** Let $x$ be the number of bacteria at time $t$.

Then the rate of increase is $\frac{dx}{dt}$ which is proportional to $x$.

$\therefore \ \frac{dx}{dt} \propto x$

$\therefore \ \frac{dx}{dt} = kx$, where $k$ is a constant

$\therefore \ \frac{dx}{x} = k\,dt$

On integrating, we get

$\int \frac{dx}{x} = k\int dt$

$\therefore \ \log x = kt + c$

Initially, i.e. when $t = 0$, $x = 1000$

$\therefore\ \log 1000 = k \times 0 + c \qquad \therefore\ c = \log 1000$

$\therefore\ \log x = kt + \log 1000$

$\therefore\ \log x - \log 1000 = kt$

$\therefore\ \log\left(\frac{x}{1000}\right) = kt \qquad \ldots (1)$

Now, when $t = 1$, $x = 2 \times 1000 = 2000$

$\therefore\ \log\left(\frac{2000}{1000}\right) = k \qquad \therefore\ k = \log 2$

$\therefore$ (1) becomes, $\log\left(\frac{x}{1000}\right) = t \log 2$

If $t = \frac{5}{2}$, then

$\log\left(\frac{x}{1000}\right) = \frac{5}{2}\log 2 = \log (2)^{\frac{5}{2}}$

$\therefore\ \left(\frac{x}{1000}\right) = (2)^{\frac{5}{2}} = 4\sqrt{2} = 4 \times 1.414 = 5.656$

$\therefore\ x = 5.656 \times 1000 = 5656$

$\therefore$ number of bacteria after $\frac{5}{2}$ hours $= 5656$.

[*Note : Answer in the textbook is incorrect.*]

**4. Find the population of a city at any time $t$, given that the rate of increase of population is proportional to the population at that instant and that in a period of 40 years, the population increased from 30,000 to 40,000.**

**Solution :** Let $P$ be the population of the city at time $t$.

Then $\frac{dP}{dt}$, the rate of increase of population, is proportional to $P$.

$\therefore\ \frac{dP}{dt} \propto P$

$\therefore\ \frac{dP}{dt} = kP$, where $k$ is a constant.

$\therefore\ \frac{dP}{P} = k\,dt$

On integrating, we get

$\int \frac{1}{P} dP = k \int dt$

$\therefore\ \log P = kt + c$

Initially, i.e. when $t = 0$, $P = 30000$

$\therefore\ \log 30000 = k \times 0 + c \qquad \therefore\ c = \log 30000$

$\therefore\ \log P = kt + \log 30000$

$\therefore\ \log P - \log 30000 = kt$

$\therefore\ \log\left(\frac{P}{30000}\right) = kt \qquad \ldots (1)$

Now, when $t = 40$, $P = 40000$

$\therefore\ \log\left(\frac{40000}{30000}\right) = k \times 40$

$\therefore\ k = \frac{1}{40}\log\left(\frac{4}{3}\right)$

$\therefore$ (1) becomes, $\log\left(\frac{P}{30000}\right) = \frac{t}{40}\log\left(\frac{4}{3}\right) = \log\left(\frac{4}{3}\right)^{\frac{t}{40}}$

$\therefore\ \frac{P}{30000} = \left(\frac{4}{3}\right)^{\frac{t}{40}}$

$\therefore\ P = 30000\left(\frac{4}{3}\right)^{\frac{t}{40}}$

$\therefore$ the population of the city at time $t = 30000\left(\frac{4}{3}\right)^{\frac{t}{40}}$.

**5. The rate of depreciation $\frac{dV}{dt}$ of a machine is inversely proportional to the square of $t+1$, where $V$ is the value of the machine $t$ years after it was purchased. The initial value of the machine was ₹ 8,00,000 and its value decreased ₹ 1,00,000 in the first year. Find the value after 6 years.**

**Solution :** Let $V$ be the value of the machine at the end of $t$ years.

Then $\frac{dV}{dt}$, the rate of depreciation, is inversly proportional to $(t+1)^2$.

$\therefore\ \frac{dV}{dt} \propto \frac{1}{(t+1)^2}$

$\therefore\ \frac{dV}{dt} = -\frac{k}{(t+1)^2}$, $k > 0$ is a constant

$\therefore\ dV = \frac{-k\,dt}{(t+1)^2}$

On integrating, we get

$\int dV = -k \int \frac{dt}{(t+1)^2}$

$\therefore\ V = -k\left[\frac{-1}{t+1}\right] + c$

$\therefore\ V = \frac{k}{t+1} + c$

Initially, i.e. when $t = 0$, $V = 800000$

$\therefore\ 800000 = \frac{k}{1} + c = k + c \qquad \ldots (1)$

Now, when $t = 1$, $V = 800000 - 100000 = 700000$

$\therefore\ 700000 = \frac{k}{1+1} + c = \frac{k}{2} + c$ ... (2)

Subtracting (2) from (1), we get

$100000 = \frac{1k}{2}$ $\quad \therefore\ k = 200000$

$\therefore$ from (1), $800000 = 200000 + c$

$\therefore\ c = 600000$

$\therefore\ V = \frac{200000}{t+1} + 600000$

When $t = 6$,

$V = \frac{200000}{7} + 600000$

$= 28571.43 + 600000$

$= 628571.43 \approx 628571$

Hence, the value of the machine after 6 years will be ₹ 6,28,571.

## ADDITIONAL SOLVED PROBLEMS–8(D)

**1. If the population of a country doubles in 60 years; in how many years will it be triple (treble) under the assumption that the rate of increase is proportional to the number of inhabitants?**

**(Given : $\log 2 = 0.6912$, $\log 3 = 1.0986$)**

**Solution :** Let $P$ be the population at time $t$ years. Then $\frac{dP}{dt}$, the rate of increase of population is proportional to $P$.

$\therefore\ \frac{dP}{dt} \propto P$

$\therefore\ \frac{dP}{dt} = kP$, where $k$ is a constant

$\therefore\ \frac{dP}{P} = k\,dt$

On integrating, we get

$\int \frac{dP}{P} = k \int dt + c$

$\therefore\ \log P = kt + c$

Initially i.e. when $t = 0$, let $P = P_0$

$\therefore\ \log P_0 = k \times 0 + c$ $\quad \therefore\ c = \log P_0$

$\therefore\ \log P = kt + \log P_0$

$\therefore\ \log P - \log P_0 = kt$

$\therefore\ \log\left(\frac{P}{P_0}\right) = kt$ ... (1)

Since, the population doubles in 60 years, i.e. when $t = 60$, $P = 2P_0$

$\therefore\ \log\left(\frac{2P_0}{P_0}\right) = 60k$ $\quad \therefore\ k = \frac{1}{60}\log 2$

$\therefore$ (1) becomes, $\log\left(\frac{P}{P_0}\right) = \frac{t}{60}\log 2$

When population becomes triple, i.e. when $P = 3P_0$, we get

$\log\left(\frac{3P_0}{P_0}\right) = \frac{t}{60}\log 2$

$\therefore\ \log 3 = \frac{t}{60}\log 2$

$\therefore\ t = 60\left(\frac{\log 3}{\log 2}\right) = 60\left(\frac{1.0986}{0.6912}\right)$

$= 60 \times 1.5894 = 95.364 \approx 95.4$ years

$\therefore$ the population becomes triple in 95.4 years (approximately)

**2. The rate of disintegration of a radioactive element at any time $t$ is proportional to its mass at that time. Find the time during which the original mass of 1.5 gm will disintegrate into its mass of 0.5 gm.**

**Solution :** Let $m$ be the mass of the radioactive element at time $t$.

Then the rate of disintegration is $\frac{dm}{dt}$ which is proportional to $m$.

$\therefore\ \frac{dm}{dt} \propto m$

$\therefore\ \frac{dm}{dt} = -km$, where $k > 0$

$\therefore\ \frac{dm}{m} = -k\,dt$

On integrating, we get

$\int \frac{1}{m}\,dm = -k \int dt$

$\therefore\ \log m = -kt + c$

Initially, i.e. when $t = 0$, $m = 1.5$

$\therefore\ \log(1.5) = -k \times 0 + c$ $\quad \therefore\ c = \log\left(\frac{3}{2}\right)$

$\therefore\ \log m = -kt + \log\left(\frac{3}{2}\right)$

$\therefore\ \log m - \log\frac{3}{2} = -kt$

$\therefore \log\left(\frac{2m}{3}\right) = -kt$

When $m = 0.5 = \frac{1}{2}$, then

$$\log\left(\frac{2 \times \frac{1}{2}}{3}\right) = -kt$$

$\therefore \log\left(\frac{1}{3}\right) = -kt$

$\therefore \log(3)^{-1} = -kt$

$\therefore -\log 3 = -kt$

$\therefore t = \frac{1}{k}\log 3$

$\therefore$ the original mass will disintegrate to 0.5 gm when $t = \frac{1}{k}\log 3$.

---

**3. The rate of decay of certain substance is directly proportional to the amount present at that instant. Initially, there are 27 gm of certain substance and three hours later it is found that 8 gm are left. Find the amount left after one more hour.**

**Solution :** Let $x$ gm be the amount of the substance left at time $t$.

Then the rate of decay is $\frac{dx}{dt}$, which is proportional to $x$.

$\therefore \frac{dx}{dt} \propto x$

$\therefore \frac{dx}{dt} = -kx$, where $k > 0$

$\therefore \frac{1}{x}dx = -k\,dt$

On integrating, we get

$$\int \frac{1}{x}dx = -k\int dt$$

$\therefore \log x = -kt + c$

Initially i.e. when $t = 0$, $x = 27$

$\therefore \log 27 = -k \times 0 + c \qquad \therefore c = \log 27$

$\therefore \log x = -kt + \log 27$

$\therefore \log x - \log 27 = -kt$

$\therefore \log\left(\frac{x}{27}\right) = -kt \qquad \ldots (1)$

Now, when $t = 3$, $x = 8$

$\therefore \log\left(\frac{8}{27}\right) = -3k$

$\therefore -3k = \log\left(\frac{2}{3}\right)^3 = 3\log\left(\frac{2}{3}\right)$

$\therefore k = -\log\left(\frac{2}{3}\right)$

$\therefore$ (1) becomes, $\log\left(\frac{x}{27}\right) = t\log\left(\frac{2}{3}\right)$

When $t = 4$, then

$\log\left(\frac{x}{27}\right) = 4\log\left(\frac{2}{3}\right) = \log\left(\frac{2}{3}\right)^4$

$\therefore \frac{x}{27} = \frac{16}{81} \qquad \therefore x = \frac{16}{3}$

$\therefore$ the amount left after 4 hours $= \frac{16}{3}$ gm.

---

**4. The rate of growth of population of a city at any time $t$ is proportional to the size of population. For a certain city it is found that the constant of proportionality is 0.04. Find the population of the city after 25 years, if the initial population is 10,000.**

**[Take $e = 2.7182$]**

**Solution :** Let $P$ be the population of the city at time $t$.

Then the rate of growth of population is $\frac{dP}{dt}$ which is proportional to $P$.

$\therefore \frac{dP}{dt} \propto P$

$\therefore \frac{dP}{dt} = kP$, where $k = 0.04$

$\therefore \frac{dP}{dt} = (0.04)P$

$\therefore \frac{1}{P}dP = (0.04)\,dt$

On integrating, we get

$$\int \frac{1}{P}dP = (0.04)\int dt$$

$\therefore \log P = (0.04)t + c$

Initially, i.e. when $t = 0$, $P = 10000$

$\therefore \log 10000 = (0.04) \times 0 + c$

$\therefore c = \log 10000$

$\therefore \log P = (0.04)t + \log 10000$

$\therefore \log P - \log 10000 = (0.04)t$

$\therefore \log\left(\frac{P}{10000}\right) = (0.04)t$

When $t = 25$, then

$$\log\left(\frac{P}{10000}\right) = (0.04) \times 25 = 1$$

$\therefore \log\left(\frac{P}{10000}\right) = \log e$ ... $[\because \log e = 1]$

$\therefore \frac{P}{10000} = e = 2.7182$

$\therefore P = 2.7182 \times 10000 = 27182$

$\therefore$ the population of the city after 25 years will be 27,182.

**5. Radium decomposes at the rate proportional to the amount present at any time. If $p$ per cent of the amount disappears in one year, what per cent of amount of radium will be left after 2 years?**

**Solution :** Let $x$ be the amount of the radium at time $t$.

Then the rate of decomposition is $\frac{dx}{dt}$ which is proportional to $x$.

$\therefore \frac{dx}{dt} \propto x$

$\therefore \frac{dx}{dt} = -kx$, where $k > 0$

$\therefore \frac{1}{x} dx = -k\, dt$

On integrating, we get

$$\int \frac{1}{x} dx = -k \int dt$$

$\therefore \log x = -kt + c$

Let the original amount be $x_0$, i.e. $x = x_0$, when $t = 0$.

$\therefore \log x_0 = -k \times 0 + c$ $\quad \therefore c = \log x_0$

$\therefore \log x = -kt + \log x_0$

$\therefore \log x - \log x_0 = -kt$

$\therefore \log\left(\frac{x}{x_0}\right) = -kt$ ... (1)

But $p\%$ of the amount disappears in one year,

$\therefore$ when $t = 1$, $x = x_0 - p\%$ of $x_0$, i.e. $x = x_0 - \frac{px_0}{100}$

$$\therefore \log\left(\frac{x_0 - \frac{px_0}{100}}{x_0}\right) = -k \times 1$$

$$\therefore k = -\log\left(1 - \frac{p}{100}\right) = -\log\left(\frac{100-p}{100}\right)$$

$\therefore$ (1) becomes, $\log\left(\frac{x}{x_0}\right) = t \log\left(\frac{100-p}{100}\right)$

When $t = 2$, then

$$\log\left(\frac{x}{x_0}\right) = 2\log\left(\frac{100-p}{100}\right) = \log\left(\frac{100-p}{100}\right)^2$$

$$\therefore \frac{x}{x_0} = \left(\frac{100-p}{100}\right)^2$$

$$\therefore x = \left(\frac{100-p}{100}\right)^2 x_0 = \left(1 - \frac{p}{100}\right)^2 x_0$$

$$\therefore \% \text{ left after 2 years} = \frac{100 \times \left(1 - \frac{p}{100}\right)^2 x_0}{x_0}$$

$$= 100\left(1 - \frac{p}{100}\right)^2 = \left[10\left(1 - \frac{p}{100}\right)\right]^2$$

$$= \left(10 - \frac{p}{10}\right)^2$$

Hence, $\left(10 - \frac{p}{10}\right)^2 \%$ of the amount will be left after 2 years.

**6. A certain population of bacteria is known to grow at a rate proportional to the amount present in a culture that provides sufficient food and space. Initially there are 500 bacteria and after seven hours 800 bacteria are observed in the culture. Find an expression for the approximate number of bacteria present in the culture at any time $t$.**

**(Given : $\log_e 1{\cdot}6 = 0{\cdot}46998$)**

**Solution :** Let $x$ be the number of bacteria at time $t$.

Then the rate of growth is $\frac{dx}{dt}$ which is proportional to $x$.

$\therefore \frac{dx}{dt} \propto x$

$\therefore \frac{dx}{dt} = kx$, where $k$ is a constant.

$\therefore \frac{dx}{x} = k\, dt$

On integrating, we get

$$\int \frac{dx}{x} = k \int dt$$

$\therefore \log x = kt + c$

Initially, i.e. when $t = 0$, $x = 500$

$\therefore \log 500 = k \times 0 + c$ $\quad \therefore c = \log 500$

$\therefore \log x = kt + \log 500$

$\therefore \log x - \log 500 = kt$

$\therefore \log\left(\frac{x}{500}\right) = kt$ ... (1)

Now, when $t = 7$, $x = 800$

$\therefore \log\left(\frac{800}{500}\right) = 7k \quad \therefore k = \frac{1}{7}\log\left(\frac{8}{5}\right)$

$\therefore$ (1) becomes, $\log\left(\frac{x}{500}\right) = \frac{t}{7}\log\left(\frac{8}{5}\right) = \frac{t}{7}\log(1.6)$

$\therefore \log\left(\frac{x}{500}\right) = \frac{t}{7} \times 0.46998 = (0.06714)t$

$\therefore \frac{x}{500} = e^{(0.06714)t}$

$\therefore x = 500\,e^{(0.06714)t}$

This is the expression for the approximate number of bacteria in the culture at any time $t$.

## EXAMPLES FOR PRACTICE 8.6

1. The rate at which the population of a city increases varies as the population. Within a period of 30 years, the population grows from 20 lakhs to 40 lakhs. Show that the population will be 56.4 lakhs after a further period of 15 years.
2. Assume that the rate of growth of the population of a certain country varies as the existing population. If the population increased by 10 % in 10 years, in how many years will the population be doubled? (Given : $\log_{10}2 = 0.3010$, $\log_{10}1.1 = 0.0414$)
3. The rate of disintegration of a radioactive element at time $t$ is proportional to its mass at that time. The original mass of 800 gm will disintegrate into its mass of 400 gm after 5 days. Find the mass remaining after 30 days.
4. The growth of a population is proportional to the number present. If the population of a colony doubles in 50 years, in how many years will the population become triple?
5. Bacteria increase at the rate proportional to the number present. If the original number $N$ doubles in 3 hours, find in how many hours the number of bacteria will be $4N$?
6. The bacteria in a culture increases at the rate proportional to the number of bacteria present at any time $t$. The number of bacteria increases from 25 to 100 in 5 hours. Find in how many hours the number of bacteria will be 200?
7. If the bacteria in culture increase continuously at a rate proportional to the number present and the initial number is $N_0$, show that the number of bacteria at time $t$ is given by $N = N_0e^{kt}$.
8. The decay rate of radium is directly proportional to its amount at any given time. If the original quantity 12 grams reduces to 6 grams in 4 hours, find the amount left after another 4 hours.
9. Radium decomposes at a rate proportional to the amount present. If half the original amount disappears in 1600 years, find the percentage lost in 100 years. (Given $\log 2 = 0.6912$ and $e^{-0.0432} = 0.9576$)
10. The rate of reduction of a person's assets is proportional to the square root of the existing assets. If the assets dwindle from 25 lakhs to 6.25 lakhs in 2 years, in how many years the person be bankrupt?

**Answers**

**2.** 72.71 years **3.** 12.5 gm **4.** $50\left(\frac{\log 3}{\log 2}\right)$ years
**5.** 6 hours **6.** $7\frac{1}{2}$ hours **8.** 3 grams **9.** 4.24 %
**10.** 4 years.

## MISCELLANEOUS EXERCISE – 8

**(Textbook pages 171 to 173)**

**(I) Choose the correct option from the given alternatives :**

1. The order and degree of $\left(\frac{dy}{dx}\right)^3 - \frac{d^3y}{dx^3} + ye^x = 0$ are respectively
   (a) 3, 1 (b) 1, 3
   (c) 3, 3 (d) 1, 1
2. The order and degree of $\left[1 + \left(\frac{dy}{dx}\right)^3\right]^{\frac{2}{3}} = 8\frac{d^3y}{dx^3}$ are respectively
   (a) 3, 1 (b) 1, 3
   (c) 3, 3 (d) 1, 1
3. The differential equation of $y = k_1 + \frac{k_2}{x}$ is
   (a) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$ (b) $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$
   (c) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0$ (d) $x\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0$

4. The differential equation of $y=k_1e^x+k_2e^{-x}$ is

(a) $\frac{d^2y}{dx^2}-y=0$ (b) $\frac{d^2y}{dx^2}+\frac{dy}{dx}=0$

(c) $\frac{d^2y}{dx^2}+y\frac{dy}{dx}=0$ (d) $\frac{d^2y}{dx^2}+y=0$

[*Note : Option (a) is modified.*]

5. The solution of $\frac{dy}{dx}=1$ is

(a) $x+y=c$ (b) $xy=c$

(c) $x^2+y^2=c$ (d) $y-x=c$

6. The solution of $\frac{dy}{dx}+\frac{x^2}{y^2}=0$ is

(a) $x^3+y^3=7$ (b) $x^2+y^2=c$

(c) $x^3+y^3=c$ (d) $x+y=c$

7. The solution of $x\frac{dy}{dx}=y\log y$ is

(a) $y=ae^x$ (b) $y=be^{2x}$

(c) $y=be^{-2x}$ (d) $y=e^{ax}$

8. Bacterial increases at the rate proportional to the number present. If the original number M doubles in 3 hours, then the number of bacteria will be 4M in

(a) 4 hours (b) 6 hours

(c) 8 hours (d) 10 hours

9. The integrating factor of $\frac{dy}{dx}+y=e^{-x}$ is

(a) $x$ (b) $-x$ (c) $e^x$ (d) $e^{-x}$

10. The integrating factor of $\frac{dy}{dx}-y=e^x$ is $e^{-x}$, then its solution is

(a) $ye^{-x}=x+c$ (b) $ye^x=x+c$

(c) $ye^x=2x+c$ (d) $ye^{-x}=2x+c$

[*Note : Question is modified.*]

### Answers

1. (a) 3, 1
2. (c) 3, 3
3. (b) $x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0$
4. (a) $\frac{d^2y}{dx^2}-y=0$
5. (d) $y-x=c$
6. (c) $x^3+y^3=c$
7. (d) $y=e^{ax}$
8. (b) 6 hours
9. (c) $e^x$
10. (a) $ye^{-x}=x+c$.

## (II) Fill in the blanks :

1. The order of highest derivative occurring in the differential equation is called ......... of the differential equation.
2. The power of the highest ordered derivative when all the derivatives are made free from negative and/or fractional indices if any is called ......... of the differential equation.
3. A solution of differential equation which can be obtained from the general solution by giving particular values to the arbitrary constants is called ......... solution.
4. Order and degree of a differential equation are always ......... integers.
5. The integrating factor of the differential equation $\frac{dy}{dx}-y=x$ is .........
6. The differential equation by eliminating arbitrary constants from $bx+ay=ab$ is .........

### Answers

1. order
2. degree
3. particular
4. positive
5. $e^{-x}$
6. $\frac{d^2y}{dx^2}=0$

## (III) State whether each of the following is *True* or *False* :

1. The integrating factor of the differential equation $\frac{dy}{dx}-y=x$ is $e^{-x}$.
2. Order and degre of a differential equation are always positive integers.
3. The degree of a differential equation is the power of the highest ordered derivative when all the derivatives are made free from negative and/or fractional indices if any.
4. The order of highest derivative occurring in the differential equation is called degree of the differential equation.

5. The power of the highest ordered derivative when all the derivatives are made free from negative and/or fractional indices if any is called order of the differential equation.

6. The degree of the differential equation

$e^{\frac{dy}{dx}} = \frac{dy}{dx} + c$ is not defined.

**Answers**

1. True 2. True 3. True 4. False
5. False 6. True.

**(IV) Solve the following :**

**1. Find the order and degree of the following differential equations :**

(i) $\left[\frac{d^3y}{dx^3} + x\right]^{\frac{3}{2}} = \frac{d^2y}{dx^2}$

(ii) $x + \frac{dy}{dx} = 1 + \left(\frac{dy}{dx}\right)^2$.

**Solution :**

(i) The given differential equation is

$$\left[\frac{d^3y}{dx^3} + x\right]^{\frac{3}{2}} = \left(\frac{d^2y}{dx^2}\right)$$

$$\therefore \left[\frac{d^3y}{dx^3} + x\right]^3 = \left(\frac{d^2y}{dx^2}\right)^2$$

This D.E. has highest order derivative $\frac{d^3y}{dx^3}$ with power 3

$\therefore$ order = **3** and degree = **3**

(ii) The given differential equation is

$$x + \frac{dy}{dx} = 1 + \left(\frac{dy}{dx}\right)^2$$

This D.E. has highest order derivative $\frac{dy}{dx}$ with power 2.

$\therefore$ order = **1**, degree = **2**.

[*Note : The answer to degree in the textbook is incorrect.*]

**2. Verify that $y = \log x + c$ is a solution of the differential equation $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$.**

**Solution :** $y = \log x + c$

Differentiating both sides w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{1}{x} + 0 = \frac{1}{x} \quad \therefore x\frac{dy}{dx} = 1$$

Differentiating again w.r.t. $x$, we get

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} \times 1 = 0$$

$$\therefore x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$$

This shows that $y = \log x + c$ is a solution of the D.E. $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$.

**3. Solve the following differential equations :**

**(i)** $\frac{dy}{dx} = 1 + x + y + xy$

**Solution :** $\frac{dy}{dx} = 1 + x + y + xy$

$$\therefore \frac{dy}{dx} = (1 + x) + y(1 + x) = (1 + x)(1 + y)$$

$$\therefore \frac{1}{1+y}dy = (1 + x)\,dx$$

Integrating, we get

$$\int \frac{1}{1+y}dy = \int (1 + x)\,dx$$

$$\therefore \log|1 + y| = x + \frac{x^2}{2} + c$$

This is the general solution.

**(ii)** $e^{dy/dx} = x$

**Solution :** $e^{dy/dx} = x \quad \therefore \frac{dy}{dx} = \log x$

$\therefore dy = \log x\,dx \quad \therefore \int 1\,dy = \int \log x\,dx$

Now $\int \log x\,dx = \int (\log x)(1)\,dx$ ... (1)

$= (\log x)\int 1\,dx - \int\left[\frac{d}{dx}(\log x)\cdot\int 1\,dx\right]dx$

$= (\log x)(x) - \int \frac{1}{x}\cdot x\,dx = x\log x - \int 1\,dx$

$= x\log x - x$

$\therefore$ from (1), the general solution is

$y = x\log x - x + c$, i.e. $y = x(\log x - 1) + c$.

**(iii)** $dr = ar\,d\theta - \theta\,dr$

**Solution :** $dr = ar\,d\theta - \theta\,dr$

$\therefore dr + \theta\,dr = ar\,d\theta$

$\therefore\ (1+\theta)\,dr = ar\,d\theta$

$\therefore\ \dfrac{dr}{r} = \dfrac{a\,d\theta}{1+\theta}$

On integrating, we get

$\int \dfrac{dr}{r} = a\int \dfrac{d\theta}{1+\theta}$

$\therefore\ \log|r| = a\log|1+\theta| + c$

This is the general solution.

**(iv) Find the differential equation of the family of curves $y = e^x(ax+bx^2)$, where $a$ and $b$ are arbitrary constants.**

**Solution :** $y = e^x(ax+bx^2)$

$\therefore\ ax+bx^2 = ye^{-x}$ ... (1)

Differentiating (1) w.r.t. $x$ twice and writing $\dfrac{dy}{dx}$ as $y_1$ and $\dfrac{d^2y}{dx^2}$ as $y_2$, we get

$a+2bx = y(-e^{-x}) + e^{-x}y_1$

$\therefore\ a+2bx = e^{-x}(y_1-y)$ ... (2)

and $a(0)+2b\times 1 = e^{-x}(y_2-y_1)+(y_1-y)(-e^{-x})$

$\therefore\ a\cdot 0+2b = e^{-x}(y_2-2y_1+y)$ ... (3)

Eliminating $a$ and $b$ from (1), (2), (3), we get

$$\begin{vmatrix} x & x^2 & e^{-x}y \\ 1 & 2x & e^{-x}(y_1-y) \\ 0 & 2 & e^{-x}(y_2-2y_1+y) \end{vmatrix} = 0$$

$$\therefore\ e^{-x}\begin{vmatrix} x & x^2 & y \\ 1 & 2x & y_1-y \\ 0 & 2 & y_2-2y_1+y \end{vmatrix} = 0$$

... [Taking $e^{-x}$ common from $C_3$]

$$\therefore\ \begin{vmatrix} x & x^2 & y \\ 1 & 2x & y_1-y \\ 0 & 2 & y_2-2y_1+y \end{vmatrix} = 0 \quad \text{... } [\because e^{-x}\neq 0]$$

$\therefore\ x[2x(y_2-2y_1+y)-2(y_1-y)] - x^2[y_2-2y_1+y-0]+y[2-0]=0$

$\therefore\ 2x^2y_2-4x^2y_1+2x^2y-2xy_1+2xy - x^2y_2+2x^2y_1-x^2y+2y=0$

$\therefore\ x^2y_2-2x^2y_1+x^2y-2xy_1+2xy+2y=0$

i.e. $x^2\dfrac{d^2y}{dx^2}-2x^2\dfrac{dy}{dx}-2x\dfrac{dy}{dx}+x^2y+2xy+2y=0$

This is the required differential equation.

**4. Solve $\dfrac{dy}{dx}=\dfrac{x+y+1}{x+y-1}$ when $x=\dfrac{2}{3}$ and $y=\dfrac{1}{3}$.**

**Solution :** $\dfrac{dy}{dx}=\dfrac{x+y+1}{x+y-1}$ ... (1)

Put $x+y=v$ $\quad\therefore\ 1+\dfrac{dy}{dx}=\dfrac{dv}{dx}$

$\therefore\ \dfrac{dy}{dx}=\dfrac{dv}{dx}-1$

$\therefore$ (1) becomes, $\dfrac{dv}{dx}-1=\dfrac{v+1}{v-1}$

$\therefore\ \dfrac{dv}{dx}=\dfrac{v+1}{v-1}+1=\dfrac{v+1+v-1}{v-1}$

$\therefore\ \dfrac{dv}{dx}=\dfrac{2v}{v-1}$

$\therefore\ \dfrac{v-1}{v}dv=2dx$

Integrating, we get

$\int\dfrac{v-1}{v}dv=2\int dx$

$\therefore\ \int\left(1-\dfrac{1}{v}\right)dv=2\int dx+c$

$\therefore\ v-\log|v|=2x+c$

$\therefore\ x+y-\log|x+y|=2x+c$

$\therefore\ \log|x+y|=y-x-c$

This is the general solution.

When $x=\dfrac{2}{3}$ and $y=\dfrac{1}{3}$, we get

$\log\left|\dfrac{2}{3}+\dfrac{1}{3}\right|=\dfrac{1}{3}-\dfrac{2}{3}-c$

$\therefore\ \log 1=-\dfrac{1}{3}-c$

$\therefore\ 0=-\dfrac{1}{3}-c \qquad \therefore\ c=-\dfrac{1}{3}$

$\therefore$ the particular solution is

$\log|x+y|=y-x+\dfrac{1}{3}$.

**5. Solve : $y\,dx-x\,dy=-\log x\,dx$.**

**Solution :** $y\,dx-x\,dy=-\log x\,dx$

$\therefore\ y\,dx-x\,dy+\log x\,dx=0$

$\therefore\ x\,dy=(y+\log x)\,dx$

$\therefore\ \dfrac{dy}{dx}=\dfrac{y+\log x}{x}=\dfrac{y}{x}+\dfrac{\log x}{x}$

$\therefore\ \dfrac{dy}{dx}-\dfrac{1}{x}\cdot y=\dfrac{\log x}{x}$ ... (1)

This is the linear differential equation of the form

$\frac{dy}{dx} + P \cdot y = Q$, where $P = -\frac{1}{x}$ and $Q = \frac{\log x}{x}$

$\therefore$ I.F. $= e^{\int P\,dx} = e^{\int -\frac{1}{x}dx} = e^{-\log x}$

$= e^{\log (x)^{-1}} = \frac{1}{x}$

$\therefore$ the solution of (1) is given by

$y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.})\,dx + c$

$\therefore y \cdot \frac{1}{x} = \int \frac{\log x}{x} \cdot \frac{1}{x}dx + c$

$\therefore \frac{y}{x} = \int \frac{\log x}{x^2}dx + c$

$\therefore \frac{y}{x} = (\log x)\int x^{-2}\,dx - \int \left[\frac{d}{dx}(\log x)\int x^{-2}\,dx\right]dx + c$

$\therefore \frac{y}{x} = (\log x) \cdot \frac{x^{-1}}{-1} - \int \frac{1}{x} \cdot \frac{x^{-1}}{-1}dx + c$

$\therefore \frac{y}{x} = -\frac{\log x}{x} + \int x^{-2}\,dx + c$

$\therefore \frac{y}{x} = -\frac{\log x}{x} + \frac{x^{-1}}{-1} + c$

$\therefore \frac{y}{x} = -\frac{\log x}{x} - \frac{1}{x} + c$

$\therefore y = -\log x - 1 + cx$

$\therefore y = cx - (1 + \log x)$

This is the general solution.

[*Note : Answer in the textbook is incorrect.*]

**6. Solve : $y \log y \cdot \frac{dx}{dy} + x - \log y = 0$.**

**Solution :** $y \log y \cdot \frac{dx}{dy} + x - \log y = 0$

$\therefore y \log y \cdot \frac{dx}{dy} = \log y - x$

$\therefore \frac{dx}{dy} = \frac{\log y - x}{y \log y}$

$\therefore \frac{dx}{dy} = \frac{1}{y} - \frac{x}{y \log y}$

$\therefore \frac{dx}{dy} + \frac{x}{y \log y} = \frac{1}{y}$ ... (1)

This is the linear differential equation of the form

$\frac{dx}{dy} + Px = Q$, where $P = \frac{1}{y \log y}$ and $Q = \frac{1}{y}$

$\therefore$ I.F. $= e^{\int P\,dy} = e^{\int \frac{1}{y \log y}dy}$

$= e^{\int \frac{(1/y)}{\log y}dy} = e^{\log|\log y|} = \log y$

$\therefore$ the solution of (1) is given by

$x \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.})\,dy + c_1$

$\therefore x \cdot \log y = \int \frac{1}{y} \cdot \log y\,dy + c_1$

$\therefore (\log y) \cdot x = \int \frac{\log y}{y}dy + c_1$

Put $\log y = t$ $\quad \therefore \frac{1}{y}dy = dt$

$\therefore (\log y) \cdot x = \int t\,dt + c_1$

$\therefore x \log y = \frac{t^2}{2} + c_1$

$\therefore x \log y = \frac{1}{2}(\log y)^2 + c_1$

$\therefore 2x \log y = (\log y)^2 + c$, where $c = 2c_1$

This is the general solution.

**7. Solve : $(x + y)\,dy = a^2 dx$.**

**Solution :** $(x + y)\,dy = a^2 dx$

$\therefore \frac{dy}{dx} = \frac{a^2}{x + y}$ ... (1)

Put $x + y = v$ $\quad \therefore 1 + \frac{dy}{dx} = \frac{dv}{dx}$

$\therefore \frac{dy}{dx} = \frac{dv}{dx} - 1$

$\therefore$ (1) becomes, $\frac{dv}{dx} - 1 = \frac{a^2}{v}$

$\therefore \frac{dv}{dx} = \frac{a^2}{v} + 1 = \frac{a^2 + v}{v}$

$\therefore \frac{v}{a^2 + v}dv = dx$

Integrating, we get

$\int \frac{v}{a^2 + v}dv = \int dx$

$\therefore \int \frac{(a^2 + v) - a^2}{a^2 + v}dv = \int dx$

$\therefore \int \left(1 - \frac{a^2}{a^2 + v}\right)dv = \int dx$

$\therefore \int 1\,dv - a^2 \int \frac{1}{a^2 + v}dv = \int dx$

$\therefore v - a^2 \log|a^2 + v| = x + c_1$

$\therefore\ x+y-a^2\log|a^2+x+y|=x+c_1$

$\therefore\ a^2\log|x+y+a^2|=y-c_1$

$\therefore\ \log|x+y+a^2|=\frac{y-c_1}{a^2}=\frac{y}{a^2}-\frac{c_1}{a^2}$

$\therefore\ x+y+a^2=e^{\left(\frac{y}{a^2}-\frac{c_1}{a^2}\right)}=e^{\frac{y}{a^2}}\cdot e^{\frac{-c_1}{a^2}}$

$\therefore\ x+y+a^2=c\cdot e^{y/a^2}$, where $c=e^{-c_1/a^2}$

This is the general solution.

[*Note : Answer in the textbook is incorrect.*]

**8. Solve : $\frac{dy}{dx}+\frac{2}{x}y=x^2$.**

**Solution :** $\frac{dy}{dx}+\frac{2}{x}y=x^2$ ... (1)

This is a linear differential equation of the form

$\frac{dy}{dx}+Py=Q$, where $P=\frac{2}{x}$, $Q=x^2$

$\therefore$ I.F. $=e^{\int P\,dx}=e^{2\int\frac{1}{x}dx}$

$=e^{2\log x}=e^{\log x^2}=x^2$

$\therefore$ the solution of (1) is given by

$y\cdot(\text{I.F.})=\int Q(\text{I.F.})\,dx+c_1$

$\therefore\ yx^2=\int x^2\cdot x^2\,dx+c_1$

$=\int x^4\,dx+c_1$

$\therefore\ yx^2=\frac{x^5}{5}+c_1$

$\therefore\ 5x^2y=x^5+c$, where $c=5c_1$

This is the general solution.

**9. The rate of growth of population is proportional to the number present. If the population doubled in the last 25 years and the present population is 1 lakh, when will the city have population of 400000?**

**Solution :** Let $P$ be the population at time $t$ years.

Then rate of growth of the population is $\frac{dP}{dt}$ which is proportional to $P$.

$\therefore\ \frac{dP}{dt}\propto P$

$\therefore\ \frac{dP}{dt}=kP$, where $k$ is a constant

$\therefore\ \frac{dP}{P}=k\,dt$

On integrating, we get

$\int\frac{dP}{P}=k\int dt$

$\therefore\ \log P=kt+c$

The population doubled in 25 years and present population is 1,00,000.

$\therefore$ initial population was 50,000

i.e. when $t=0$, $P=50000$

$\therefore\ \log 50000=k\times 0+c \qquad \therefore\ c=\log 50000$

$\therefore\ \log P=kt+\log 50000$

When $t=25$, $P=100000$

$\therefore\ \log 100000=k\times 25+\log 50000$

$\therefore\ 25k=\log 100000-\log 50000=\log\left(\frac{100000}{50000}\right)$

$\therefore\ k=\frac{1}{25}\log 2$

$\therefore\ \log P=\frac{t}{25}\log 2+\log 50000$

If $P=400000$, then

$\log 400000=\frac{t}{25}\log 2+\log 50000$

$\therefore\ \log 400000-\log 50000=\frac{t}{25}\log 2$

$\therefore\ \log\left(\frac{400000}{50000}\right)=\log(2)^{t/25}$

$\therefore\ \log 8=\log(2)^{t/25} \qquad \therefore\ 8=(2)^{t/25}$

$\therefore\ (2)^{t/25}=(2)^3$

$\therefore\ \frac{t}{25}=3 \qquad \therefore\ t=75$

$\therefore$ the population will be 400000 in $(75-25)=50$ years.

[*Note : Question is modified.*]

**10. The resale value of a machine decreases over a 10 years period at a rate that depends on the age of the machine. When the machine is $x$ years old, the rate at which its value is changing is ₹ 2200 $(x-10)$ per year. Express the value of the machine as a function of its age and initial value. If the machine was originally worth ₹ 1,20,000 how much will it be worth when it is 10 years old?**

**Solution :** Let $V$ be the value of the machine after $x$ years.

Then rate of change of the value is $\frac{dV}{dx}$ which is $2200(x-10)$.

$\therefore \ \dfrac{dV}{dx} = 2200\,(x-10)$

$\therefore \ dV = 2200(x-10)\,dx$

On integrating, we get

$\int dV = 2200 \int (x-10)\,dx$

$\therefore \ V = 2200\left[\dfrac{x^2}{2} - 10x\right] + c$

Initially, i.e. at $x = 0$, $V = 120000$

$\therefore \ 120000 = 2200 \times 0 + c = c$

$\therefore \ c = 120000$

$\therefore \ V = 2200\left[\dfrac{x^2}{2} - 10x\right] + 120000 \qquad \ldots (1)$

This gives value of the machine in terms of initial value and age $x$.

We have to find $V$ when $x = 10$.

When $x = 10$, from (1)

$V = 2200\left[\dfrac{100}{2} - 100\right] + 120000$

$= 2200[-50] + 120000$

$= -110000 + 120000$

$= 10000$

Hence, the value of the machine after 10 years will be ₹ 10000.

**11. Solve : $y^2 dx + (xy + x^2) dy = 0$.**

**Solution :** *Refer to the solution of Q. 2. of Exercise 8.4.*

**Ans.** $xy^2 = c^2 (x + 2y)$.

**12. Solve : $x^2 y\, dx - (x^3 + y^3)\, dy = 0$.**

**Solution :** *Refer to the solution of Q. 3. of Exercise 8.4.*

**Ans.** $\log|y| - \dfrac{x^3}{2y^3} = c$

**13. Solve : $yx\dfrac{dy}{dx} = x^2 + 2y^2$**

**Solution :** *Refer to the solution of Q. 6. of Exercise 8.4.*

**Ans.** $x^2 + y^2 = cx^4$.

**14. Solve : $(x + 2y^3)\dfrac{dy}{dx} = y$.**

**Solution :** $(x + 2y^3)\dfrac{dy}{dx} = y$

$\therefore \ x + 2y^3 = y\dfrac{dx}{dy}$

$\therefore \ \dfrac{x}{y} + 2y^2 = \dfrac{dx}{dy}$

$\therefore \ \dfrac{dx}{dy} - \dfrac{x}{y} = 2y^2 \qquad \ldots (1)$

This is a linear differential equation of the form

$\dfrac{dx}{dy} + Px = Q$, where $P = -\dfrac{1}{y}$, $Q = 2y^2$

$\therefore \ \text{I.F.} = e^{\int P\,dy} = e^{\int -\frac{1}{y}dy} = e^{-\int \frac{1}{y}dy}$

$= e^{-\log y} = e^{\log\left(\frac{1}{y}\right)} = \dfrac{1}{y}$

$\therefore$ the solution of (1) is given by

$x \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.})\,dy + c$

$\therefore \ \dfrac{x}{y} = \int 2y^2 \cdot \dfrac{1}{y}\,dy + c = 2\int y\,dy + c$

$= 2\left(\dfrac{y^2}{2}\right) + c$

$\therefore \ x = y(y^2 + c)$

This is the general solution.

**15. Solve : $y\,dx - x\,dy + \log x\,dx = 0$.**

**Solution :** $y\,dx - x\,dy + \log x\,dx = 0$

$\therefore \ (y + \log x)\,dx = x\,dy$

$\therefore \ \dfrac{y + \log x}{x} = \dfrac{dy}{dx}$

$\therefore \ \dfrac{y}{x} + \dfrac{\log x}{x} = \dfrac{dy}{dx}$

$\therefore \ \dfrac{dy}{dx} - \dfrac{1}{x}\cdot y = \dfrac{\log x}{x} \qquad \ldots (1)$

This is a linear differential equation of the form

$\dfrac{dy}{dx} + Py = Q$, where $P = -\dfrac{1}{x}$, $Q = \dfrac{\log x}{x}$

$\therefore \ \text{I.F.} = e^{\int P\,dx} = e^{\int -\frac{1}{x}dx} = e^{-\int \frac{1}{x}dx}$

$= e^{-\log x} = e^{\log\left(\frac{1}{x}\right)} = \dfrac{1}{x}$

$\therefore$ the solution of (1) is given by

$y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.})\,dx + c$

$\therefore \ \dfrac{y}{x} = \int \dfrac{\log x}{x} \cdot \dfrac{1}{x}\,dx + c$

$= \int \log x \cdot x^{-2}\,dx + c$

$= (\log x)\int x^{-2}\,dx - \int\left[\left\{\dfrac{d}{dx}(\log x)\right\}\int x^{-2}\,dx\right]dx$

$$= (\log x)\left(\frac{x^{-1}}{-1}\right) - \int \frac{1}{x}\cdot\left(\frac{x^{-1}}{-1}\right)dx + c$$

$$= -\frac{\log x}{x} + \int x^{-2}\,dx + c$$

$$\therefore \frac{y}{x} = -\frac{\log x}{x} - \frac{1}{x} + c$$

$$\therefore y = cx - (1 + \log x)$$

This is the general solution.

**16. Solve : $\frac{dy}{dx} = \log x$.**

**Solution :** $\frac{dy}{dx} = \log x$

$\therefore dy = \log x\,dx$

On integrating, we get

$\int dy = \int \log x \cdot 1\,dx$

$$\therefore y = (\log x)\int 1\,dx - \int\left[\left\{\frac{d}{dx}(\log x)\right\}\cdot\int 1\,dx\right]dx$$

$$= (\log x)\cdot x - \int \frac{1}{x}\cdot x\,dx$$

$$= x\log x - \int 1\,dx$$

$$\therefore y = x\log x - x + c$$

This is the general solution.

**17. $y\log y\frac{dx}{dy} = \log y - x$.**

**Solution :** *Refer to the solution of Q. 6.*

**Ans.** $x\log y = \frac{1}{2}(\log y)^2 + c$.

[*Note : Question is modified.*]

## ACTIVITIES | Textbook page 173

**Complete the following activities :**

**1.** The equation $\frac{dy}{dx} - y = 2x$ is of the form

$\boxed{\frac{dy}{dx} + Py = Q}$,

where $P = \boxed{-1}$ and $Q = \boxed{2x}$

$\therefore$ I.F. $= e^{\int P\,dx} = \boxed{e^{-x}}$

$\therefore$ the solution of the linear D.E. is

$$y\,\boxed{\text{I.F.}} = \int 2x\,(\text{I.F.})\,dx + c$$

$$\therefore ye^{-x} = \int 2x\,\boxed{e^{-x}}\,dx + c$$

$$\therefore ye^{-x} = 2\int x\,\boxed{e^{-x}}\,dx + c$$

$$= 2\left\{x\int e^{-x}\,dx - \int\boxed{e^{-x}}\,dx + \frac{d}{dx}\boxed{x}\,dx\right\} + c$$

$$= 2\left\{x\frac{e^{-x}}{\boxed{(-1)}} - \int\frac{e^{-x}}{\boxed{(-1)}}\,dx\right\} + c$$

$$\therefore ye^{-x} = -2xe^{-x} + 2\int\boxed{e^{-x}}\,dx + c$$

$$= -2xe^{-x} + 2\boxed{\frac{e^{-x}}{-1}} + c$$

$y + \boxed{2x} + \boxed{2} = ce^{x}$ is the required general solution of the differential equation.

**2. Verify that $y = a + \frac{b}{x}$ is a solution of $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$.**

**Solution :** $y = a + \frac{b}{x}$

$$\therefore \frac{dy}{dx} = \boxed{-\frac{b}{x^2}}$$

$$\frac{d^2y}{dx^2} = \boxed{\frac{2b}{x^3}}$$

Consider $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx}$

$$= x\frac{2b}{x^3} + 2\boxed{-\frac{b}{x^2}}$$

$$= \boxed{0}$$

$\therefore y = a + \frac{b}{x}$ is a solution of $\boxed{x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0}$

## ACTIVITIES FOR PRACTICE

**Complete the following activities :**

**1. Find the order and degree of the D.E.**

$$\left[\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2}\right]^{\frac{5}{3}} = \left(\frac{d^4y}{dx^4}\right)^2$$

**Solution :** The given D.E. is

$\boxed{\phantom{xx}} = \boxed{\phantom{xx}}$

On cubing both the sides, we get

$$\left(\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2}\right)^{\square} = \left(\frac{d^4y}{dx^4}\right)^{\square}$$

$\therefore$ it follows that the D.E. has order $\boxed{\phantom{x}}$ and degree $\boxed{\phantom{x}}$.

**2. Verify that $y = ae^{2x} + be^{-2x}$ is a solution of the D.E.**

$$\frac{d^2y}{dx^2} = 4y.$$

**Solution :** $y = ae^{2x} + be^{-2x}$

$\therefore \frac{dy}{dx} = \square ae^{2x} + (-2)b\square$ and

$\frac{d^2y}{dx^2} = 4\square + 4\square$

$= 4\square$

$= 4y$

$\therefore y = ae^{2x} + be^{-2x}$ is a solution of the D.E. $\square$

**3. Form the D.E. by the eliminating arbitrary constants $a, b, c$ from $y = a + bx + cx^2$.**

**Solution :** $y = a + bx + cx^2$

$\therefore \frac{dy}{dx} = \square + b\square + c\square$

$\therefore \frac{d^2y}{dx^2} = \square + \square c$

$\therefore \frac{d^3y}{dx^3} = \square$

$\therefore \frac{d^3y}{dx^3} = \square$ is the required D.E.

**4. Solve : $x\,dy + y\,dx = 0$ given that $x = 3$, when $y = 2$.**

**Solution :** $x\,dy + y\,dx = 0$

$\therefore \frac{dy}{\square} + \frac{\square}{x} = 0$

On integrating, we get

$\int \frac{1}{\square} dy + \int \frac{1}{x} \square = \log c$

$\therefore \log \square + \log x = \log c$

$\therefore \log \square = \log \square$

$\therefore x\square = c$ is the general solution

When $x = 3, y = 2$

$\therefore \square \times \square = c$

$\therefore \square = 6$ is the required solution.

## OBJECTIVE SECTION

### MULTIPLE CHOICE QUESTIONS

**Select and write the correct answer from the given alternatives in each of the following questions :**

1. The order and degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^{\frac{1}{6}} - \left(\frac{dy}{dx}\right)^{\frac{1}{3}} = 0$ are respectively

(a) 3, 2 (b) 2, 3 (c) 6, 3 (d) 3, 1

2. The order and degree of the differential equation $\left[1 + \left(\frac{dy}{dx}\right)^3\right]^{\frac{7}{3}} = 7\frac{d^2y}{dx^2}$ are respectively

(a) 2, 3 (b) 3, 2 (c) 7, 2 (d) 3, 7

3. The order and degree of the differential equation $\sqrt{1 + \frac{1}{\left(\frac{dy}{dx}\right)^2}} = \left(\frac{d^2y}{dx^2}\right)^{\frac{3}{2}}$ are respectively

(a) 2, 1 (b) 2, 3 (c) 1, 2 (d) 2, 2

4. The differential equation of $y = c^2 + \frac{c}{x}$ is

(a) $x^4\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} = y$ (b) $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

(c) $x^3\left(\frac{dy}{dx}\right)^2 + x\frac{dy}{dx} = y$ (d) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0$

5. The differential equation of $y = A(x - A)^2$ is

(a) $\left(\frac{dy}{dx}\right)^2 = x\frac{dy}{dx} - 2y$

(b) $\left(\frac{dy}{dx}\right)^2 = 4y\left(x\frac{dy}{dx} - 2y\right)$

(c) $\left(\frac{dy}{dx}\right)^3 + 4xy + 8y^2 = 0$

(d) $\left(\frac{dy}{dx}\right)^3 = 4y\left[x\frac{dy}{dx} - 2y\right]$

6. The differential equation of $(x - a)^2 + y^2 = a^2$ is

(a) $2xy\frac{dy}{dx} = x^2 - y^2$ (b) $2xy\frac{dy}{dx} = y^2 - x^2$

(c) $2xy\frac{dy}{dx} = x^2 + y^2$ (d) $2xy\frac{dy}{dx} + x^2 + y^2 = 0$

7. The differential equation of $xy = Ae^x + Be^{-x} + x^2$ is

(a) $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy + x^2 - 2 = 0$

(b) $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + xy - x^2 + 2 = 0$

(c) $x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=y$

(d) $x\frac{d^2y}{dx^2}+2\frac{dy}{dx}-xy-x^2-2=0$

**8.** The differential equation of the family of curves $y=c_1e^{5x}+c_2e^{-5x}$ is

(a) $\frac{d^2y}{dx^2}+25y=0$ (b) $\frac{d^2y}{dx^2}-25y=0$

(c) $\frac{d^2y}{dx^2}+5=0$ (d) $\frac{d^2y}{dx^2}-5=0$

**9.** $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is a solution of

(a) $xy\frac{dy}{dx}+y\frac{d^2y}{dx^2}=0$

(b) $xy\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^2-y\frac{dy}{dx}=0$

(c) $\frac{d^2y}{dx^2}+yx+\left(\frac{dy}{dx}\right)^2=0$

(d) $y\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2+y=0$

**10.** $x^2+y^2=r^2$ is a solution of

(a) $y=x\frac{dy}{dx}+r\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

(b) $y=x\sqrt{1+\left(\frac{dy}{dx}\right)^2}+r^2y$

(c) $\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0$

(d) $\frac{d^2y}{dx^2}=(x+1)\frac{dy}{dx}$

**11.** $y=ae^x+be^{-3x}$ is a solution of

(a) $\frac{d^2y}{dx^2}+y=0$ (b) $\frac{d^2y}{dx^2}+xy\frac{dy}{dx}+y=0$

(c) $\frac{d^2y}{dx^2}+2\frac{dy}{dx}-3y=0$ (d) $\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$

**12.** The solution of $\log\frac{dy}{dx}=ax+by$ is

(a) $ae^{ax}-be^{-bx}=c$ (b) $e^{ax}-e^{-by}=c$

(c) $ae^{ax}+\frac{e^{by}}{b}=c$ (d) $\frac{e^{ax}}{a}+\frac{e^{-by}}{b}=c$

**13.** The solution of differential equation $y-x\frac{dy}{dx}=3\left(1+x^2\frac{dy}{dx}\right)$ is

(a) $(y-3)(3x+1)=cx$

(b) $(y-3)+(3x+1)=cx$

(c) $y-3=c(3x+1)$

(d) $(y-3)-(3x-1)=cx$

**14.** The solution of $2xy\frac{dy}{dx}=1-x^2+y^2-x^2y^2$ is

(a) $\log(1+y^2)=\log|x|+\frac{x^2}{2}+c$

(b) $\log(1+y^2)=\log|x|-\frac{x^2}{2}+c$

(c) $\log(1+y^2)=\frac{1}{2}\log|x|-\frac{x^2}{4}+c$

(d) $\log(1+y^2)=\frac{x^2}{2}-\log|x|+c$

**15.** The particular solution of $\frac{dy}{dx}=xe^{y-x}$, when $x=0$, $y=0$ is

(a) $e^{x+y}=x+1$ (b) $e^x+e^y=x+1$

(c) $e^{x-y}=x+1$ (d) $e^{x-y}=x-1$

**16.** The particular solution of $y(1+\log x)\frac{dx}{dy}-x\log x=0$, when $x=e$, $y=e^2$ is

(a) $x\log x=y$ (b) $x\log x=ey$

(c) $ex\log x=y$ (d) $x\log x=e+y$

**17.** The solution of $x^2y\,dx-(x^3+y^3)\,dy=0$ is

(a) $\log|y|-\frac{x^3}{2y^3}=c$ (b) $\log|y|+\frac{x^3}{2y^3}=c$

(c) $\log|y|-\frac{x^3}{4y^3}=c$ (d) $\log|y|-\frac{x^3}{y^3}=c$

**18.** The solution of $(x+2y+1)\,dx-(2x+4y+3)\,dy=0$ is

(a) $\log|4x+8y+5|=4x-8y+c$

(b) $\log|4x+8y+5|=4x+8y+c$

(c) $\log|4x+8y+5|=8x-4y+c$

(d) $\log|4x+8y+5|=8x+4y+c$

**19.** The solution of $\frac{dy}{dx}-y=e^x$, $y(0)=1$, is

(a) $y=(x+1)e^x$ (b) $y=(x-1)e^x$

(c) $y=(x^2+1)e^x$ (d) $y=(1-x)e^x$

**20.** The general solution of the differential equation $e^xdy+(ye^x+2x)\,dx=0$ is

(a) $ye^y+x^2=c$ (b) $xe^y+y^2=c$

(c) $ye^x+x^2=c$ (d) $ye^y-x^2=c$

21. The solution of $(x+2y^3)dy = y\,dx$ is
(a) $x = y^2 + cy$ (b) $x^2 = y^3 + cy$
(c) $x = y^3 + c$ (d) $x = y^3 + cy$

22. Integrating factor of linear differential equation $x\dfrac{dy}{dx} + 2y = x^2 \log x$ is
(a) $\dfrac{1}{x^2}$ (b) $\dfrac{1}{x}$ (c) $x$ (d) $x^2$

23. The integrating factor of $y\log y \cdot \dfrac{dx}{dy} + x - \log y = 0$ is
(a) $\dfrac{1}{y}$ (b) $\log y$ (c) $2\log y$ (d) $y\log y$

24. Bacteria increases at the rate proportional to the number present. If the original number $N$ doubles in 3 hours, then the number of bacteria will be $4N$ in
(a) 4 hours (b) 5 hours
(c) 8 hours (d) 6 hours

25. The decay rate of certain substance is directly proportional to the amount present at that instant. Initially there are 27 grams of the substance and three hours later it is found that 8 grams are left. The amount left after one more hour is
(a) $5\frac{1}{3}$ grams (b) $5\frac{2}{3}$ grams
(c) 5 grams (d) 5.1 grams

## Answers

1. (d) 3, 1
2. (a) 2, 3
3. (b) 2, 3
4. (a) $x^4\left(\dfrac{dy}{dx}\right)^2 - x\dfrac{dy}{dx} = y$
5. (d) $\left(\dfrac{dy}{dx}\right)^3 = 4y\left[x\dfrac{dy}{dx} - 2y\right]$
6. (b) $2xy\dfrac{dy}{dx} = y^2 - x^2$
7. (a) $x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - xy + x^2 - 2 = 0$
8. (b) $\dfrac{d^2y}{dx^2} - 25y = 0$
9. (b) $xy\dfrac{d^2y}{dx^2} + 2\left(\dfrac{dy}{dx}\right)^2 - y\dfrac{dy}{dx} = 0$
10. (a) $y = x\dfrac{dy}{dx} + r\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$
11. (c) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - 3y = 0$
12. (d) $\dfrac{e^{ax}}{a} + \dfrac{e^{-by}}{b} = c$
13. (a) $(y-3)(3x+1) = cx$
14. (b) $\log(1+y^2) = \log|x| - \dfrac{x^2}{2} + c$
15. (c) $e^{x-y} = x + 1$
16. (c) $ex\log x = y$
17. (a) $\log|y| - \dfrac{x^3}{2y^3} = c$
18. (a) $\log|4x + 8y + 5| = 4x - 8y + c$
19. (a) $y = (x+1)e^x$
20. (c) $ye^x + x^2 = c$
21. (d) $x = y^3 + cy$
22. (d) $x^2$
23. (b) $\log y$
24. (d) 6 hours
25. (a) $5\frac{1}{3}$ grams.

## TRUE OR FALSE

**State whether the following statements are *True* or *False* :**

1. In a D.E. of order $m$ and degree $n$, $m \leqslant n$.
2. $f(x)\dfrac{dy}{dx} + g(x)y = h(x)$ is a linear D.E.
3. $y = e^x$ is a general solution of $\dfrac{dy}{dx} = y$.
4. I.F. of $\dfrac{dy}{dx} + \dfrac{y}{x^2 - 1} = x$ is $\sqrt{\dfrac{x-1}{x+1}}$.
5. $y = ae^{2x} + be^{-3x}$ is a solution of the D.E. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + 6y = 0$.

## Answers

1. False 2. True 3. False 4. True 5. False.

## FILL IN THE BLANKS

**Fill in the following blanks with an appropriate words/numbers :**

1. Degree of the D.E. $\frac{d^m y}{dx^m} + \left(\frac{dy}{dx}\right)^m = x^m \ (m > 1)$ is .........
2. The D.E. $\frac{dy}{dx} = \frac{y}{x} + \frac{1}{e^{-y/x}}$ can be solved by using the substitution .........
3. If $\frac{dy}{dx} = ky$, then $y = c$ .........
4. I.F. of $dx = y(x+y)\,dy$ is a function of .........
5. The D.E. obtained by eliminating $a$, $b$ from $y = a + bx$ is .........

### Answers

**1.** 1 **2.** $y = vx$ **3.** $e^{kx}$

**4.** $y$ **5.** $\frac{d^2y}{dx^2} = 0$.

* * *