Mathematics and Statistics (Part 2) Standard XII

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1 COMMISSION, BROKERAGE AND DISCOUNT

CHAPTER OUTLINE

IMPORTANT FORMULAE

Abbreviations :

Invoice Price : IP
List Price (catalogue price) : LP
Present Worth : PW or P
Sum Due/Face Value : SD or FV
True Discount : TD
Banker's Gain : BG
Banker's Discount : BD
Cash Value : CV

Notations :

Period (in Years) : n
Rate of Interest (p.a.) : r

(1) Invoice price = List price – Trade discount [IP = LP – TD]

(2) The selling price/net selling price = Invoice price – Cash discount [NP = IP – CD]

(3) Profit = Net selling price – Cost price [P = NP – CP]

(4) Loss = Cost price – Net selling price [L = CP – NP]

(5) $\text{SD} = \text{PW} + \text{TD}$

(6) $\text{TD} = \dfrac{\text{PW} \times n \times r}{100}$

(7) $\text{BD} = \dfrac{\text{SD} \times n \times r}{100}$

(8) $\text{BG} = \text{BD} - \text{TD}$

(9) $\text{BG} = \dfrac{\text{TD} \times n \times r}{100}$

(10) $\text{BD} = \text{TD}\left(1 + \dfrac{nr}{100}\right)$

(11) Cash value = SD – BD

1.1 : COMMISSION AND BROKERAGE AGENTS

1. **Principal :** An individual party or parties participating in a transaction is referred to as principal.

2. **Commission :** The charges paid to an agent, for doing the work on behalf of some other person, is called commission. The commission or remuneration paid to an agent is generally fixed as some percentage of the value of the transaction.

3. **Different types of agents according to their specialization :**

(1) **Commission Agent :** A commission agent is a person who buys or sells goods on behalf of his principal and gets commission for his service.

(2) **Broker :** A broker is an agent who brings together the buyer and seller for the purpose of purchase or sale. The commission the broker gets is called brokerage. The brokerage is charged to both the parties.

(3) **Auctioneer :** An agent who sells goods by auction is called an auctioneer. He sells goods to the highest bidder. The name of the principal may not be disclosed in the transaction.

(4) **Factor :** An agent who is given the possession of goods and enters a contract for sale in his/her own name is called a factor.

(5) **Del Credere Agent :** A del credere agent gives guarantee to his principal that the party to whom he/she sells the goods will pay the sale price of goods. If a buyer is unable to pay after the transaction is completed, del credere agent is liable for the payment.

Agent gets additional commission other than the usual commission for this. This is known as del credere commission.

4. Discount : Discount is the reduction in the price of an article, allowed by the seller to the purchaser. It is expressed in terms of percentage.

5. Types of discount :

(1) **Trade discount :** The discount allowed by one trader to another is called trader discount. It is given on the catalogue price, list price or marked price of the goods. Also known as true discount (TD).

(2) **Cash discount :** Cash discount is allowed in consideration of ready cash payment (CD).
The buyer may be allowed both of these discounts. The trade discount is first calculated on list (catalogue) price. The cash discount is then calculated on the price obtained after deducting the trade discount from the list price. This is known as the **invoice** price.

EXERCISE 1.1 Textbook pages 5 and 6

1. An agent charges 12% commission on the sales. What does he earn, if the total sale amounts to ₹ 48,000? What does the seller get?

Solution :

Agent's commission at 12% on ₹ 48,000

$$= ₹\ 48000 \times \frac{12}{100} = ₹\ 5760$$

Seller gets sale amount − commission

$$= ₹\ (48000 - 5760) = ₹\ 42{,}240.$$

Agent's commission is ₹ 5760; Seller gets ₹ 42,240.

2. A salesman receives 3% commission on the sales up to ₹ 50,000 and 4% commission on the sales over ₹ 50,000. Find his total income on the sale of ₹ 2,00,000.

Solution :

3% commission on the sales up to ₹ 50,000

$$= ₹\ 50000 \times \frac{3}{100} = ₹\ 1500 \qquad \dots (1)$$

4% commission on the sales over ₹ 50,000

Sales over ₹ 50,000

$$= ₹\ (200000 - 50000) = ₹\ 1{,}50{,}000$$

i.e. 4% commission on ₹ 1,50,000

$$= ₹\ 150000 \times \frac{4}{100} = ₹\ 6000 \qquad \dots (2)$$

From (1) and (2),

total commission $= ₹\ (1500 + 6000) = ₹\ 7500$

Salesman's total income is ₹ 7500

3. Ms Saraswati was paid ₹ 88,000 as commission on the sale of computers at the rate of 12.5%. If the price of each computer was ₹ 32,000, how many computers did she sell?

Solution :

Commission at 12.5% on a computer costing ₹ 32,000

$$= ₹\ 32000 \times \frac{12.5}{100} = 4000$$

Ms Saraswati received ₹ 88,000 as commission on selling computers.

The number of computers sold

$$= \frac{\text{total commission}}{\text{commission on 1 computer}}$$

$$= \frac{88000}{4000} = 22$$

Ms Saraswati sold 22 computers.

4. Anita is allowed 6.5% commission on the total sales made by her, plus a bonus of $\frac{1}{2}$% on the sale over ₹ 20,000. If her total commission amount to ₹ 3400. Find the sales made by her.

Solution :

Let the total sales made by Anita be ₹ x.

Commission at 6.5% on total sales

$$= ₹\ x \times \frac{6.5}{100} = ₹\ \frac{6.5x}{100} \qquad \dots (1)$$

Sale exceeding ₹ 20,000 = ₹ $(x - 20000)$

Bonus at $\frac{1}{2}$% on sale over ₹ 20,000

$$= ₹\ (x - 20000) \times \frac{1}{2} \times \frac{1}{100} = ₹\ (x - 20000) \times \frac{1}{200} \quad \dots (2)$$

Total commission amounts to ₹ 3400. ... (Given) ... (3)

From (1), (2) and (3),

$$\frac{6.5x}{100} + \frac{x - 20000}{200} = 3400$$

$\therefore\ 13x + x - 20000 = 680000$

... (Multiplying both the sides by 200)

$\therefore\ 14x = 680000 + 20000$

$\therefore\ 14x = 700000 \qquad \therefore\ x = 50000$

Total sales made by Anita is ₹ 50,000.

5. Priya gets salary of ₹ 15,000 per month and commission at 8 % on the sales over ₹ 50,000. If she gets ₹ 17,400 in a certain month, find the sales made by her in that month.

Solution :

Total income = monthly salary + commission

$\therefore$ ₹ 17400 = ₹ 15000 + commission

$\therefore$ commission = ₹ (17400 − 15000) = ₹ 2400

Commission at 8 % on the sale over ₹ 50,000

$\therefore$ sale over ₹ 50,000 $= \frac{2400}{8\,\%} = \frac{2400 \times 100}{8} =$ ₹ 30,000

$\therefore$ the sales made by Priya in a certain month
= ₹ (50000 + 30000) = ₹ 80,000.

[*Alternative Method :*

Let the sales made by Priya in a month be ₹ x over ₹ 50,000.

Commission at 8 % on sales over ₹ 50,000

= ₹ $x \times \frac{8}{100}$ = ₹ $0.08\,x$ Salary ₹ 15,000

$\therefore$ Priya receives ₹ $(15000 + 0.08x)$

This is given to be ₹ 17,400.

$\therefore$ $15000 + 0.08x = 17400$ $\therefore$ $0.08x = 17400 - 15000$

$\therefore$ $0.08x = 2400$ $\therefore$ $x = \frac{2400}{0.08}$ $\therefore$ $x = 30{,}000$

$\therefore$ total sales = ₹ (50000 + 30000) = ₹ 80,000

The sales made by Priya in a month is ₹ 80,000.]

6. The income of a broker remains unchanged though the rate of commission is increased from 4 % to 5 %. Find the percentage reduction in the value of the business.

Solution :

Let the initial value of the business be ₹ 100.

Then the original income of the broker at 4 % = ₹ 4.

Let the new value of the business be ₹ x.

Commission at 5 % on ₹ x = ₹ $x \times \frac{5}{100}$ = ₹ $\frac{x}{20}$

It is given that the income remains unchanged.

$\therefore$ $4 = \frac{x}{20}$ $\therefore$ $x = 80$

The original value of the business is ₹ 100.

The new value of the business is ₹ 80.

$\therefore$ the reduction in the value of the business is ₹ 20.

This reduction is over ₹ 100.

$\therefore$ the reduction in the value of the business is 20%.

7. Mr Pavan is paid a fixed weekly salary plus commission based on percentage of sales made by him. If on the sale of ₹ 68,000 and ₹ 73,000 in two successive weeks, he received in all ₹ 9880 and ₹ 10,180, find his weekly salary and the rate of commission paid to him.

Solution :

Income of Mr Pavan = weekly salary + commission on sales

Salary + commission on ₹ 68,000 = ₹ 9880 … (1)

Salary + commission on ₹ 73,000 = ₹ 10,180 … (2)

Subtracting (1) from (2),

Commission on ₹ 5000 [₹ 73000 − ₹ 68000]
= ₹ 300 [₹ 10180 − ₹ 9880]

$\therefore$ the rate of commission $= \frac{300}{5000} \times 100 = 6\,\%$.

Commission on ₹ 68,000 at 6%

= ₹ $68000 \times \frac{6}{100}$ = ₹ 4080 … (3)

From (1) and (3),

Salary = ₹ (9880 − 4080) = ₹ 5800

Fixed weekly salary is ₹ 5800.

The rate of commission is 6 %.

8. Deepak's salary was increased from ₹ 4000 to ₹ 5000. The sales being the same, due to reduction in the rate of commission from 3 % to 2 %, his income remained unchanged. Find his sales.

Solution :

Let Deepak's sale be ₹ x.

Commission at 3 % = ₹ $x \times \frac{3}{100}$ = ₹ $\frac{3x}{100}$

$\therefore$ his income is ₹ $\left(4000 + \frac{3x}{100}\right)$ … (1)

Now, the commission is 2 %

$\therefore$ commission = ₹ $x \times \frac{2}{100}$ = ₹ $\frac{2x}{100}$.

But now the salary is ₹ 5000.

$\therefore$ his income is ₹ $(5000 + \frac{2x}{100})$ … (2)

There is no change in his income.

$\therefore$ $\left(4000 + \frac{3x}{100}\right) = \left(5000 + \frac{2x}{100}\right)$ … [From (1) and (2)]

$$\therefore \frac{3x}{100}-\frac{2x}{100}=5000-4000 \qquad \therefore \frac{x}{100}=1000$$

$\therefore x=1{,}00{,}000$

Deepak's sales is ₹ 1,00,000.

9. An agent is paid a commission of 7% on cash sales and 5% on credit sales made by him. If on the sale of ₹ 1,02,000 the agent claims a total commission of ₹ 6420, find his cash sales and credit sales.

Solution :

Let the agent's cash sales be ₹ x.

Commission at 7% on cash sales

$$= ₹\ x \times \frac{7}{100} = ₹\ \frac{7x}{100} \qquad \ldots (1)$$

Total sales is ₹ 1,02,000

$\therefore$ agent's credit sales is ₹ $(102000-x)$

Commission at 5% on credit sales

$$= ₹\ (102000-x)\times\frac{5}{100}=\frac{₹\ (510000-5x)}{100} \qquad \ldots (2)$$

Total commission is given to be ₹ 6420. … (3)

From (1), (2) and (3),

$$\frac{7x}{100}+\frac{(510000-5x)}{100}=6420$$

$\therefore 7x+510000-5x=642000$

$\therefore 2x=132000 \qquad \therefore x=66000$

The agent's cash sales is ₹ 66,000 and his credit sales is ₹ (102000 − 66000) = ₹ 36,000.

10. Three cars were sold through an agent for ₹ 2,40,000, ₹ 2,22,000 and ₹ 2,25,000 respectively. The rates of commission were 17.5% on the first, 12.5% on the second. If the agent overall received 14% commission on the total sales, find the rate of commission paid on the third car.

Solution :

Commission at 17.5% on ₹ 2,40,000

$$= ₹\ 240000\times\frac{17.5}{100}= ₹\ 42{,}000 \qquad \ldots (1)$$

Commission at 12.5% on ₹ 2,22,000

$$= ₹\ 222000\times\frac{12.5}{100}= ₹\ 27{,}750 \qquad \ldots (2)$$

Let the commission on third car be x%.

Commission at x% on ₹ 2,25,000

$$= ₹\ 225000\times\frac{x}{100}= ₹\ 2250x \qquad \ldots (3)$$

Overall commission is 14%

$$=14\%\times ₹\ (240000+222000+225000)$$

$$=\frac{14}{100}\times ₹\ 687000 = ₹\ 96{,}180 \qquad \ldots (4)$$

From (1), (2), (3) and (4),

$42000+27750+2250x=96180$

$\therefore 2250x=96180-42000-27750$

$\therefore 2250x=26430 \qquad \therefore x=\frac{26430}{2250}$

$\therefore x\approx 11.75$

The commission paid on the third car is 11.75%

11. Swatantra Distributors allows 15% discount on the list price of washing machine. Further 5% discount is given for cash payment. Find the list price of the washing machine, if it was sold for the net amount of ₹ 38,356.25

Solution :

Let the list price of the washing machine be ₹ x.

$$15\% \text{ discount} = ₹\ x\times\frac{15}{100}= ₹\ \frac{15x}{100}$$

Invoice price = List price − Trade discount

$$\therefore \text{invoice price} = ₹\left(x-\frac{15x}{100}\right)= ₹\ \frac{85x}{100}$$

5% discount on invoice price

$$= ₹\ \frac{85x}{100}\times\frac{5}{100}= ₹\ \frac{425x}{10000}$$

$$\therefore \text{net amount} = ₹\ x - ₹\left(\frac{5x}{100}+\frac{425x}{10000}\right)$$

$$= ₹\ x - ₹\ \frac{925x}{10000} = ₹\ \frac{8075x}{10000}$$

Net amount is given to be ₹ 38,356.25

$$\therefore \frac{8075x}{10000}=38356.25 \qquad \therefore x=\frac{38356.25\times 10000}{8075}$$

$\therefore x=4.75\times 10000 \qquad \therefore x=47500$

The list price of the washing machine is ₹ 47,500.

12. A bookseller received ₹ 1530 as 15% commission on list price. Find list price of the books.

Solution :

Let the list price of the books be ₹ x.

Commission at 15 % on ₹ x

$$= ₹\ x \times \frac{15}{100} = ₹\ \frac{15x}{100}$$

The commission given is ₹ 1530

$$\therefore\ \frac{15x}{100} = 1530 \qquad \therefore\ x = \frac{1530 \times 100}{15} \qquad \therefore\ x = 10200$$

The list price of the books is ₹ 10,200.

13. A retailer sold a suit for ₹ 8832 after allowing 8 % discount on marked price and further 4% cash discount. If he made 38% profit, find the cost price and the marked price of the suit.

Solution :

Let the marked price of the suit be ₹ x.

$$8\ \%\ \text{discount} = ₹\ x \times \frac{8}{100} = ₹\ \frac{8x}{100} \qquad \dots (1)$$

IP = LP − TD

$$\therefore\ \text{invoice price} = ₹\ x - ₹\ \frac{8x}{100} = ₹\ \frac{92x}{100}$$

$$4\ \%\ \text{discount of invoice price} = ₹\ \frac{92x}{100} \times \frac{4}{100} = ₹\ \frac{368x}{100} \dots (2)$$

From (1) and (2), total discount

$$= ₹\ \frac{8x}{100} + ₹\ \frac{368x}{100} = ₹\ \frac{1168x}{10000}$$

$$\therefore\ \text{selling price} = ₹\ x - ₹\ \frac{1168x}{10000}$$

$$\therefore\ x - \frac{1168x}{10000} = 8832$$

$$\therefore\ \frac{8832x}{10000} = 8832 \qquad \therefore\ x = 10{,}000$$

∴ marked price of the suit is ₹ 10,000

38 % profit on selling price ₹ 8832

$$\therefore\ \text{cost price} = \frac{8832}{138} \times 100 = ₹\ 6400$$

Cost price of the suit is ₹ 6400.

Marked price of the suit is ₹ 10,000.

14. An agent charges 10% commission plus 2% del credere. If he sells goods worth ₹ 37,200, find his total earnings.

Solution :

10% commission on sale ₹ 37,200

$$= ₹\ 37200 \times \frac{10}{100} = ₹\ 3720$$

2% del credere on ₹ 37,200

$$= ₹\ 37200 \times \frac{2}{100} = ₹\ 744$$

Agent's total earning = ₹ (3720 + 744) = ₹ 4464.

15. A wholesaler allows 25% trade discount and 5% cash discount. What will be the net price of an article marked at ₹ 1600.

Solution :

25% trade discount on ₹ 1600

$$= ₹\ 1600 \times \frac{25}{100} = ₹\ 400$$

∴ invoice price = ₹ (1600 − 400) = ₹ 1200

5% cash discount on invoice price

$$= ₹\ 1200 \times \frac{5}{100} = ₹\ 60$$

∴ net price of the article

= ₹ 1600 − ₹ (400 + 60) = ₹ (1600 − 460) = ₹ 1140

The net price of the article is ₹ 1140.

EXAMPLES FOR PRACTICE 1.1

1. An agent charges 12.5% commission on the sales. What does he earn, if the total sales amount to ₹ 36,000? What does the seller get?
2. A salesman receives 4% commission on the sales up to ₹ 10,000 and 5% commission on the sales over ₹ 10,000. Find his total income on the sale of ₹ 40,000.
3. A salesman is allowed 10% commission on the total sales made by him plus a bonus of 1.25% on the sales over ₹ 15,000. If his total earnings is ₹ 2175, find the sales made by him.
4. A medical representative is paid a fixed monthly salary plus a commission based on percentage of sales. If on the sales of ₹ 12,500 and ₹ 13,200 in two successive months he received ₹ 8737.50 and ₹ 8790 respectively, find his monthly salary and the rate of commission paid on sales.
5. A wholesaler allows 25% trade discount and 5% cash discount. What will be the net price of an article, if it was sold for the net amount of ₹ 1140.
6. A furniture dealer deals a chair for ₹ 7219.20 after allowing 6% trade discount and 4% cash discount. If he loses 4%, find the cost price and the marked price of the chair.

7. The income of an agent remains unchanged though the rate of commission is increased from 5% to 6.25%. Find the percentage reduction in the value of business.
8. The price of a TV set is ₹ 17,000. An agent charges at 3% and earns ₹ 25,500. Find the number of TV sets sold by him.
9. A merchant gives his agent 12% commission plus $2\frac{3}{4}$% del credere on a sale of goods worth ₹ 52,800. How much does the merchant receive after paying the agent's total commission?
10. The income of an agent remains unchanged though the rate of commission is increased from 6% to 8%. Find the percentage reduction in the value of business.

Answers

1. ₹ 4500; ₹ 31,500 **2.** ₹ 1900 **3.** ₹ 21,000
4. Monthly salary ₹ 7800; 7.5% **5.** ₹ 3562.50
6. Cost price = ₹ 7520, Marked price = ₹ 8000.
7. 20% **8.** 50 TV sets **9.** ₹ 45,012 **10.** 25%.

1.2 : DISCOUNT

1. Present worth, Sum due, True discount : When goods are sold on credit, the price quoted for goods includes a sufficient margin of interest for the period of credit allowed.

Let the goods be worth ₹ 100, if the payment is made on the spot. If a credit of 3 months is allowed, then the businessman will quote the price by adding interest for 3 months. If the rate of interest is 16% p.a. then the interest for 3 months will be ₹ 4. Hence, the customer has to pay ₹ 104 after 3 months.

So, ₹ 104 due after 3 months at 16% p.a. is equivalent to ₹ 100 today.

₹ 100 is known as **present worth** (PW) of ₹ 104 due after 3 months.

₹ 104 is known as sum due (SD) and ₹ 4 is known as the **true discount** (TD) on the sum due.

Sum due (SD) = Present worth (PW) + True discount (TD).

$$\text{TD} = \frac{\text{PW} \times n \times r}{100} \qquad \therefore\ \text{SD} = \text{PW} + \frac{\text{PW} \times n \times r}{100}$$

$$\therefore\ \text{SD} = \text{PW}\left[1 + \frac{n \times r}{100}\right]$$

2. Drawer and Drawee : A person who draws the bill is called the **drawer** (or creditor or payee).

A person on whom the bill is drawn is called the **drawee** (or debtor or acceptor).

3. Date of bill and Face Value : The date on which the bill is drawn is called **date of bill.** The amount of which the bill is drawn is called **face value** (FV) of the bill. FV is the sum due on the present worth.

Period of the bill is the time after completion of which the drawer receives the payment of the bill.

4. Nominal Due Date and Legal Due Date : The date on which the period of bill expires is called the **nominal due date.** The buyer has to make the payment to the seller on this date.

The buyer is allowed to pay the amount even 3 days later. These 3 days are called the days of grace. The date obtained after adding 3 days of grace to the nominal due date is known as the **legal due date.**

5. Discounting a Bill : If the drawer of the bill wants money before the legal due date, then there is a facility available at the bank or with an agent who can discount the bill and pay the amount to the drawer. In this case some amount from the face value of the bill is deducted. This is called **discounting the bill.**

6. Banker's Discount, Cash Value, Banker's Gain : When a bill is discounted in a bank, the banker will deduct the amount from the face value of the bill at the given rate of interest for the period from the date of discounting to the legal due date and pay the balance to the drawer. This amount is known as **Banker's Discount** (BD).

The amount paid to the holder of the bill after deducting banker's discount is known as **Cash Value** (CV) of the bill paid on the date of discounting.

The banker's discount is called **commercial discount.**

True discount is calculated on present worth and the banker's discount is calculated on face value (sum due). Hence, the banker's discount is always higher than the true discount.

The difference between the banker's discount and the true discount is called **Banker's Gain** (BG). It is equal to the interest on true discount.

EXERCISE 1.2 | Textbook page 11

1. What is the present worth of a sum of ₹ 10,920 due six months hence at 8% p.a. simple interest?

Solution :

SD = ₹ 10,920, $n = 6$ months $= \frac{1}{2}$ years, $r = 8\%$

$$SD = PW\left(1 + \frac{nr}{100}\right)$$

$$\therefore\ 10920 = PW\left(1 + \frac{\frac{1}{2} \times 8}{100}\right)$$

$$\therefore\ 10920 = PW\left(1 + \frac{4}{100}\right)$$

$$\therefore\ 10920 = PW\left(1 + \frac{1}{25}\right)$$

$$\therefore\ 10920 = PW\left(\frac{26}{25}\right) \qquad \therefore\ PW = \frac{10920 \times 25}{26} = 420 \times 25$$

$\therefore$ PW = ₹ 10,500

The present worth is ₹ 10,500.

[*Alternative Method* :

$n = 6$ months $= \frac{1}{2}$ years, $r = 8\%$.

Let the present worth be ₹ x.

$$TD = \frac{PW \times n \times r}{100} = \frac{x \times \frac{1}{2} \times 8}{100} = \frac{4x}{100} = \frac{x}{25}$$

SD = PW + TD

$$\therefore\ 10920 = x + \frac{x}{25} \qquad \therefore\ 10920 = \frac{26x}{25} \qquad \therefore\ x = \frac{10920 \times 25}{26}$$

$\therefore\ x = 10500$

The present worth is ₹ 10,500.]

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2. What is sum due of ₹ 8000 due 4 months hence at 12.5% simple interest?

Solution :

PW = ₹ 8000, $n = 4$ months $= \frac{1}{3}$ years, $r = 12.5\%$

$$SD = PW\left(1 + \frac{nr}{100}\right)$$

$$\therefore\ SD = 8000\left(1 + \frac{\frac{1}{3} \times 12.5}{100}\right)$$

$$= 8000\left(1 + \frac{125}{3000}\right)$$

$$= 8000\left(1 + \frac{1}{24}\right)$$

$$= 8000 \times \frac{25}{24} = \frac{200000}{24} = 8333.33$$

Sum due is ₹ 8333.33.

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3. The true discount on the sum due 8 months hence at 12% p.a. is ₹ 560. Find the sum due and present worth of the bill.

Solution :

True discount = ₹ 560, $r = 12\%$,

$n = 8$ months $= \frac{8}{12} = \frac{2}{3}$ years

$$\text{True discount} = \frac{PW \times n \times r}{100}$$

$$\therefore\ 560 = \frac{PW \times \frac{2}{3} \times 12}{100}$$

$\therefore\ 560 \times 100 = PW \times 8$

$$\therefore\ PW = \frac{560 \times 100}{8} = ₹\ 7000$$

$\therefore$ PW = ₹ 7000

Present worth of the bill is ₹ 7000.

Sum due = PW + TD

= 7000 + 560

= ₹ 7560

The sum due of the bill is ₹ 7560.

Present worth of the bill is ₹ 7000.

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4. The true discount on a sum is $\frac{3}{8}$ of the sum due at 12% p.a. Find the period of the bill.

Solution :

True discount = $\frac{3}{8}$ (sum due), $r = 12\%$

Sum due = PW + TD

$$\therefore\ \text{sum due} = PW + \frac{3}{8}(\text{sum due})$$

$$\therefore\ \text{sum due} - \frac{3}{8}(\text{sum due}) = PW$$

$$\therefore\ PW = \frac{5}{8}(\text{sum due})$$

$$\text{True discount} = \frac{PW \times n \times r}{100}$$

$$\therefore \frac{3}{8}(\text{sum due}) = \frac{\frac{5}{8}(\text{sum due}) \times n \times 12}{100}$$

$$\therefore \frac{3}{8}(\text{sum due}) \times \frac{8}{5(\text{sum due})} = \frac{12n}{100}$$

$$\therefore \frac{3}{5} = \frac{12n}{100}$$

$$\therefore \frac{3 \times 100}{5 \times 12} = n$$

$$\therefore n = 5$$

The period of the bill is 5 years.

5. 20 copies of a book can be purchased for a certain sum payable at the end of 6 months and 21 copies for the same sum in ready cash. Find the rate of interest.

Solution :

Let the cost of one book be ₹ x.

Then the cost of 20 books = ₹ $20x$.

Now, cost of 20 books is same as the cost of 21 books.

∴ interest on ₹ $20x$ for 6 months at the rate $r\%$ will be same as the cost of one book.

$$\therefore x = 20x \times \frac{1}{2} \times \frac{r}{100} \qquad \therefore r = \frac{x \times 2 \times 100}{20x}$$

$$\therefore r = 10$$

The rate of interest is 10% per annum.

6. Find the true discount, banker's discount and banker's gain on a bill of ₹ 4240 due 6 months hence at 9% p.a.

Solution :

Sum due = ₹ 4240,

$n = 6$ months $= \frac{1}{2}$ years, $r = 9\%$

Banker's Discount :

$$\text{BD} = \frac{\text{SD} \times n \times r}{100}$$

$$= \frac{4240 \times \frac{1}{2} \times 9}{100}$$

$$= \frac{4240 \times 9}{200}$$

$$= \frac{38160}{200}$$

$$\therefore \text{BD} = 190.80$$

Banker's discount is ₹ 190.80.

True Discount :

Let TD be ₹ x.

Then BD = TD + Interest on TD for $\frac{1}{2}$ years at 9%

$$= x + \left(x \times \frac{1}{2} \times \frac{9}{100}\right)$$

$$\therefore 190.80 = x + \frac{9x}{200}$$

$$\therefore 190.80 = \frac{209x}{200}$$

$$\therefore x = \frac{190.80 \times 200}{209}$$

$$\therefore x = \frac{38160}{209}$$

$$\therefore x = 182.58 \approx 182.60$$

True discount is ₹ 182.60.

Banker's Gain :

BG = BD − TD

= ₹ (190.80 − 182.60)

∴ BG = ₹ 8.20

Hence, Banker's gain is ₹ 8.20.

7. True discount on a bill is ₹ 2200 and banker's discount is ₹ 2310. If the bill is due 10 months hence, find the rate of interest.

Solution :

TD = ₹ 2200, BD = ₹ 2310,

$n = 10$ months $= \frac{10}{12} = \frac{5}{6}$ years

BD = TD + Interest on TD for $\frac{5}{6}$ years at the rate $r\%$.

$$\therefore 2310 = 2200 + \left(2200 \times \frac{5}{6} \times \frac{r}{100}\right)$$

$$\therefore 2310 - 2200 = \frac{55}{3}r$$

$$\therefore 110 = \frac{55}{3}r \qquad \therefore r = \frac{110 \times 3}{55}$$

$$\therefore r = 6$$

The rate of interest is 6%.

8. A bill of ₹ 6395 drawn on 19th January 2015 for 8 months was discounted on 28th February 2015 at 8% p.a. interest. What is the banker's discount? What is the cash value of the bill?

Solution :

FV of the bill = ₹ 6395, $r = 8\%$.

Date of drawing the bill = 19th January 2015

Period of the bill = 8 months

Nominal due date = 19th September 2015

Legal due date = 22th September 2015

Date of discount = 28th February 2015

∴ number of days from the date of discounting to the legal due date is as follows :

Mar.	Apr.	May	June	July	Aug.	Sept.	Total
31	30	31	30	31	31	22	206

$\therefore$ period $n = \frac{206}{365}$

BD = Interest on FV ₹ 6395 for 206 days at 8%

$\therefore \text{BD} = \frac{\text{FV} \times n \times r}{100}$

$= 6395 \times \frac{206}{365} \times \frac{8}{100}$

$\therefore$ BD = 288.74*

∴ banker's discount is ₹ 288.74.

Cash Value (CV) of the bill :

CV = FV − BD

= ₹ (6395 − 288.74)

= ₹ 6106.26*

The cash value of the bill is ₹ 6106.26.

[* **Note :** *Answer given in the textbook is incorrect.*]

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9. A bill of ₹ 8000 drawn on 5th January 1998 for 8 months was discounted for ₹ 7680 on a certain date. Find the date on which it was discounted at 10% p.a.

Solution :

Face Value (FV) (SD) = ₹ 8000, $r = 10\%$,

CV = ₹ 7680

Now, BD = SD − CV

= 8000 − 7680

= ₹ 320

Also, $\text{BD} = \frac{\text{SD} \times n \times r}{100}$

$\therefore 320 = \frac{8000 \times n \times 10}{100}$

$\therefore 320 = 800n$

$\therefore n = \frac{320}{800} = \frac{2}{5}$ years

$\therefore n = \frac{2}{5} \times 365$

$\therefore n = 146$ days

Date of drawing the bill = 5th January 1998

Period of the bill = 8 months

Nominal due date = 5th September 1998

Legal due date = 8th September 1998

The number of days from the date of discounting to the legal date is 146 days.

∴ the date of discounting the bill is obtained by deducting 146 days from the legal due date, i.e. 8th September 1998.

Sept.	Aug.	July	June	May	April	Total
8	31	31	30	31	15	146

∴ the bill was discounted on 15th April 1998.

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10. A bill drawn on 5th June for 6 months was discounted at the rate of 5% p.a. on 19th October. If the cash value of the bill is ₹ 43,500, find the face value of the bill.

Solution :

Cash value = ₹ 43500, $r = 5\%$

Date of drawing = 5th June

Period of the bill = 6 months

Nominal due date = 5th December

Legal due date = 8th December

Date of discounting = 19th October

∴ number of days from the date of discounting to the legal due date is as follows :

Oct.	Nov.	Dec.	Total
12	30	8	50

$\therefore$ period $n = \frac{50}{365}$ years

Let the face value (SD) of the bill be ₹ x.

$\text{BD} = \frac{\text{FV} \times n \times r}{100}$

$= x \times \frac{50}{365} \times \frac{5}{100} = \frac{x}{146}$

Also, BD = FV − CV = $x - 43500$

$\therefore \frac{x}{146} = x - 43500$

$\therefore 43500 = x - \frac{x}{146}$

$\therefore 43500 = \frac{146x - x}{146}$

$\therefore\ 43500 \times 146 = 145x$

$\therefore\ x = \dfrac{43500 \times 146}{145}$

$\therefore\ x = 43800$

The face value of the bill is ₹ 43,800.

11. A bill was drawn on 14th April for ₹ 7000 and was discounted on 6th July at 5% p.a. The Banker paid ₹ 6930 for the bill. Find the period of the bill.

Solution :

Face Value (FV) or SD = ₹ 7000, $r = 5\%$,

Cash Value (CV) = ₹ 6930

BD = SD − CV

= 7000 − 6930

= ₹ 70

Also, BD $= \dfrac{\text{SD} \times n \times r}{100}$

$\therefore\ 70 = \dfrac{7000 \times n \times 5}{100}$

$\therefore\ 70 = 350n$

$\therefore\ n = \dfrac{70}{350} = \dfrac{1}{5}$ years

$\therefore\ n = \dfrac{1}{5} \times 365 = 73$ days

To find the legal due date, 73 days are to be counted from the date of discounting, i.e. 6th July.

July	Aug.	Sept.	Total
25	31	17	73

Hence, the legal due date is 17th September

$\therefore$ nominal due date is 14th September

Now, date of drawing is 14th April.

Hence, the period of the bill is from 14th April to 14th September, i.e. 5 months.

12. If difference between true discount and banker's discount on a sum due 4 months hence is ₹ 20, find true discount, banker's discount and amount of the bill, the rate of simple interest charged being 5% p.a.

Solution :

True Discount :

Let TD be ₹ x.

Banker's Gain (BG) = BD − TD

= Interest on TD for 4 months at 5% p.a.

$\therefore\ 20 = x \times \dfrac{4}{12} \times \dfrac{5}{100}$

$\therefore\ 20 = \dfrac{x}{60}$

$\therefore\ 20 \times 60 = x$

$\therefore\ x = 1200$

True discount is ₹ 1200

Banker's Discount :

BD = BG + TD

= 20 + 1200

$\therefore$ BD = 1220

Banker's discount is ₹ 1220.

Amount of Bill :

Let amount of the bill (Face Value) be ₹ y.

BD = Interest on FV for 4 months at 5% p.a.

$\therefore\ 1220 = y \times \dfrac{4}{12} \times \dfrac{5}{100}$

$\therefore\ 1220 = \dfrac{y}{60}$

$\therefore\ 1220 \times 60 = y$

$\therefore\ y = 73200$

The amount of the bill is **₹ 73,200.**

13. A bill of ₹ 51,000 was drawn on 18th February 2010 for 9 months. It was encashed on 28th June 2010 at 5% p.a. Calculate the banker's gain and true discount.

Solution :

FV of the bill = ₹ 51,000, $r = 5\%$

Date of drawing the bill = 18th February 2010

Period of the bill = 9 months

Nominal due date = 18th November 2010

Legal due date = 21st November 2010

Date of discounting = 28th June 2010

$\therefore$ number of days from the date of discounting to the legal due date is as follows :

June	July	Aug.	Sept.	Oct.	Nov.	Total
2	31	31	30	31	21	146

$\therefore$ period $n = \dfrac{146}{365} = \dfrac{2}{5}$ years

$$\text{BD} = \frac{\text{FV} \times n \times r}{100}$$

$$= 51000 \times \frac{2}{5} \times \frac{5}{100}$$

$$= ₹\ 1020$$

Also, BD = TD + Interest on TD for $\frac{2}{5}$ years at 5%.

Let TD be ₹ x.

$$\therefore\ 1020 = x + \left(x \times \frac{2}{5} \times \frac{5}{100}\right)$$

$$\therefore\ 1020 = x + \frac{x}{50}$$

$$\therefore\ 1020 = \frac{50x + x}{50}$$

$$\therefore\ 1020 \times 50 = 51x$$

$$\therefore\ x = \frac{1020 \times 50}{51}$$

$$\therefore\ x = 1000$$

True discount is ₹ 1000.

Banker's Gain :

$$\text{BG} = \text{BD} - \text{TD}$$

$$= 1020 - 1000$$

$$\therefore\ \text{BD} = 20$$

Banker's gain is ₹ 20*.

[***Note :** *Answer given in the textbook is incorrect.*]

14. A certain sum due 3 months hence is $\frac{21}{20}$ of the present worth. What is the rate of interest?

Solution :

$$\text{SD} = \frac{21}{20}\ (\text{PW}),\ n = 3 \text{ months} = \frac{1}{4} \text{ years}$$

$$\text{SD} = \text{PW} + \text{TD}$$

$$\therefore\ \text{PW} + \text{TD} = \frac{21}{20}\ \text{PW}$$

$$\therefore\ \text{TD} = \frac{21}{20}\ \text{PW} - \text{PW}$$

$$\therefore\ \text{TD} = \text{PW}\left(\frac{21}{20} - 1\right)$$

$$\therefore\ \text{TD} = \text{PW}\left(\frac{21-20}{20}\right)$$

$$\therefore\ \text{TD} = \frac{1}{20} \times \text{PW}$$

$$\text{Also, TD} = \frac{\text{PW} \times n \times r}{100}$$

$$= \text{PW} \times \frac{1}{4} \times \frac{r}{100}$$

$$\therefore\ \frac{1}{20} \times \text{PW} = \text{PW} \times \frac{1}{4} \times \frac{r}{100}$$

$$\therefore\ \frac{1}{20} = \frac{r}{400}$$

$$\therefore\ \frac{400}{20} = r$$

$$\therefore\ r = 20$$

The rate of interest is 20% p.a.

15. A bill of a certain sum drawn on 28th February 2007 for 8 months was encashed on 26th March 2007 for ₹ 10,992 at 14% p.a. Find the face value of the bill.

Solution :

Cash Value (CV) = ₹ 10,992, $r = 14\%$

Let the Face Value (FV) or SD of the bill be ₹ x.

Date of drawing the bill = 28th February 2007

Period of the bill = 8 months

Nominal due date = 28th October 2007

Legal due date = 31st October 2007

Date of discounting = 26th March 2007

∴ number of days from the date of discounting to the legal due date is as follows :

Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Total
5	30	31	30	31	31	30	31	219

$$\therefore\ \text{period } n = \frac{219}{365} \text{ years}$$

$$\text{BD} = \text{FV} - \text{CV}$$

$$= x - 10992$$

$$\text{Also, BD} = \frac{\text{FV} \times n \times r}{100}$$

$$= x \times \frac{219}{365} \times \frac{14}{100} = \frac{3066x}{36500} = 0.084x$$

$$\therefore\ x - 10992 = 0.084x$$

$$\therefore\ x - 0.084x = 10992$$

$$\therefore\ 0.916x = 10992$$

$$\therefore\ x = \frac{10992}{0.916}$$

$$\therefore\ x = 12000$$

The face value of the bill is ₹ 12,000.

EXAMPLES FOR PRACTICE 1.2

1. What is the present worth of a sum of ₹ 7488 due six months hence at 8% p.a. simple interest?
2. What is the sum due of ₹ 5000 due 8 months hence at 12% p.a. simple interest?
3. True discount on the sum due 9 months hence at 16% p.a. is ₹ 600. Find the sum due and present worth of the bill.
4. The true discount on a sum is $\frac{3}{8}$ of the sum due at 15% p.a. Find the period of the bill.
5. A bill drawn on 8th March 2010 for 6 months was discounted on 18th April 2010 at 9% p.a. for ₹ 2410. Find the face value of the bill.
6. A bill drawn for ₹ 2190 for 7 months was discounted for ₹ 2142 on 5th August 2015. If the rate of interest is 8% p.a., find the date on which the bill was drawn.
7. A bill of ₹ 6000 drawn on 4th January 2016 for 5 months was discounted on 26th March 2016. If the cash value was ₹ 5810, find the rate of interest.
8. A bill of ₹ 7500 was discounted for ₹ 7290 at a bank on 28th October 2014. If the rate of interest was 14% p.a., what is the legal due date of the bill?
9. A Banker's discount calculated for 1 year is 26 times his gain. Find the rate of interest.
10. The present worth of the sum of ₹ 5830, due 9 months hence is ₹ 5500. Find the rate of interest.
11. A bill of ₹ 2000 drawn on 15th February 2015 for 10 months was discounted on 13th May 2015 at $3\frac{3}{4}\%$ p.a. Calculate the banker's discount.
12. Find the true discount, banker's discount and banker's gain on a bill of ₹ 36,600 due 4 months hence at 5% p.a.
13. A bill of ₹ 2000 was drawn on 7th March 2017 for 6 months and was discounted on 17th April 2017 at 8% p.a. Find the banker's charge. How much will the holder receive?
14. Find the true discount, banker's discount and banker's gain on a bill of ₹ 6300 due 6 months hence at 10% p.a.
15. The difference between true discount and banker's discount on a sum due 4 months hence at the rate of 9% p.a. is ₹ 18. Find (i) the true discount (ii) banker's discount and (iii) the amount of the bill.

Answers

1. ₹ 7200 **2.** ₹ 5400 **3.** SD : ₹ 5600; PW : ₹ 5000
4. 4 years **5.** ₹ 2500 **6.** 10th April 2015 **7.** 7.5%
8. 73 days, legal due date : 9th January 2015 **9.** 4%
10. 8% **11.** ₹ 45 **12.** TD = ₹ 600, BD = ₹ 610, BG = ₹ 10.
13. Banker's charge : ₹ 64, The holder receives ₹ 1936
14. TD = ₹ 300; BD = ₹ 315; BG = ₹ 15.
15. (i) TD = ₹ 600 (ii) BD = ₹ 618 (iii) ₹ 20600.

MISCELLANEOUS EXERCISE – 1

(Textbook pages 11 to 13)

I. Choose the correct alternatives :

1. An agent who gives guarantee to his principal that the party will pay the sale price of goods is called
   (a) auctioneer (b) del credere agent
   (c) factor (d) broker
2. An agent who is given the possession of goods to be sold is known as
   (a) factor (b) broker
   (c) auctioneer (d) del credere agent
3. The date on which the period of the bill expires is called
   (a) legal due date (b) grace date
   (c) nominal due date (d) date of drawing
4. The payment date after adding 3 days of grace period is known as
   (a) the legal due date (b) the nominal due date
   (c) days of grace (d) date of drawing
5. The sum due is also called as
   (a) face value (b) present value
   (c) cash value (d) true discount
6. P is the abbreviation for
   (a) face value (b) present worth
   (c) cash value (d) true discount
7. Banker's gain is the simple interest on
   (a) banker's discount (b) face value
   (c) cash value (d) true discount
8. The marked price is also called as
   (a) cost price (b) selling price
   (c) list price (d) invoice price

9. When only one discount is given then
   (a) List price = Invoice price
   (b) Invoice price = Net selling price
   (c) Invoice price = Cost price
   (d) Cost price = Net selling price

10. The difference between face value and present worth is called
   (a) banker's discount (b) true discount
   (c) banker's gain (d) cash value

Answers

1. (b) del credere agent **2.** (a) factor **3.** (c) nominal due date **4.** (a) the legal due date **5.** (a) face value **6.** (b) present worth **7.** (d) true discount **8.** (c) list price **9.** (b) Invoice price = Net selling price **10.** (b) true discount.

II. Fill in the blanks :

1. A person who draws the bill is called ……… .
2. An ……… is an agent who sells the goods by auction.
3. Trade discount is allowed on the ……… price.
4. The banker's discount is also called ……… .
5. The banker's discount is always ……… than the true discount.
6. The difference between the banker's discount and the true discount is called ……… .
7. The date by which the buyer is legally allowed to pay the amount is known as ……… .
8. A ……… is an agent who brings together the buyer and the seller.
9. If buyer is allowed both trade and cash discounts, ……… discount is first calculated on ……… price.
10. ……… = List price (catalogue Price) – Trade Discount.

Answers

1. *the drawer **2.** auctioneer **3.** list **4.** commercial discount **5.** higher **6.** banker's gain **7.** legal due date **8.** broker **9.** trade, list **10.** Invoice price.

[***Note :** *Answer given in the textbook is incorrect.*]

III. State whether each of the following is *True* or *False* :

1. Broker is an agent who gives a guarantee to seller that the buyer will pay the sale price of goods.
2. Cash discount is allowed on list price.
3. Trade discount is allowed on catalogue price.
4. The buyer is legally allowed 6 days grace period.
5. The date on which the period of the bill expires is called the nominal due date.
6. The difference between the banker's discount and true discount is called sum due.
7. The banker's discount is always lower than the true discount.
8. The bankers discount is also called as commercial discount.
9. In general, cash discount is more than trade discount.
10. A person can get both, trade discount and cash discount.

Answers

1. False **2.** False **3.** True
4. False **5.** True **6.** False
7. False **8.** True **9.** False
10. True.

IV. Solve the following problems :

1. A salesman gets a commission of 6.5% on the total sales made by him and bonus of 1% on sales over ₹ 50,000. Find his total income on a turnover of ₹ 75,000.

Solution :

Turnover is ₹ 75,000.

Rate of commission is 6.5%

Commission on ₹ 75,000 at 6.5%

$$= 75000 \times \frac{6.5}{100} = ₹\ 4875$$

Sales in excess over ₹ 50,000 = ₹ (75000 – 50000)

$$= ₹\ 25000$$

Bonus on ₹ 25,000 at 1%

$$= 25{,}000 \times \frac{1}{100} = ₹\ 250$$

Total income of the salesman = ₹ (4875 + 250) = ₹ 5125.

2. A shop is sold at 30% profit. The amount of brokerage at the rate of $\frac{3}{4}\%$ amounts to ₹ 73,125. Find the cost of the shop.

Solution :

Let the sale price of the shop be ₹ x.

Brokerage is ₹ 73,125.

Brokerage at $\frac{3}{4}\%$ on ₹ $x = x \times \frac{3}{4} \times \frac{1}{100}$

$$\therefore\ 73125 = \frac{3x}{400}$$

$\therefore\ x = \dfrac{73125 \times 400}{3}$

$\therefore\ x = ₹\ 97{,}50{,}000$

The selling price of the shop is ₹ 97,50,000. ... (1)

Let the cost price of the shop be ₹ y.

Sold at 30% profits.

$\therefore$ profit $= y \times \dfrac{30}{100} = \dfrac{3y}{100}$

$\therefore$ selling price $= ₹\left(y + \dfrac{3y}{10}\right)$

$= ₹\ \dfrac{13y}{10}$ ... (2)

From (2) and (1),

$\dfrac{13y}{10} = 9750000$

$\therefore\ y = \dfrac{9750000 \times 10}{13}$

$= 7500000$

The cost of the shop is ₹ 75,00,000.

3. A merchant gives 5% commission and 1.5% del credere to his agent. If the agent sells goods worth ₹ 30,600 how much does he get? How much does the merchant receive?

Solution :

Agent's commission at 5% on ₹ 30,600

$= 30600 \times \dfrac{5}{100} = ₹\ 1530.$

Amount of del credere at 1.5% on ₹ 30,600

$= 30600 \times \dfrac{1.5}{100} = ₹\ 459$

Agent's total commission = ₹ (1530 + 459)

= ₹ 1989

Merchant receives ₹ (30600 − 1989) = ₹ 28,611.

Agent gets ₹ 1989; Merchant receives ₹ 28,611.

4. After deducting commission at $7\frac{1}{2}$% on first ₹ 50,000 and 5% on balance of sales made by him, an agent remits ₹ 93,750 to his principal. Find the value of goods sold by him.

Solution :

Let the value of the goods sold by the agent be ₹ x.

Commission at $7\frac{1}{2}$% on first ₹ 50,000

$= 50000 \times \dfrac{15}{2} \times \dfrac{1}{100} = ₹\ 3750$ ... (1)

Balance sale = ₹ $(x - 50000)$

Commission at 5% on balance sale

$= (x - 50000) \times \dfrac{5}{100} = \dfrac{x - 50000}{20}$ ... (2)

The agent's total commission

$= ₹\left(3750 + \dfrac{x - 50000}{20}\right)$ ... [From (1) and (2)]

$= \dfrac{3750 \times 20 + x - 50000}{20}$

$= \dfrac{75000 + x - 50000}{20}$

$= ₹\ \dfrac{25000 + x}{20}$ ... (3)

Agent's remitance to the principal

= The value of the goods sold − commission

$\therefore\ 93750 = x - \dfrac{25000 + x}{20}$

$\therefore\ 93750 \times 20 = 20x - 25000 - x$

$\therefore\ 1875000 = 19x - 25000$

$\therefore\ 19x = 1875000 + 25000$

$\therefore\ 19x = 1900000$

$\therefore\ x = \dfrac{1900000}{19} = 100000$

The value of the goods sold by the agent is ₹ 1,00,000.

5. The present worth of ₹ 11,660 due 9 months hence is ₹ 11,000. Find the rate of interest.

Solution :

The present worth is ₹ 11,000.

Sum due is ₹ 11,660.

$n = 9$ months $= \dfrac{9}{12} = \dfrac{3}{4}$ years,

$r = ?$

SD = PW + TD

$= \text{PW} + \dfrac{\text{PW} \times n \times r}{100}$

$= \text{PW}\left(1 + \dfrac{n \times r}{100}\right)$

$\therefore\ 11660 = 11000\left(1 + \dfrac{\frac{3}{4} \times r}{100}\right)$

$\therefore\ \dfrac{11660}{11000} = 1 + \dfrac{3r}{400}$

$\therefore\ 1.06 - 1 = \dfrac{3r}{400}$

$\therefore\ 0.06 = \dfrac{3r}{400}$

$\therefore\ r = \frac{0.06 \times 400}{3} = \frac{24}{3} = 8$

The rate of interest is 8 % p.a.

6. An article is marked at ₹ 800. A trader allows a discount of 2.5% and gains 20% on the cost. Find the cost price of the article.

Solution :

List (marked) price = ₹ 800.

Rate of discount is 2.5%.

$\therefore$ trade discount $= 800 \times \frac{2.5}{100} =$ ₹ 20

$\therefore$ selling price = ₹ (800 − 20)

= ₹ 780

Let the cost price of the article be ₹ x.

Trader gains 20%.

$\therefore$ the selling price = ₹ $\frac{120x}{100}$ = ₹ $\frac{6x}{5}$

The selling price is ₹ 780.

$\therefore\ \frac{6x}{5} = 780 \qquad \therefore\ x = \frac{780 \times 5}{6}$

$\therefore\ x = 650$

The cost price of the article is ₹ 650.

7. A salesman is paid a fixed monthly salary plus a commission on the sales. If on the sales of ₹ 96,000 and ₹ 1,08,000 in two successive months he receives in all ₹ 17,600 and ₹ 18,800 respectively, find his monthly salary and rate of commission paid to him.

Solution :

Let the fixed monthly salary of the salesman be ₹ x and the rate of commission be r%.

Now, the receipt on the first month's sale of ₹ 96,000 is ₹ 17,600.

$\therefore\ 17600 = x + 96000 \times \frac{r}{100}$

$\therefore\ 17600 = x + 960r \qquad \ldots (1)$

The receipt on the second month's sales of ₹ 1,08,000 is ₹ 18,800.

$\therefore\ 18800 = x + 108000 \times \frac{r}{100}$

$\therefore\ 18800 = x + 1080r \qquad \ldots (2)$

Subtracting (1) from (2), we get

$1200 = 120r$

$\therefore\ r = \frac{1200}{120} \qquad \therefore\ r = 10\%$

Substituting $r = 10$ in (1), we get

$17600 = x + 960 \times 10$

$\therefore\ 17600 = x + 9600$

$\therefore\ x = 17600 - 9600$

$\therefore\ x = 8000$

The fixed monthly salary of the saleman is ₹ 8000 and the rate of commission is 10 %.

8. A merchant buys some mixers at 15% discount on catalogue price. The catalogue price is ₹ 5500 per piece of mixer. The freight charges amount to $2\frac{1}{2}\%$ on the catalogue price. The merchant sells each mixer at 5% discount on catalogue price. His net profit is ₹ 41,250. Find number of mixers.

Solution :

The catalogue price per piece of mixer is ₹ 5500.

Discount on catalogue price is 15 %.

$\therefore$ the discount per piece of mixer

= ₹ $5500 \times \frac{15}{100}$ = ₹ 825.

The freight charges is $2\frac{1}{2}\%$.

$\therefore$ the freight charges = ₹ $5500 \times \frac{2.5}{100}$ = ₹ 137.50

The buying price per mixer

= catalogue price − discount + the freight charges

= ₹ (5500 − 825 + 137.50)

= ₹ 4812.50 … (1)

The merchant sells the mixer at 5% discount on the catalogue price.

5% discount on catalogue price = ₹ $5500 \times \frac{5}{100}$ = ₹ 275

$\therefore$ the selling price per mixer

= ₹ (5500 − 275)

= ₹ 5225 … (2)

From (1) and (2), the profit of the merchant per mixer

= ₹ (5225 − 4812.50)

= ₹ 412.50

The number of mixers $= \frac{\text{Net profit}}{\text{profit per mixer}}$

$$= \frac{41250}{412.50}$$

$$= 100$$

The merchant bought 100 mixers.

9. A bill is drawn for ₹ 7000 on 3rd May for 3 months and is discounted on 25th May at 5.5%. Find the present worth.

Solution :

Sum due (Face value) = ₹ 7000, $r = 5.5\%$

Date of drawing the bill = 3rd May

Period of the bill = 3 months

Nominal due date = 3rd August.

Legal due date = 6th August.

Date of discount = 25th May.

Number of days from the date of discounting to the legal due date is as follows :

May	June	July	August	Total
6	30	31	6	73

$\therefore$ period $n = \frac{73}{365} = \frac{1}{5}$ years

$$\text{SD} = \text{PW}\left(1 + \frac{n \times r}{100}\right)$$

$$\therefore\ 7000 = \text{PW}\left(1 + \frac{1}{5} \times \frac{55}{10} \times \frac{1}{100}\right)$$

$$\therefore\ 7000 = \text{PW}\left(1 + \frac{11}{1000}\right)$$

$$\therefore\ 7000 = \text{PW}\left(\frac{1011}{1000}\right)$$

$$\therefore\ \frac{7000 \times 1000}{1011} = \text{PW}$$

$$\therefore\ \text{PW} = 6923.84$$

The present worth of the bill is ₹ 6923.84.

10. A bill was drawn on 14th April 2005 for ₹ 3500 and was discounted on 6th July 2005 at 5% per annum. The banker paid ₹ 3465 for the bill. Find the period of the bill.

Solution :

Face value of the bill = ₹ 3500.

Cash value of the bill = ₹ 3465.

Banker's discount

$$\text{BD} = \text{FV} - \text{CV}$$

$$= ₹\ (3500 - 3465)$$

$$= ₹\ 35$$

BD = Interest on FV for n years at 5%

$$\therefore\ 35 = 3500 \times n \times \frac{5}{100}$$

$$\therefore\ 35 = 175n$$

$$\therefore\ n = \frac{35}{175} \text{ years}$$

$$\therefore\ n = \frac{35}{175} \times 365 \text{ days}$$

$$\therefore\ n = 73 \text{ days}$$

The bill was discounted on 6th July 2005.

$\therefore$ legal due date is 73 days after 6th July 2005.

July	Aug.	Sept.	Total
25	31	17	73

$\therefore$ legal due date is 17th September 2005.

$\therefore$ nominal due date is 14th September 2005.

Now, date of drawing the bill is 14th April 2005.

$\therefore$ the period of the bill is from 14th April 2005 to 14th September 2005, i.e. 5 months.

11. The difference between true discount and bankers discount on a bill 6 months hence at 4% p.a. is ₹ 80. Find the true discount, banker's discount and amount of the bill.

Solution :

Let TD be ₹ x. $\quad n = 6$ months $= \frac{1}{2}$ years, $r = 4\%$.

BG = BD − TD

= Interest on TD for 6 months at 4% p.a.

$$\therefore\ 80 = x \times \frac{1}{2} \times \frac{4}{100}$$

$$\therefore\ 80 = \frac{x}{50}$$

$$\therefore\ x = 80 \times 50$$

$$\therefore\ x = ₹\ 4000$$

$\therefore$ true discount (TD) is ₹ 4000 … (1)

BD = BG + TD

= 80 + 4000 = 4080

$\therefore$ banker's discount is ₹ 4080. … (2)

Let amount of the bill (FV) be ₹ y.

BD = Interest on FV for 6 months at 4% p.a.

$\therefore\ 4080 = y \times \frac{1}{2} \times \frac{4}{100}$

$\therefore\ 4080 = \frac{y}{50}$

$\therefore\ y = 4080 \times 50 \qquad \therefore\ y = ₹\ 2{,}04{,}000$

∴ the amount of the bill is ₹ 2,04,000. ... (3)

---

12. A manufacturer makes clear profit of 30% on cost after allowing 35% discount. If the cost of production rises by 20%, by what percentage should he reduce the rate of discount so as to make the same rate of profit keeping his list prices unaltered?

Solution :

Let the list price be ₹ 100.

Discount at 35% on ₹ 100 = ₹ 35.

∴ selling price = ₹ (100 − 35) = ₹ 65

Profit at 30% on ₹ 100 = ₹ 30

∴ selling price = ₹ (100 + 30) = ₹ 130

When selling price is ₹ 130, the list price = ₹ 100,

then if the selling price is ₹ 65,

the list price $= \frac{100 \times 65}{130} = ₹\ 50$

If the production cost rises by 20%, then the list price will be $₹\left(50 + 50 \times \frac{20}{100}\right)$

$= ₹\ (50 + 10) = ₹\ 60$

To make the profit at 30% on the list price ₹ 60,

$= ₹\left(60 + 60 \times \frac{30}{100}\right)$

$= ₹\ (60 + 18) = ₹\ 78.$

The list price is unaltered, i.e. it should remain ₹ 100, but the new list price is ₹ 78.

∴ discount = ₹ (100 − 78) = ₹ 22, i.e. 22% discount is allowed on the list price.

Now, 35% discount was allowed on the list price.

∴ reduction in the rate of discount

$= 35 - 22 = 13\%$

The percentage reduction in the rate of discount is 13%.

---

13. A trader offers 25% discount on the catalogue price of a radio and yet makes 20% profit. If he gains ₹ 160 per radio, what must be the catalogue price of the radio?

Solution :

Let the catalogue price of a radio be ₹ x.

Discount at 25% on ₹ $x = x \times \frac{25}{100} = ₹\ \frac{x}{4}$.

∴ selling price of the radio = catalogue price − discount

$= ₹\left(x - \frac{x}{4}\right) = ₹\ \frac{3x}{4}$

If the cost price is ₹ 100, the selling price is ₹ 120 making 20% profit.

When the selling price is ₹ 120, the cost price is ₹ 100.

∴ when the selling price is ₹ $\frac{3x}{4}$, the cost price is $= \frac{100 \times \frac{3x}{4}}{120} = ₹\ \frac{75x}{120}$

He gains profit of ₹ 160 per radio.

Profit = Selling price − Cost price

$\therefore\ 160 = \frac{3x}{4} - \frac{75x}{120}$

$\therefore\ 160 = \frac{90x - 75x}{120}$

$\therefore\ 160 \times 120 = 15x$

$\therefore\ x = \frac{160 \times 120}{15}$

$\therefore\ x = ₹\ 1280$

The catalogue price of the radio is ₹ 1280.

---

14. A bill of ₹ 4800 was drawn on 9th March 2006 at 6 months and was discounted on 19th April 2006 at $6\frac{1}{4}\%$ p.a. How much does the banker charge and how much does the holder receive?

Solution :

Face value of the bill = ₹ 4800, $r = 6\frac{1}{4} = \frac{25}{4}\%$

Date of drawing the bill = 9th March 2006

Period of the bill = 6 months

Nominal due date = 9th September 2006

Legal due date = 12th September 2006

Date of discount = 19th April 2006

Number of days from the date of discounting to the legal due date is as follows :

April	May	June	July	Aug.	Sept.	Total
11	31	30	31	31	12	146

$\therefore$ period $n = \frac{146}{365} = \frac{2}{5}$ years

Banker's charge :

BD = Interest on FV of ₹ 4800 at $\frac{25}{4}$% for $\frac{2}{5}$ years

$= 4800 \times \frac{2}{5} \times \frac{25}{4} \times \frac{1}{100}$

= ₹ 120

The holder receives the amount = FV – BD

= ₹ (4800 – 120)

= ₹ 4680.

Banker's charge is ₹ 120; the holder receives ₹ 4680.

15. A bill of ₹ 65,700 drawn on July 10 for 6 months was discounted for ₹ 65,160 at 5% p.a. On what day was the bill discounted?

Solution :

Face value = ₹ 65700

Cash value = ₹ 65160

BD = FV – CV

= ₹ (65700 – 65160)

= ₹ 540

Date of drawing the bill = 10th July

Period = 6 months

Nominal due date = 10th January (next year)

Legal due date = 13th January (next year)

BD = Interest on FV for n years at 5% p.a.

$\therefore\ 540 = 65700 \times n \times \frac{5}{100}$

$\therefore\ 540 = 3285n$

$\therefore\ n = \frac{540}{3285}$ years

$\therefore\ n = \frac{540}{3285} \times 365$ days

$\therefore\ n = 60$ days

$\therefore$ date of discounting the bill is 60 days before the legal date 13th January.

Jan.	Dec.	Nov.	Total
13	31	16	60

The date of discounting the bill is November 30 – 16 = 14th November.*

[* **Note :** *Answer in the textbook is incorrect.*]

16. An agent sold a car and charged 3% commission on the sale value. If the owner of the car received ₹ 48,500, find the sale value of the car. If the agent charged 2% to the buyer, find his total remuneration.

Solution :

Let the sale value of the car be ₹ 100.

3% commission is charged on the sale value.

$\therefore$ the owner receives ₹ (100 – 3) = ₹ 97

When the owner of the car receives ₹ 97, the sale value of the car is ₹ 100, then

when the owner of the car receives ₹ 48500, the sale value of the car

$= \frac{100 \times 48500}{97}$

= 100 × 500 = ₹ 50000

$\therefore$ the sale value of the car is ₹ 50,000

Agent charged 2% to the buyer on ₹ 50,000

$\therefore$ agent's charge $= 50000 \times \frac{2}{100} =$ ₹ 1000

Agent's commission at the rate of 3% of ₹ 50,000

$= 50000 \times \frac{3}{100} =$ ₹ 1500

$\therefore$ agent's total remuneration

$= \begin{pmatrix}\text{Commission from} \\ \text{buyer}\end{pmatrix} + \begin{pmatrix}\text{Commission from} \\ \text{seller}\end{pmatrix}$

= ₹ (1000 + 1500) = ₹ 2500

The value of the car is ₹ 50,000; Agent's total remuneration is ₹ 2500.

17. An agent is paid a commission of 4% on cash sales and 6% on credit sales made by him. If on the sale of ₹ 51,000, the agent claims a total commission of ₹ 2700, find the sales made by him for cash and on credit.

Solution :

Total sales = ₹ 51,000.

Let cash sales be ₹ x.

Then credit sales = ₹ $(51000 - x)$

Commission at 4% on cash sales of ₹ x

$= x \times \frac{4}{100} =$ ₹ $\frac{x}{25}$

Commission at 6% on credit sales of ₹ $(51000 - x)$

$= (51000 - x) \times \frac{6}{100}$

$= ₹ \left(3060 - \frac{6x}{100}\right)$

Total commission paid to the agent

$= ₹ \left(\frac{x}{25} + 3060 - \frac{6x}{100}\right)$

$= ₹ \left(3060 + \frac{4x - 6x}{100}\right)$

$= ₹ \left(3060 - \frac{2x}{100}\right)$

But the agent claims a total commission of ₹ 2700.

$\therefore\ 2700 = 3060 - \frac{2x}{100}$

$\therefore\ \frac{2x}{100} = 3060 - 2700 \quad \therefore\ \frac{x}{50} = 360 \quad \therefore\ x = 50 \times 360$

$\therefore\ x = ₹\ 18000$

The sales made by the agent for cash is ₹ 18,000.

∴ the sales made by the agent on credit

$= ₹\ (51000 - 18000)$

$= ₹\ 33{,}000.$

The sales made by the agent for cash is ₹ 18,000 and on credit ₹ 33,000.

ACTIVITIES Textbook pages 14 and 15

(Answers are given directly.)

1. The value of the goods sold = ₹ x

Commission @7.5% on first ₹ 10,000 = ₹ $\boxed{750}$

Commission @5% on the balance

$₹\ (x - 10000) = \frac{5}{100} \times \boxed{x - 10000}$

$= ₹ \boxed{\frac{5(x - 10000)}{100}}$

An agent remits ₹ 33,950 to his Principal

$\therefore\ x - \boxed{750} - \boxed{\frac{5(x - 10000)}{100}} = 33950$

$\frac{95x}{100} = 33950 + \boxed{250}$

$\frac{19x}{\boxed{20}} = 34200$

$x = ₹\ \boxed{36000}.$

2. Rate of discount = 15% and other charges = 2.5% on list price.

List price of tricycle in Mumbai = ₹ 600

Net selling price = List Price − Discount $\boxed{15\%}$ +

Other charges $= 600 - \frac{\boxed{15}}{100} \times 600 + \frac{2.5}{100} \times \boxed{600}$

$= ₹\ 525$

List price of tricycle in Nashik = ₹ 750

Rate of discount = 10%

Net selling price = List Price − Discount

$= \boxed{750} - \frac{\boxed{10}}{100} \times 750$

$= ₹\ 675$

A merchant bought tricycles from Mumbai and sold it in Nashik and made a profit of ₹ 13,500

$\therefore$ profit per tricycle $= 675 - \boxed{525}$

$= ₹\ 150$

$\text{Number of tricyles bought} = \frac{\text{Total Profit}}{\text{Profit per tricycles}}$

$= \frac{13500}{\boxed{150}}$

$= \boxed{90}$

3. Cost Price = ₹ 100

A manufacturer makes a profit of 30% on cost after allowing 35% discount.

Selling price $= \boxed{\text{CP}}$ + profit

$= 100 + \frac{30}{100} \times \boxed{100}$

$= ₹\ 130$

Selling price = List price − Discount

$\therefore\ 130 = \text{List price} - \frac{35}{100} \times \boxed{\text{List price}}$

$\therefore\ 130 = \frac{65}{100} \times \boxed{\text{List price}}$

$\therefore$ list price $= \dfrac{130 \times 100}{\boxed{65}}$

$= ₹\ 200$

Now, the cost of production rises by 20%.

$\therefore$ new cost price $= 100 + \dfrac{\boxed{20}}{100} \times 100$

$= ₹\ 120$

New list price $= ₹\ 200$

Rate of discount $= x\,\%$

Selling price $= \boxed{\text{CP}} +$ profit

$= 120 + \dfrac{\boxed{30}}{100} \times 120$

$= ₹\ 156$

List price = Selling price + Discount

$\therefore\ \boxed{200} = \boxed{156} + \dfrac{x}{100} \times 200$

$\therefore\ 2x = 200 - \boxed{156}$

$\therefore\ 2x = 44$

$\therefore\ x = \boxed{22}$

Reduction in the rate of discount

$= \boxed{35} - 22$

$= 13\,\%$.

4. Face Value (SD) = ₹ 4015, $r = 8\%$ p.a.

Date of drawing bill = 19th Janurary 2018

Period of the bill = 8 months

Nominal Due date = $\boxed{\textbf{19}^{\textbf{th}}\ \textbf{September 2018}}$

Legal Due date = 22nd September 2018

Date of discounting the bill = 28th February 2018

Number of days from date of discounting to legal due date

Mar.	Apr.	May	June	July	Aug.	Sept.	Total
31	$\boxed{30}$	$\boxed{31}$	30	$\boxed{31}$	31	$\boxed{22}$	206 days

$\therefore$ BD $= \dfrac{\text{SD} \times n \times r}{100} = 4015 - \boxed{181.30}$

$= ₹\ 3833.70.$

5. Face value (SD) = ₹ 7300, $r = 12\%$ p.a.

Cash value (CV) = ₹ 7108

BD = SD $- \boxed{\text{CV}} =$ ₹ 192

Date of drawing the bill = 7th June 2017

Date of discounting the bill = 22nd October 2017

Number of days from date of discounting to legal due date = x

$\therefore$ BD $= \dfrac{\text{SD} \times n \times r}{100}$

$\therefore\ \boxed{192} = \boxed{7300} \times \dfrac{x}{\boxed{365}} \times \dfrac{12}{100}$

$\therefore\ x = \boxed{80}$ days

$\therefore$ legal due date is 80 days after the date of discounting the bill.

Oct.	Nov.	Dec.	Jan.	Total
$\boxed{9}$	30	$\boxed{31}$	$\boxed{10}$	80 days

Legal due date = 10th January $\boxed{2018}$

Nominal due date = 7th January $\boxed{2018}$

$\therefore$ period of the bill = $\boxed{7}$ months.

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2 INSURANCE AND ANNUITY

CHAPTER OUTLINE

INTRODUCTION

Life is full of risk due to uncertainties. We try to avoid risk by taking necessary steps. But it is not possible to completely prevent risks.

Insurance offers protection against loss to the life of a person or to property, vehicle or valuables due to uncertain events like fire, earthquakes, floods, burglary, etc. The loss due to these events is minimised by insurance.

Insurance is a contract between the insurance company (called the insurer) and the person who insures (called the insured). Insurance policy is a legal document of the agreement or contract between insurer and insured.

Insurance is mainly of two types :

(1) Life Insurance : It is a contract between the insurance company and the person under which a specific amount is payable by the insurance company on the death of the person or on the person attaining a particular age.

- **Premium :** A person who is insured for life agrees to pay a certain instalment of money periodically (i.e. monthly, quarterly, yearly) to the insurance company. This amount of instalment is called premium.
- **Policy value :** The amount received by a person from the insurance company on maturity or on his death or on attaining a certain age is called the policy value.

(2) General Insurance : General insurance covers all types of insurance except the life insurance. It offers protection against loss to the property, vehicle or valuables of a person due to fire, earthquake, flood, etc.

In case of loss or damage, the insurance company guarantees to pay compensation in the proportion that exists between the policy value and property value.

All contracts of general insurance are governed by the principle of indemnity by which the insurer undertakes to pay only the actual amount of loss suffered by the insured. Thus, the insured cannot make a profit out of insurance.

IMPORTANT FORMULAE

1. Insurance :

(1) Premium = Rate of premium × Policy value

(2) General Insurance : $\text{Claim} = \dfrac{\text{Policy value}}{\text{Property value}} \times \text{Loss}$

2. Annuity :

(1) An immediate annuity : The accumulated value A of an immediate annuity for n annual payments of an amount C at an interest rate r per cent per annum, compounded annually, is given by

- $A = \dfrac{C}{i}\left[(1+i)^n - 1\right]$, where $i = \dfrac{r}{100}$

Also, the present value P of such an immediate annuity is given by

- $P = \dfrac{C}{i}[1-(1+i)^{-n}]$

The present value and the future value of an annuity have the following relations :

(i) $A = P(1+i)^n$,

(ii) $\frac{1}{P} - \frac{1}{A} = \frac{i}{C}$.

(2) **Basic formula for an annuity due :** Let C denote the amount paid at the beginning of each of n years and let r denote the rate of interest per cent per annum.

Let $i = \frac{r}{100}$

The accumulated value A' is given by

$$A' = \frac{C(1+i)}{i}\,[(1+i)^n - 1]$$

The present value P' is given by

$$P' = \frac{C(1+i)}{i}\,[1-(1+i)^{-n}]$$

A' and P' have the following relations

$$A' = P'(1+i)^n$$

$$\frac{1}{P'} - \frac{1}{A'} = \frac{i}{C(1+i)}$$

[**Note :** We can use the formula of immediate annuity for annuity due only by replacing C by $C(1+i)$.]

2.1 : FIRE, MARINE AND ACCIDENT INSURANCE

1. **Fire Insurance :** Under this type of insurance, property such as buildings, godowns containing goods, factories, etc. can be insured against loss due to fire.

   (1) **Property value :** The value of entire property is called property value.

   (2) **Policy value :** The value of the property insured is called policy value.

   (3) **Premium :** The amount paid to insurance company to insure the property is called premium. Premium rates are expressed as percentage of the value of the property insured.

   Premium = Rate of premium × Policy value.

   (4) **Period :** The duration of contract for which property is insured is called period.

   Note : The period of fire insurance is one year.

   (5) **Loss :** The value of damage to property is called loss.

   (6) **Claim :** The amount that the insured can demand under the policy is called claim. It is calculated as follows :

   $$\text{Claim} = \frac{\text{Policy value}}{\text{Property value}} \times \text{Loss}$$

2. **Accident Insurance :** Personal accident insurance is a policy that can reimburse your medical costs, provide compensation in case of disability or death caused by accidents. It covers the protection against damage to vehicles like cars, trucks, two wheelers, etc. due to accidents. The policy also covers the liability of the insured to third parties involved in the accident.

   Period : The period of accident insurance is one year.

3. **Marine Insurance :** It covers the risk of damage in the transport of goods by sea. The premium of the policy is determined on the basis of the value of the ship and its contents.

   Shipments are protected from the time they leave the seller's warehouse till the time they reach the buyer's warehouse.

   Period : The journey period of the ship is called the period of the policy.

EXERCISE 2.1 | Textbook page 20

1. Find the premium on a property worth ₹ 25,00,000 at 3% if (i) the property is fully insured (ii) the property is insured to the extent of 80% of its value.

Solution :

Property value = ₹ 25,00,000

Rate of premium = 3%

(i) The property is fully insured :

$$\text{Amount of premium} = 2500000 \times \frac{3}{100}$$

$$= ₹\ 75000$$

(ii) The property is insured to the extent of 80% :

$$\text{Policy value} = 2500000 \times \frac{80}{100} = ₹\ 2000000$$

Amount of premium at the rate 3%

$$= 2000000 \times \frac{3}{100} = ₹\ 60000.$$

(i) ₹ 75,000 (ii) ₹ 60,000.

2. A shop is valued at ₹ 3,60,000 for 75% of its value. If the rate of premium is 0.9%, find the premium paid by the owner of the shop. Also, find the agent's commission, if the agent gets commission at 15% of the premium.

Solution :

The value of shop = ₹ 3,60,000.

Policy value of the shop is 75% of the value of the shop.

∴ policy value of the shop

$$= 360000 \times \frac{75}{100}$$

$$= ₹\ 270000$$

Rate of premium is 0.9%,

∴ amount of premium

$$= 270000 \times \frac{0.9}{100}$$

$$= ₹\ 2430$$

Premium paid by the owner is ₹ 2430.

Commission of an agent :

Commission at 15% of the premium of ₹ 2430

$$= 2430 \times \frac{15}{100} = \frac{36450}{100} = ₹\ 364.50$$

Agent's commission is ₹ 364.50.

---

3. A person insures his office valued at ₹ 5,00,000 for 80% of its value. Find the rate of premium, if he pays ₹ 13,000 as premium. Also, find agent's commission at 11%.

Solution :

Value of office = ₹ 5,00,000.

Policy value is 80% of the value.

$$\therefore \text{ policy value} = 500000 \times \frac{80}{100} = ₹\ 400000$$

Let the rate of the premium be r%.

$$\therefore \text{ amount of premium} = 400000 \times \frac{r}{100}$$

But, the premium paid is ₹ 13000

$$\therefore\ 13000 = 400000 \times \frac{r}{100}$$

$$\therefore\ r = \frac{13000 \times 100}{400000}$$

$$\therefore\ r = 3.25\%$$

Rate of premium is 3.25%

Agent's Commission :

Amount of premium = ₹ 13000

Rate of commission = 11%

$$\therefore \text{ agent's commission} = 13000 \times \frac{11}{100}$$

Agent's commission = ₹ 1430.

---

4. A building is insured for 75% of its value. The annual premium at 0.70 per cent amounts to ₹ 2625. If the building is damaged to the extent of 60% due to fire, how much can be claimed under the policy?

Solution :

Let the value of the building be ₹ x.

$$\text{Then the policy value of the building} = x \times \frac{75}{100}$$

$$= ₹\ \frac{3x}{4}$$

Rate of premium is 0.70%

Premium amount = ₹ 2625

Amount of the premium = policy value × rate of premium

$$\therefore\ 2625 = \frac{3x}{4} \times \frac{0.70}{100}$$

$$\therefore\ x = \frac{2625 \times 4 \times 100}{3 \times 0.70}$$

$$\therefore\ x = 500000$$

∴ the value of the building = ₹ 5,00,000

Damage = 60% of the value of the building

$$= 500000 \times \frac{60}{100} = ₹\ 3,00,000$$

$$\text{Policy value of the building} = \frac{3x}{4}$$

$$= \frac{3}{4} \times 500000 = ₹\ 3,75,000$$

$$\text{Claim} = \frac{\text{Policy value}}{\text{Property value}} \times \text{Loss}$$

$$= \frac{375000}{500000} \times 300000 = ₹\ 225000$$

∴ claim = ₹ 2,25,000.

---

5. A stock worth ₹ 7,00,000 was insured for ₹ 4,50,000. Fire burnt stock worth ₹ 3,00,000 completely and damaged the remaining stock to the extent of 75% of its value. What amount can be claimed under the policy?

Solution :

Value of the stock = ₹ 7,00,000

Policy value of the stock = ₹ 4,50,000

Value of the stock burnt = ₹ 3,00,000

The value of the remaining stock $= (\text{₹ } 700000 - 3{,}00{,}000)$

$= \text{₹ } 4{,}00{,}000$

Damage to the remaining stock

$$= 400000 \times \frac{75}{100} = \text{₹ } 3{,}00{,}000$$

Total loss $= \text{₹ } (300000 + 300000)$

$= \text{₹ } 6{,}00{,}000$

$$\text{Claim} = \frac{\text{Policy value}}{\text{Stock value}} \times \text{Loss}$$

$$= \frac{450000}{700000} \times 600000$$

$$= 385714.28 \approx 385714.30$$

Amount to be claimed is ₹ 3,85,714.30.

---

6. A cargo of rice was insured at 0.625% to cover 80% of its value. The premium paid was ₹ 5250. If the price of rice is ₹ 21 per kg, find the quantity of rice (in kg) in the cargo.

Solution :

Let the value of a cargo of rice be ₹ x.

The policy value of the cargo is 80% of x.

$\therefore$ the policy value of the cargo $= \text{₹ } x \times \frac{80}{100} = \text{₹ } \frac{4x}{5}$

The rate of premium = 0.625%

The premium paid = ₹ 5250

Amount of premium = Policy value × Rate of premium

$$\therefore\ 5250 = \frac{4x}{5} \times 0.625 \times \frac{1}{100}$$

$$\therefore\ 5250 = \frac{x}{200}$$

$\therefore\ x = 5250 \times 200 \qquad \therefore\ x = \text{₹ } 1050000$

The value of the cargo is ₹ 10,50,000

If the value of rice is ₹ 21 per kg, then the cargo contains

$\frac{1050000}{21} = 50000$ kg of rice.

The quantity of rice in the cargo is 50,000 kg.

---

7. 60000 articles costing ₹ 200 per dozen were insured against fire for ₹ 2,40,000. If 20% of the articles were burnt and 7200 of the remaining articles were damaged to the extent of 80% of their value, find the amount that can be claimed under the 'policy.

[**Note :** *Question has been modified.*]

Solution :

The number of articles = 60000

$$= \frac{60000}{12} = 5000 \text{ dozen}$$

Cost of one dozen articles = ₹ 200

$\therefore$ the cost of 5000 dozen articles

$= \text{₹ } 200 \times 5000 = \text{₹ } 10{,}00{,}000$

Policy value of the articles = ₹ 2,40,000

The number of articles completely burnt

$$= 60000 \times \frac{20}{100} \times 12000 = \frac{12000}{12} = 1000 \text{ dozen}$$

The value of the burnt articles

$= \text{₹ } 200 \times 1000 = \text{₹ } 2{,}00{,}000 \qquad \ldots (1)$

80% of 7200 articles were damaged.

$$7200 \times \frac{80}{100} = 5760. \quad \frac{5760}{12} = 480 \text{ dozen}$$

The value of the damaged articles

$= \text{₹ } 200 \times 480 = \text{₹ } 96{,}000 \qquad \ldots (2)$

Total loss $= \text{₹ } (200000 + 96000)$ … [From (1) and (2)]

$= \text{₹ } 2{,}96{,}000.$

$$\text{Claim} = \frac{\text{Policy value}}{\text{Property value}} \times \text{Loss}$$

$$= \frac{240000}{1000000} \times 296000$$

$$= \text{₹ } 71040$$

₹ 71,040 can be claimed.

---

8. The rate of premium is 2% and other expenses are 0.75%. A cargo worth ₹ 3,50,100 is to be insured so that all its value and the cost of insurance will be recovered in the event of total loss.

[**Note :** *Question has been modified.*]

Solution :

Let the policy value of the cargo be ₹ 100, including the premium and other expenses.

The rate of premium is 2%.

Other expenses are 0.75%

$\therefore$ the premium ₹ 2 and other expenses ₹ 0.75.

$\therefore$ the value of the cargo

$= \text{₹ } [100 - (2 + 0.75)] = \text{₹ } 97.25$

When the value of the cargo is ₹ 97.25, the policy value is ₹ 100.

Therefore, when the value of the cargo is ₹ 3,50,100, the policy value of the cargo

$$= \frac{100 \times 350100}{97.25}$$

$$= 360000$$

The policy value of the cargo is ₹ 3,60,000.

9. A property worth ₹ 4,00,000 is insured with three companies, A, B and C. The amounts insured with these companies are ₹ 1,60,000, ₹ 1,00,000 and ₹ 1,40,000 respectively. Find the amount recoverable from each company in the event of a loss to the extent of ₹ 9000.

Solution :

Company	Amount insured
A	₹ 1,60,000
B	₹ 1,00,000
C	₹ 1,40,000
Total	₹ 4,00,000

$$\text{Claim} = \frac{\text{Policy value}}{\text{Property value}} \times \text{Loss}$$

Claim from company A :

$$= \frac{160000}{400000} \times 9000 = ₹\ 3600$$

Claim from company B :

$$= \frac{100000}{400000} \times 9000 = ₹\ 2250$$

Claim from company C :

$$\frac{140000}{400000} \times 9000 = ₹\ 3150$$

10. A car valued at ₹ 8,00,000 is insured for ₹ 5,00,000. The rate of premium is 5% less 20%. How much will the owner bear including the premium, if value of the car is reduced to 60% of its original value.

Solution :

The value of a car = ₹ 8,00,000

The policy value of a car = ₹ 5,00,000.

The rate of premium = 5% – (20% of 5)

= 5% – 1% = 4%

Amount of premium = Policy value × Rate of premium

$$= ₹\ 500000 \times \frac{4}{100} = ₹\ 20000$$

The value of the car is reduced to 60% of its original value.

$$\therefore \text{ the value of the car} = ₹\ 800000 \times \frac{60}{100}$$

$$= ₹\ 4{,}80{,}000$$

Loss = ₹ (800000 – 480000) = ₹ 3,20,000

= ₹ 3,20,000

$$\text{Claim} = \frac{\text{Insured value}}{\text{Value of the car}} \times \text{Loss}$$

$$= \frac{500000}{800000} \times 320000 = ₹\ 2{,}00{,}000$$

Loss the owner bears = loss – claim + premiuim

= ₹ (320000 – 200000 + 20000)

= ₹ 1,40,000.

11. A shop and a godown worth ₹ 1,00,000 and ₹ 2,00,000 respectively were insured through an agent who was paid 12% of the total premium. If the shop was insured for 80% and the godown for 60% of their respective values, find the agent's commission, given that the rate of premium was 0.80% less 20%.

Solution :

The value of the shop = ₹ 1,00,000.

The value of the godown = ₹ 2,00,000

Insured value of the shop

$$= ₹\ 100000 \times \frac{80}{100}$$

= ₹ 80,000 … (1)

Insured value of the godown

$$= ₹\ 200000 \times \frac{60}{100}$$

= ₹ 1,20,000 … (2)

Total policy value = ₹ (80000 + 120000)

… [From (1) and (2)]

= ₹ 2,00,000

The rate of premium = 0.80% less 20%

= 0.80% – (20% of 0.80)

= 0.80% – 0.16% = 0.64%

Amount of premium on ₹ 2,00,000

$$= ₹\ 200000 \times \frac{0.64}{100} = ₹\ 1280$$

Agent's commission at 12% of the premium

$$= ₹\ 1280 \times \frac{12}{100} = ₹\ 153.60$$

Agent's commission is ₹ 153.60.

12. The rate of premium on a policy of ₹ 1,00,000 is ₹ 56 per thousand per annum. A rebate of ₹ 0.75 per thousand is permitted, if the premium is paid annually. Find the net amount of premium payable, if the policy holder pays the premium annually.

Solution :

The policy value = ₹ 1,00,000.

The rate of premium = ₹ 56 per thousand per annum.

If a rebate of ₹ 0.75 per thousand is permitted when the premium is paid annually, then the rate of premium

$= ₹\ (56 - 0.75) = ₹\ 55.25$

When the policy value is ₹ 1000, the premium is ₹ 55.25, then for the policy value of ₹ 1,00,000

$$\text{net premium} = \frac{100000}{1000} \times 55.25$$

$$= ₹\ 5525$$

The policy holder pays ₹ 5525* premium annually.

[***Note :** *Answer given in the textbook is incorrect.*]

13. A warehouse valued at ₹ 40,000 contains goods worth ₹ 2,40,000. The warehouse is insured against fire for ₹ 16,000 and the goods to the extent of 90% of their value. Goods worth ₹ 80,000 are completely destroyed, while the remaining goods are destroyed to 80% of their value due to a fire. The damage to the warehouse is to the extent of ₹ 8000. Find the total amount that can be claimed.

Solution :

The value of the warehouse = ₹ 40,000.

Goods in warehouse worth ₹ 2,40,000

Goods is insured for 90% of its value

$$= ₹\ 240000 \times \frac{90}{100} = ₹\ 2,16,000$$

This is the policy value.

Goods worth ₹ 80,000 are completely destroyed.

∴ loss is ₹ 80,000.

$$\text{Claim} = \frac{\text{Policy value}}{\text{Value of good}} \times \text{Loss}$$

$$= \frac{216000}{240000} \times 80000$$

$$= ₹\ 72,000 \qquad \dots (1)$$

The remaining goods, i.e. goods worth ₹ (240000 − 80000) = ₹ 1,60,000 was destroyed to 80%.

∴ the value of the remaining goods

$$= ₹\ 160000 \times \frac{80}{100}$$

$$= ₹\ 1,28,000.$$

Loss incurred = ₹ 1,28,000

$$\text{Claim} = \frac{\text{Policy value}}{\text{Value of the goods}} \times \text{Loss}$$

$$= \frac{216000}{240000} \times 128000$$

$$= ₹\ 1,15,200 \qquad \dots (2)$$

The damage to the warehouse = ₹ 8000

The value of the warehouse = ₹ 40,000

Policy value of the warehouse = ₹ 16,000

$$\text{Claim for the warehouse} = \frac{16000}{40000} \times 8000$$

$$= ₹\ 3200 \qquad \dots (3)$$

Total claim = ₹ (72000 + 115200 + 3200)

... [From (1), (2) and (3)]

= ₹ 1,90,400

The total amount that can be claimed is ₹ 1,90,400.

14. A person takes a life policy for ₹ 2,00,000 for a period of 20 years. He pays premium for 10 years during which bonus was declared at an average rate of ₹ 20 per year per thousand. Find the paid up value of the policy, if he discontinues paying premium after 10 years.

Solution :

Policy value = ₹ 2,00,000.

Rate of bonus = ₹ 20 per year per thousand

Bonus for 1 year on the policy of ₹ 2,00,000

$$= \frac{200000}{1000} \times 20 = ₹\ 4000$$

∴ bonus for 10 years = ₹ 4000 × 10 = ₹ 40,000.

Period of policy 20 years

$$\therefore \text{ yearly premium to be paid} = \frac{200000}{20} = ₹\ 10,000$$

∴ premium paid for 10 years = ₹ 10000 × 10

= ₹ 1,00,000

Paid up value of the policy = premium paid + bonus

= ₹ (100000 + 40000)

= ₹ 1,40,000

The paid up value of the policy is ₹ 1,40,000.

EXAMPLES FOR PRACTICE 2.1

1. A car worth ₹ 8,10,000 is insured for ₹ 6,75,000. The car is damaged to the extent of ₹ 3,60,000 in an accident. Find the amount of compensation that can be claimed under the policy.
2. A shopkeeper insures his shop valued ₹ 30 lakh for 80% of its value. He pays a premium ₹ 1,20,000. Find the rate of premium. If the agent gets commission at 12%, find the agent's commission.
3. A building worth ₹ 50,00,000 is insured for $\frac{3}{5}$th of its value at a premium of 5%. Find the amount of premium. Find the commission of the agent, if the rate of commission is 4%.
4. A property worth ₹ 8,00,000 is insured with three companies, A, B and C for amounts ₹ 2,40,000, ₹ 1,60,000 and ₹ 2,00,000 respectively. A fire caused a loss of ₹ 4,80,000. Calculate the amount that can be claimed from three companies.
5. For what amount must a car valued at ₹ 2,92,500 be insured at 2.5% so as to recover its value and the cost of premium in case of loss.
6. A merchant insures fully his warehouse worth ₹ 2,40,000 and goods in it. If the total amount of premium paid at the rate of 2.0% is ₹ 7200, find the value of the goods in the warehouse.
7. A trolly loaded with wheat worth ₹ 80,000 is insured for ₹ 64,000. In an accident it was damaged to the extent of ₹ 15,000. Find the amount of compensation that can be claimed under the policy.
8. (i) If the policy value and the property value are in the ratio 2 : 5 and the loss is ₹ 2,750, find the amount to be claimed under the policy.
   (ii) If the policy of a cargo covers 80% of its value and the amount claimed is ₹ 1,240, find the loss.
9. Find the agent's commission at 15% on the first premium if he places insurance for ₹ 200000 on the life of a person, the premium being at the rate of ₹ 35 per thousand, per annum, paid annually.

Answers

1. ₹ 3,00,000 **2.** The rate of premium : 5%; Agent commission : ₹ 14,400 **3.** The amount of premium : ₹ 1,50,000; Agent's commission : ₹ 6,000
4. Company A : ₹ 1,44,000; Company B : ₹ 96,000; Company C : ₹ 1,20,000 **5.** ₹ 3,00,000 **6.** ₹ 1,20,000
7. ₹ 12,000 **8.** (i) ₹ 1100 (ii) ₹ 1550 **9.** ₹ 1050.

2.2 : ANNUITY

A rupee today is worth more than a rupee after one year. Money has a time value. The time value of money explains why interest is paid or received.

An annuity is a sequence of payments of equal amounts with a fixed frequency.

A life annuity pays out an income at regular intervals until the death.

1. Terminology of Annuity :

Annuitant : A person who receives an annuity is called the annuitant.

Issuer : A company (usually an insurance company) that issues an annuity.

Owner : An individual or an entity that buys an annuity from the issuer of the annuity and makes contributions to the annuity.

Beneficiary : A person who receives a death benefit from an annuity at the death of the annuitant.

2. Two phases of an annuity :

Accumulation phase : The time period when money is added to the annuity is called accumulation phase.

An annuity can be purchased in one single lump sum (single premium annuity) or by making investments periodically over time.

Distribution phase : The distribution phase is when the annuitant receiving distributions from the annuity.

Options : (i) Withdraw some or all of the money in the annuity in lump sums. (ii) A guaranteed income for a specific period of time or the entire lifetime of the annuitant.

3. Types of Annuities : There are three types of annuities :

(i) Annuity Certain : An annuity certain is an investment that provides a series of payments for a set period of time to a person or to the person's beneficiary. It is an investment in retirement income offered by insurance companies. The annuity may be taken as a lump sum. Annuity certain generally pays a higher rate of return than lifetime annuity.

(ii) Contingent Annuity : Contingent annuity is a form of annuity contract that provides payments at the time when the named contingency occurs. Upon death of one spouse, the surviving spouse will begin to receive monthly payments.

(iii) Perpetual Annuity or Perpetuity : A perpetual annuity promises to pay a certain amount of money to its owner forever. Though a perpetuity may promise to pay you forever, its value is not infinite.

4. **Classification of Annuity Certain :** Annuity certain is classified on the basis of payment, interval and time of payment. There are three types of annuity certain :

(i) **Annuity Immediate OR Ordinary Annuity :** If the payment is made at the end of each period, it is called an Annuity Immediate OR Ordinary Annuity.

(ii) **Annuity Due :** If the periodic payment is made at the commencement of each period, it is called Annuity Due.

(iii) **Deferred OR Reversionary Annuity :** If the deposit is allowed to accumulate for a certain period and payments begin after the lapse of that period, it is called Deferred OR Reversionary Annuity.

We shall study only immediate annuity and annuity due.

5. **Present Value of an Annuity :** The sum of the present values of the different instalments of an annuity is called the present value of an annuity. Thus, it is the total present worth of all periodic future payments.

6. **Future Value of an Annuity :** It is the sum of all the accumulated values of each payment calculated at the end of last period of the annuity certain at a given rate of compound interest. Thus, if each periodic payment is invested at compound interest, the final accumulated sum with interest would be the amount of that annuity.

[Notes :

(1) We consider only uniform and certain annuities.

(2) If the type of an annuity is not mentioned, we assume that the annuity is immediate annuity.

(3) If there is no mention of the type of interest, then it is assumed that the interest is compounded per annum.
If payments are made half yearly (that is, twice per year), then r is replaced by $\frac{r}{2}$ (the compounding rate) and n is replaced by $2n$ (the number of time periods).
If payments are made quarterly (that is, four times per year), then r is replaced by $\frac{r}{4}$ (the compounding rate) and n is replaced by $4n$ (the number of time periods).
If payments are made monthly (that is, 12 times per year), then r is replaced by $\frac{r}{12}$ (the compounding rate) and n is replaced by $12n$ (the number of time periods).]

7. **Immediate Annuity :** Payments are made at the end of every time period in immediate annuity.

8. **Annuity Due :** Payments are made at the beginning of every time period in annuity due.

EXERCISE 2.2 | **Textbook pages 27 and 28**

1. Find the accumulated (future) value of annuity of ₹ 800 for 3 years at interest rate 8% compounded annually. [Given : $(1.08)^3 = 1.2597$]

Solution :

Here, $C = ₹\ 800,\ n = 3,\ r = 8\%$

$\therefore\ i = \frac{r}{100} = \frac{8}{100} = 0.08;\ (1.08)^3 = 1.2597$ ... (Given)

$$A = \frac{C}{i}[(1+i)^n - 1]$$

$$= \frac{800}{0.08}[(1+0.08)^3 - 1]$$

$$= 10000[(1.08)^3 - 1]$$

$$= 10000[1.2597 - 1]$$

$$= 10000 \times 0.2597$$

$$= 2597.$$

Hence, the accumulated value of annuity of ₹ 800 is ₹ 2597.*

[* **Note :** *Answer given in the textbook is incorrect.*]

2. A person invested ₹ 5000 every year in a finance company that offered him interest compounded at 10% p.a. What is the amount accumulated after 4 years. [Given : $(1.1)^4 = 1.4641$]

खहेयअंहृ :

Here, $C = ₹\ 5000,\ n = 4,\ r = 10\%$

$$i = \frac{r}{100} = \frac{10}{100} = 0.1. \quad (1.1)^4 = 1.4641 \qquad \ldots \text{(Given)}$$

$$A = \frac{C}{i}[(1+i)^n - 1]$$

$$= \frac{5000}{0.1}[(1+0.1)^4 - 1]$$

$$= 50000\,[(1.1)^4 - 1]$$

$$= 50000\,[1.4641 - 1]$$

$$= 50000 \times 0.4641$$

$$= 23{,}205$$

Hence, the amount accumulated is ₹ 23,205.

3. Find the amount accumulated after 2 years, if a sum of ₹ 24,000 is invested every 6 months at 12% p.a. compounded half yearly.

[Given : $(1.06)^4 = 1.2625$]

Solution :

Here, $C = ₹\ 24000,\ n = 2$ years. Term is of 6 months.

$$\therefore\ n = 2 \times 2 = 4,\ r = \frac{12}{2}\% = 6\%$$

$$i = \frac{r}{100} = \frac{6}{100} = 0.06; \quad (1.06)^4 = 1.2625 \qquad \ldots \text{(Given)}$$

$$A = \frac{C}{i}[(1+i)^n - 1]$$

$$= \frac{24000}{0.06}[(1+0.06)^4 - 1]$$

$$= 400000\,[(1.06)^4 - 1]$$

$$= 400000\,[1.2625 - 1]$$

$$= 400000 \times 0.2625$$

$$= 105000$$

Hence, the amount accumulated is ₹ 1,05,000.*

[* **Note :** *Answer given in the textbook is incorrect.*]

4. Find accumulated value after 1 year of an annuity immediate in which ₹ 10,000 is invested every quarter at 16% p.a. compounded quarterly.

[Given : $(1.04)^4 = 1.1699$]

Solution :

Here, $C = ₹\ 10000$

$n = 1$ year. Term is every quarter.

$$\therefore\ n = 1 \times 4 = 4 \text{ and } r = \frac{16\%}{4} = 4\%$$

$$i = \frac{r}{100} = \frac{4}{100} = 0.04; \quad (1.04)^4 = 1.1699 \qquad \ldots \text{(Given)}$$

$$A = \frac{C}{i}[(1+i)^n - 1]$$

$$= \frac{10000}{0.04}[(1+0.04)^4 - 1]$$

$$= 250000\,[(1.04)^4 - 1]$$

$$= 250000\,[1.1699 - 1]$$

$$= 250000 \times 0.1699$$

$$= 42475$$

Hence, the accumulated value is ₹ 42475.*

[* **Note :** *Answer given in the textbook is incorrect.*]

5. Find the present value of an annuity immediate of ₹ 36,000 p.a. for 3 years at 9% p.a. compounded annually. [Given : $(1.09)^{-3} = 0.7722$]

Solution :

Here, $C = ₹\ 36000,\ n = 3,\ r = 9\%$

$$i = \frac{r}{100} = \frac{9}{100} = 0.09; \ (1.09)^{-3} = 0.7722 \qquad \ldots \text{(Given)}$$

$$\text{Now, } P = \frac{C}{i}[1 - (1+i)^{-n}]$$

$$= \frac{36000}{0.09}[1 - (1+0.09)^{-3}]$$

$$= 400000\,[1 - (1.09)^{-3}]$$

$$= 400000\,[1 - 0.7722]$$

$$= 400000 \times 0.2278 = 91120$$

Hence, the present value of an annuity immediate of ₹ 36000 is ₹ 91,120.

6. Find the present value of an ordinary annuity of ₹ 63,000 per annum for 4 years at 14% p.a., compounded annually. [Given : $(1.14)^{-4} = 0.5921$]

Solution :

Here, $C = ₹\ 63000,\ n = 4,\ r = 14\%$

$$i = \frac{r}{100} = \frac{14}{100} = 0.14; \ (1.14)^{-4} = 0.5921 \qquad \ldots \text{(Given)}$$

$$\text{Now, } P = \frac{C}{i}[1 - (1+i)^{-n}]$$

$$= \frac{63000}{0.14}[1 - (1+0.14)^{-4}]$$

$$= 450000\,[1 - (1.14)^{-4}]$$

$$= 450000\,[1 - 0.5921]$$

$$= 450000 \times 0.4079$$

$$= 183555$$

Hence, the present value of an ordinary annuity of ₹ 63000 is ₹ 1,83,555.

7. A lady plans to save for her daughter's marriage. She wishes to accumulate a sum of ₹ 4,64,100 at the end of 4 years. What amount should she invest every year, if she can get interest of 10% p.a. compounded annually? [Given : $(1.1)^4 = 1.4641$]

Solution :

Here, $A = ₹\ 4{,}64{,}100,\ n = 4,\ r = 10\%,\ C = ?$

$$i = \frac{r}{100} = \frac{10}{100} = 0.1; \quad (1.1)^4 = 1.4641 \qquad \ldots \text{(Given)}$$

$$A = \frac{C}{i}[(1+i)^n - 1]$$

$$\therefore\ 464100 = \frac{C}{0.1}[(1+0.1)^4 - 1]$$

$$\therefore\ 464100 \times 0.1 = C[(1.1)^4 - 1]$$

$$\therefore\ 46410 = C[1.4641 - 1]$$

$$\therefore\ 46410 = C \times 0.4641$$

$$\therefore\ C = \frac{46410}{0.4641}$$

$$\therefore\ C = ₹\ 100000$$

Hence, the lady should invest ₹ 1,00,000 every year for 4 years to get ₹ 4,64,100 at the end of 4 years.

8. A person wants to create a fund of ₹ 6,96,150 after 4 years at the time of his retirement. He decides to invest a fixed amount at the end of every year in a bank that offers him interest of 10% p.a. compounded annually. What amount should he invest every year? [Given : $(1.1)^4 = 1.4641$]

Solution :

Here, $A = ₹\ 6{,}96{,}150,\ n = 4,\ r = 10\%,\ C = ?$

$$i = \frac{r}{100} = \frac{10}{100} = 0.1; \quad (1.1)^4 = 1.4641 \qquad \ldots \text{(Given)}$$

$$A = \frac{C}{i}[(1+i)^n - 1]$$

$$\therefore\ 696150 = \frac{C}{0.1}[(1+0.1)^4 - 1]$$

$$\therefore\ 696150 \times 0.1 = C[(1.1)^4 - 1]$$

$$\therefore\ 696150 = C[1.4641 - 1]$$

$$\therefore\ 69615 = C \times 0.4641$$

$$\therefore\ C = \frac{69615}{0.4641}$$

$$\therefore\ C = ₹\ 1{,}50{,}000$$

Hence, the person should invest a sum of ₹ 1,50,000 every year.

9. Find the rate of interest compounded annually, if an annuity immediate at ₹ 20,000 per year amounts to ₹ 2,60,000 in 3 years.

Solution :

Here, $C = ₹\ 20{,}000,\ A = ₹\ 2{,}60{,}000,\ n = 3,\ r = ?$

$$A = \frac{C}{i}[(1+i)^n - 1]$$

$$\therefore\ 260000 = \frac{20000}{i}[(1+i)^3 - 1]$$

$$\therefore\ \frac{260000}{20000} = \frac{1}{i}[(1+i)^3 - 1]$$

$$\therefore\ 13 = \frac{1 + 3i + 3i^2 + i^3 - 1}{i}$$

$$\therefore\ 13 = \frac{i^3 + 3i^2 + 3i}{i}$$

$$\therefore\ 13 = i^2 + 3i + 3$$

$$\therefore\ i^2 + 3i + 3 - 13 = 0$$

$$\therefore\ i^2 + 3i - 10 = 0$$

$$\therefore\ i^2 + 5i - 2i - 10 = 0$$

$$\therefore\ i(i+5) - 2(i+5) = 0$$

$$\therefore\ (i+5)(i-2) = 0$$

$$\therefore\ (i+5) = 0 \ \text{ or } \ i - 2 = 0$$

$$\therefore\ i = -5 \quad \text{or } \ i = 2$$

$i = -5$ not acceptable, $\because\ i > 0$

$$\therefore\ i = 2$$

$$i = \frac{r}{100}$$

$$\therefore\ 2 = \frac{r}{100} \qquad \therefore\ r = 200\%$$

Hence, the rate of interest is 200%.

10. Find the number of years for which an annuity of ₹ 500 is paid at the end of every year, if the accumulated amount works out to be ₹ 1655 when interest compounded at 10% p.a.?

Solution :

Here, $C = ₹\ 500,\ A = ₹\ 1665,\ r = 10\%,\ n = ?$

$$i = \frac{r}{100} = \frac{10}{100} = 0.1$$

$$A = \frac{C}{i}[(1+i)^n - 1]$$

$\therefore\ 1665=\dfrac{500}{0.1}[(1+0.1)^n-1]$

$\therefore\ 1665=5000[(1.1)^n-1]$

$\therefore\ \dfrac{1665}{5000}=(1.1)^n-1$

$\therefore\ 0.331=(1.1)^n-1$

$\therefore\ 0.331+1=(1.1)^n$

$\therefore\ 1.331=(1.1)^n$

$\therefore\ (1.1)^3=(1.1)^n$

$\therefore\ 3=n$ ... ($\because$ base is same)

Hence, the number of years for which an annuity of ₹ 500 is paid at the end of every year is 3 years.

11. Find the accumulated value of annuity due of ₹ 1000 p.a. for 3 years at 10% p.a. compounded annually. [Given : $(1.1)^3=1.331$]

Solution :

Here, $C=$ ₹ 1000, $n=3$, $r=10\%$, $A'=?$

$i=\dfrac{r}{100}=\dfrac{10}{100}=0.1;\ \ (1.1)^3=1.331$ ... (Given)

$A'=\dfrac{C(1+i)}{i}[(1+i)^n-1]$

$A'=\dfrac{1000(1+0.1)}{0.1}[(1+0.1)^3-1]$

$=\dfrac{1000\times 1.1}{0.1}[(1.1)^3-1]$

$=11000[1.331-1]$

$=11000\times 0.331$

$=3641$

Hence, the accumulated value of annuity due of ₹ 1000 is ₹ 3641.

12. A person plans to put ₹ 400 at the beginning of each year for 2 years in a deposit that gives interest at 2% p.a. compounded annually. Find the amount that will be accumulated at the end of 2 years?

Solution :

Here, $C=$ ₹ 400, $n=2$, $r=2\%$, $A'=?$

$i=\dfrac{r}{100}=\dfrac{2}{100}=0.02.$

$A'=\dfrac{C(1+i)}{i}[(1+i)^n-1]$

$\therefore\ A'=\dfrac{400(1+0.02)}{0.02}[(1+0.02)^2-1]$

$=\dfrac{400(1.02)}{0.02}[(1.02)^2-1]$

$=20400[1.0404-1]$

$=20400[0.0404]\quad =824.16$

Hence, the accumulated amount after 2 years will be ₹ 824.16.

13. Find the present value of an annuity due of ₹ 600 to be paid quarterly, at 32% p.a. compounded quarterly for 1 year. [Given : $(1.08)^{-4}=0.7350$]

Solution :

Here, $C=$ ₹ 600, $P'=?$

Term of payment is quarterly

$\therefore\ n=1\times 4=4$

$r=32\%$ p.a.

$\therefore\ r=\dfrac{32}{4}=8\%$ quarterly, $i=\dfrac{r}{100}=\dfrac{8}{100}=0.08$

Now, $P'=\dfrac{C(1+i)}{i}[1-(1+i)^{-n}]$

$=\dfrac{600(1+0.08)}{0.08}[1-(1+0.08)^{-4}]$

$=\dfrac{600\times 1.08}{0.08}[1-(1.08)^{-4}]$

$=\dfrac{600\times 1.08}{0.08}[1-0.7350]$

$=8100[0.265]$

$=2146.50$

Hence, the present value of an annuity due of ₹ 600 is ₹ 2146.50.

14. An annuity immediate is to be paid for some years, at 12% p.a. The present value of the annuity is ₹ 10,000 and the accumulated value is ₹ 20,000. Find the amount of each annuity payment?

Solution :

Here, $P=$ ₹ 10,000, $A=$ ₹ 20,000, $r=12$ $\therefore\ i=\dfrac{12}{100}=0.12$

We have to find the amount of each annuity C.

$\dfrac{1}{P}-\dfrac{1}{A}=\dfrac{i}{C}$

$\therefore\ \dfrac{1}{10000}-\dfrac{1}{20000}=\dfrac{0.12}{C}$

$\therefore\ 0.0001-0.00005=\dfrac{0.12}{C}$

$\therefore\ 0.00005\times C=0.12$

$\therefore\ C=\dfrac{0.12}{0.00005}$

$\therefore\ C = 2400$

Hence, the amount of each annuity payment is ₹ 2400.

15. For an annuity immediate paid for 3 years with interest compounded at 10% p.a., the present value is ₹ 24,000. What will be the accumulated value after 3 years? [Given : $(1.1)^3 = 1.331$]

Solution :

Here, $P = ₹\ 24000,\ n = 3,\ r = 10\%,\ A = ?$

$i = \frac{r}{100} = \frac{10}{100} = 0.1;\quad (1.1)^3 = 1.331$ ... (Given)

$A = P(1+i)^n$

$= 24000\,(1+0.1)^3$

$= 24000\,(1.1)^3$

$= 24000 \times 1.331$

$= 31944$

Hence, the accumulated value of an annuity immediate after three years will be ₹ 31,944.

16. A person sets up a sinking fund in order to have ₹ 1,00,000 after 10 years. What amount should be deposited biannually in the account that pays him 5% p.a. compounded semiannually? [Given : $(1.025)^{20} = 1.675$]

Solution :

Here, $A = ₹\ 1,00,000,\ n = 10,\ r = 5\%$

Amount should be set aside semiannually

$\therefore\ n = 10 \times 2 = 20,\ r = \frac{5}{2} = 2.5$

$i = \frac{r}{100} = \frac{2.5}{100} = 0.025,\ C = ?$

$A = \frac{C}{i}[(1+i)^n - 1]$

$\therefore\ 100000 = \frac{C}{0.025}[(1+0.025)^{20} - 1]$

$\therefore\ 100000 \times 0.025 = C[(1.025)^{20} - 1]$

$\therefore\ 2500 = C[1.675 - 1]$ $[\because\ (1.025)^{20} = 1.675]$

$\therefore\ 2500 = C[0.675]$

$\therefore\ C = \frac{2500}{0.675}$

$\therefore\ C = 3703.70$

Hence, ₹ 3703.70 should be deposited semiannually into an account.

EXAMPLES FOR PRACTICE 2.2

1. Find the accumulated value after 3 years of an immediate annuity of ₹ 5000 at interest rate 5% p.a. compounded annually. [Given : $(1.05)^3 = 1.1576$]
2. Mr Desai plans to accumulate a sum of ₹ 5,00,000 in 5 years for higher education of his son. How much should he save every year, if he gets interest compounded at 10% p.a.? [Given : $(1.10)^5 = 1.6105$]
3. Find the rate of interest compounded annually, if an immediate annuity of ₹ 20,000 per year amounts to ₹ 41,000 in 2 years.
4. Find the present value of an immediate annuity of ₹ 5000 per years for 8 years at the rate of 5%. [Given : $(1.05)^{-8} = 0.6768$]
5. A person buys a machine at ₹ 2,00,000. Its estimated life is 12 years. After this period when the new machine is to be bought, the person will have to pay the double the amount of its original price. If the rate of interest is 15%, what amount should be set aside at the end of each year so that at the end of 12 years, it would amount to a balance sufficient to replace the machine? [Given : $(1.15)^{12} = 5.351$]
6. A company sets aside ₹ 1000 at the end of every years to create a sinking fund. What will be the amount at the end of 10 years at 9% per annum? [Given : $(1.09)^{10} = 2.366$]
7. A loan of ₹ 25,000 is repaid in 4 equal annual instalments including principal and interest. Find the amount of annual instalment at the rate of 5% per annum. [Given : $(1.05)^4 = 1.215$]
8. A sum of ₹ 500 is deducted at the end of every month from the salary of a person. What amount would be credited to his PF account, at the end of 25 years of his service, if the rate of interest is 12%. [Given : $(1.01)^{300} = 19.50$]
9. Raghu purchases a car by paying ₹ 25,000 cash. He pays ₹ 1500 at the end of every month for 10 years. If the rate of interest is 12%, find the cash value of the car. [Given : $(1.01)^{120} = 3.281$]
10. If the present value of an annuity immediate of ₹ 360 is ₹ 1200 and its amount at the same rate for the same period is ₹ 1500, find the rate of interest.

11. Find the amount of an annuity due of ₹ 400 per quarter payable for 6 years at 8% per annum.
[Given : $(1.02)^{24} = 3.147$]

12. For an immediate annuity paid for 3 years with interest compounded at 10% p.a., its present value is ₹ 10,000. What is its accumulated value after 3 years?
[Given : $(1.1)^3 = 1.331$]

13. Find the accumulated value after 3 years of an immediate annuity of ₹ 2000 with interest compounded at 10% p.a. [Given : $(1.1)^3 = 1.331$]

Answers

1. ₹ 15,760 **2.** ₹ 81,900.08 **3.** 5%
4. ₹ 32,320 **5.** ₹ 13,790 **6.** ₹ 15177.78
7. ₹ 7064 **8.** ₹ 9,25,000 **9.** ₹ 1,29,282.23
10. 6% **11.** ₹ 43,798.80 **12.** ₹ 13,310
13. ₹ 6620.

MISCELLANEOUS EXERCISE – 2

(Textbook pages 29 to 33)

I. Choose the correct alternative :

1. "A contract that pledges payment of an agreed upon amount to the person (or his/her nominee) on the happening of an event covered against" is technically known as
(a) death coverage (b) savings for future
(c) life insurance (d) provident fund

2. Insurance companies collect a fixed amount from their customers at a fixed interval of time. This amount is called
(a) EMI (b) instalment
(c) contribution (d) premium

3. Following are different types of insurance.
I. Life insurance
II. Health insurance
III. Liability insurance
(a) Only I (b) Only II
(c) Only III (d) All the three

4. By taking insurance, an individual
(a) reduces the risk of an accident
(b) reduces the cost of an accident
(c) transfers the risk to someone else
(d) converts the possibility of large loss to certainty of a small one

5. You get payments of ₹ 8000 at the beginning of each year for five years at 6% , what is the value of this annuity?
(a) ₹ 34,720 (b) ₹ 39,320
(c) ₹ 35,720 (d) ₹ 40,000

6. In an ordinary annuity, payments or receipts occur at
(a) beginning of each period
(b) end of each period
(c) mid of each period
(d) quarterly basis

7. Amount of money today which is equal to series of payments in future is called
(a) normal value of annuity
(b) sinking value of annuity
(c) present value of annuity
(d) future value of annuity

8. Rental payment for an apartment is an example of
(a) annuity due (b) perpetuity
(c) ordinary annuity (d) instalment

9. .... is a series of constant cashflows over a limited period of time.
(a) Perpetuity (b) Annuity
(c) Present value (d) Future value

10. A retirement annuity is particularly attractive to someone who has
(a) a severe illness
(b) risk of low longevity
(c) large family
(d) chance of high longevity

Answers

1. (c) life insurance
2. (d) premium
3. (d) all the three
4. (d) converts the possibility of large loss to certainty of a small one.
5. (c) ₹ 35,720
6. (b) end of each period
7. (c) present value of annuity
8. (b) perpetuity
9. (b) Annuity
10. (d) chance of high longevity.

II. Fill in the blanks :

1. An instalment of money paid for insurance is called ………
2. General insurance covers all risks except ……… .
3. The value of insured property is called ……… .
4. The proportion of property value to insured value is called ……… .
5. The person who receives annuity is called ……… .
6. The payment of each single annuity is called ……… .
7. The intervening time between payment of two successive instalments is called as ……… .
8. An annuity where payments continue forever is called ……… .
9. If payments of an annuity fall due at the beginning of every period, the series is called ……… .
10. If payments of an annuity fall due at the end of every period, the series is called ……… .

Answers

1. premium **2.** life **3.** property value
4. policy value **5.** annuitant **6.** instalment
7. payment period **8.** perpetuity **9.** annuity due
10. immediate annuity or ordinary annuity.

III. State whether each of the following is *True* or *False* :

1. General insurance covers life, fire and theft.
2. The amount of claim cannot exceed the amount of loss.
3. Accident insurance has a period of five years.
4. Premium is the amount paid to the insurance company every month.
5. Payment of every annuity is called an instalment.
6. Annuity certain begins on a fixed date and ends when an event happens.
7. Annuity contingent begins and ends on certain fixed dates.
8. The present value of an annuity is the sum of the present value of all instalments.
9. The future value of an annuity is the accumulated values of all instalments.
10. Sinking fund is set aside at the beginning of a business.

Answers

1. False **2.** True **3.** False **4.** True **5.** False
6. True **7.** False **8.** True **9.** False **10.** True.

IV. Solve the following problems :

1. A house valued at ₹ 8,00,000 is insured at 75% of its value. If the rate of premium is 0.80%. Find the premium paid by the owner of the house. If agent's commission is 9% of the premium, find agent's commission.

Solution :

The value of the house = ₹ 8,00,000.

The policy value of the house

$$= ₹ \left(800000 \times \frac{75}{100}\right) = ₹\ 6,00,000$$

The rate of premium = 0.80%

$$\text{Amount of premium} = ₹\ 600000 \times \frac{0.80}{100} = ₹\ 4800$$

Agent's commission at 9% of the premium

$$= ₹\ 4800 \times \frac{9}{100} = ₹\ 432$$

The premium paid by the owner = ₹ 4800;

Agent's commission = ₹ 432.

2. A shopkeeper insures his shop and godown valued at ₹ 5,00,000 and ₹ 10,00,000 respectively for 80% of their values. If the rate of premium is 8%, find the total annual premium.

Solution :

The value of the shop = ₹ 5,00,000

The value of the godown = ₹ 10,00,000

∴ total value = ₹ 15,00,000

∴ insured value of the shop and godown

$$= ₹\ 1500000 \times \frac{80}{100} = ₹\ 120000.$$

The rate of premium = 8%

∴ amount of the premium

$$= ₹\ 1200000 \times \frac{8}{100} = ₹\ 96000.$$

The total premium is ₹ 96,000.

3. A factory building is insured for $\left(\frac{5}{6}\right)^{\text{th}}$ of its value at a rate of premium of 2.50%. If the agent is paid a commission of ₹ 2812.50, which is 7.5% of the premium, find the value of the building.

Solution :

Let the value of the building be ₹ x.

Then insured value $= ₹\ \frac{5}{6}x$

The rate of premium $= 2.50\%$

Amount of premium on $₹\ \frac{5x}{6}$ at 2.50%

$$= \frac{5x}{6} \times \frac{250}{100} \times \frac{1}{100} = ₹\ \frac{x}{48}$$

The rate of agent's commission is 7.5% of the premium.

$\therefore$ agent's commission on $₹\ \frac{x}{48}$ at 7.5% p.a.

$$= \frac{x}{48} \times \frac{75}{10} \times \frac{1}{100} = \frac{x}{640}$$

$\therefore\ 2812.50 = \frac{x}{640}$

$\therefore\ x = 2812.50 \times 640$

$\therefore\ x = ₹\ 1800000$

The value of the building is ₹ 18,00,000.

4. A merchant takes out fire insurance policy to cover 80% of the value of his stock. Stock worth ₹ 80,000 was completely destroyed in a fire, while the rest of stock was reduced to 20% of its value. If the proportional compensation under the policy was ₹ 67,200, find the value of the stock.

Solution :

Let the book value of the stock be ₹ x.

Insured value of the stock $= ₹\left(x \times \frac{80}{100}\right) = ₹\ \frac{4x}{5}$

Stock of ₹ 80,000 completely destroyed in a fire and balance ₹ $(x - 80000)$ is reduced to 20%.

$\therefore$ damage to balance stock

$$= ₹\ (x - 80000) \times \frac{20}{100}$$

$$= ₹\ (x - 80000) \times \frac{1}{5}$$

$\therefore$ total loss $= ₹\left[80000 + \frac{x - 80000}{5}\right]$

$$= \left[80000 + \frac{x}{5} - 16000\right]$$

$$= ₹\left[\frac{x}{5} + 64000\right]$$

Compensation (claim) received under the policy is ₹ 67,200.

$$\text{Claim} = \frac{\text{Insured Value}}{\text{Value of the stock}} \times \text{Loss}$$

$\therefore\ 67200 = \frac{\frac{4x}{5}}{x} \times \left[\frac{x}{5} + 64000\right]$

$\therefore\ 67200 = \frac{4}{5}\left[\frac{x}{5} + 64000\right]$

$\therefore\ 67200 \times \frac{5}{4} = \frac{x}{5} + 64000$

$\therefore\ 84000 - 64000 = \frac{x}{5}$

$\therefore\ 20000 = \frac{x}{5} \qquad \therefore\ x = 20000 \times 5$

$\therefore\ x = 100000$

The value of the stock is ₹ 1,00,000.*

[* **Note :** *Answer given in the textbook is incorrect.*]

5. A 35 years old person takes a policy for ₹ 1,00,000 for a period of 20 years. The rate of premium is ₹ 76 and the average rate of bonus is ₹ 7 per thousand per annum. If he dies after paying 10 annual premiums, what amount will his nominee receive?

Solution :

Policy value $= ₹\ 1{,}00{,}000$.

Period of policy $= 20$ years.

Rate of premium $= ₹\ 76$ per thousand

$\therefore$ amount of premium $= \frac{76}{1000} \times 100000$

$$= ₹\ 7600$$

He pays for 10 annual premiums.

$\therefore$ total premium paid $= ₹\ (10 \times 7600)$

$$= ₹\ 76000$$

The average rate of bonus $= ₹\ 7$ per thousand p.a. of the policy value.

$\therefore$ on policy of ₹ 1,00,000 bonus for one year

$$= \frac{7}{1000} \times 100000 = ₹\ 700$$

$\therefore$ bonus for 10 years $= ₹\ 10 \times 700$

$$= ₹\ 7000$$

He dies after paying 10 annual premiums.

$\therefore$ his nomineee will receive the amount

$= \text{Policy value} + \text{Bonus earned}$

$= ₹\ (100000 + 7000)$

$= ₹\ 1{,}07{,}000.$

6. 15,000 articles costing ₹ 200 per dozen were insured against fire for ₹ 1,00,000. If 20% of the articles were burnt completely and 2400 of other articles were damaged to the extent of 80% of their value, find the amount that can be claimed under the policy.

Solution :

Number of articles $= \frac{15000}{12} = 1250$ dozens.

Cost of articles per dozen = ₹ 200

$\therefore$ total cost of articles = ₹ (200 × 1250)

= ₹ 2,50,000

Insured value of articles = ₹ 1,00,000

20% of the articles completely burnt.

$\therefore$ the burnt articles $= 1250 \times \frac{20}{100} = 250$ dozens

$\therefore$ cost of burnt articles = ₹ (200 × 250)

= ₹ 50,000

Number of damaged articles in dozens :

$2400 = \frac{2400}{12} = 200$ dozens articles.

$\therefore$ cost of damaged articles = ₹ (200 × 200)

= ₹ 40,000

$\therefore$ loss of damaged articles $= 40000 \times \frac{80}{100}$

= ₹ 32,000

$\therefore$ total loss = ₹ (50000 + 32000)

= ₹ 82,000

$$\text{Claim} = \frac{\text{Insured value}}{\text{Total value}} \times \text{Loss}$$

$$= \frac{100000}{250000} \times 82000 = ₹\ 32800$$

The amount claimed under the policy is ₹ 32,800.

7. For what amount should a cargo worth ₹ 25,350 be insured so that in the event of total loss, its value as well as the cost of insurance may be recovered, when the rate of premium is 2.5%.

Solution :

Let the policy value be ₹ 100 which includes the premium of ₹ 2.50.

$\therefore$ value of cargo = ₹ (100 − 2.50) = ₹ 97.50

When the value of cargo is ₹ 97.50,

the policy value = ₹ 100.

$\therefore$ when the value of cargo is ₹ 25350, the policy value $= \frac{100 \times 25350}{97.50} =$ ₹ 26000

The cargo should be insured for ₹ 26,000.

8. A cargo of grain is insured at $\left(\frac{3}{4}\right)$% to cover 70% of its value. ₹ 1008 is the amount of premium paid. If the grain is worth ₹ 12 per kg, how many kilograms of the grain did the cargo contain?

Solution :

Let the value of a cargo containing grains be ₹ x.

Then the policy value of the cargo

$= ₹\ x \times \frac{70}{100} = ₹\ \frac{7x}{10}$

The rate of premium $= \frac{3}{4}\%$.

The premium paid = ₹ 1008.

Amount of premium = Policy value × rate of premium

$\therefore\ 1008 = \frac{7x}{10} \times \frac{3}{4} \times \frac{1}{100}$

$\therefore\ 1008 = \frac{21x}{4000}$

$\therefore\ \frac{1008 \times 4000}{21} = x$

$\therefore\ x = 192000.$

$\therefore$ the value of the cargo with grains is ₹ 1,92,000.

The cargo contains grains $= \frac{192000}{12} = 16000.$

The cargo contains 16,000 kg of grain.

9. 4000 bedsheets worth ₹ 6,40,000 were insured for $\left(\frac{3}{7}\right)^{\text{th}}$ of their value. Some of the bedsheets were damaged in the rainy season and were reduced to 40% of their value. If the amount recovered against damage was ₹ 32,000, find the number of damaged bedsheets.

Solution :

The value of 4000 bedsheets = ₹ 6,40,000.

Insured value $= ₹\ 640000 \times \frac{3}{7}$

$= ₹\ \frac{1920000}{7}$

Cost of one bedsheet $= ₹\ \frac{640000}{4000} =$ ₹ 160

Let the damaged bedsheets in the rainy season be x.

The cost of damaged bedsheets $= ₹\ 160x$

$\therefore$ the value of damaged bedsheets

$$= ₹\ 160x \times \frac{40}{100} = ₹\ 64x$$

$\therefore$ loss $= ₹\ 64x$

$$\text{Claim} = \frac{\text{Insured value}}{\text{Total value}} \times \text{Loss}$$

$$\therefore\ 32000 = \frac{1920000}{7} \times \frac{1}{640000} \times 64x$$

$$\therefore\ x = \frac{32000 \times 7 \times 640000}{1920000 \times 64}$$

$\therefore\ x = 1166.67 \approx 1167$

The number of damaged bedsheets is 1167.*

[* **Note :** *Answer given in the textbook is incorrect.*]

10. A property valued at ₹ 7,00,000 is insured to the extent of ₹ 5,60,000 at $\left(\frac{5}{8}\right)\%$ less 20%. Calculate the saving made in the premium. Find the amount of loss that the owner must bear including premium, if the property is damaged to the extent of 40% of its value.

Solution :

The value of property $= ₹\ 7{,}00{,}000$.

Insured value $= ₹\ 5{,}60{,}000$.

The rate of premium $= \frac{5}{8}\%$.

$\therefore$ the amount of premium

$$= ₹ \left(560000 \times \frac{5}{8} \times \frac{1}{100}\right) = ₹\ 3500$$

$$\text{New rate of premium} = \frac{5}{8} - \left(20\%\text{ of } \frac{5}{8}\right)$$

$$= \frac{5}{8} - \frac{5}{8} \times \frac{20}{100}$$

$$= \frac{5}{8} - \frac{1}{8} = \frac{4}{8} = \frac{1}{2}\%$$

$\therefore$ the amount of premium at $\frac{1}{2}\%$

$$= ₹ \left(560000 \times \frac{1}{2} \times \frac{1}{100}\right) = ₹\ 2800$$

$\therefore$ saving made in the premium $= ₹\ (3500 - 2800) = ₹\ 700$

Damage to the property

$$= 700000 \times \frac{40}{100} = ₹\ 2{,}80{,}000$$

$\therefore$ loss $= ₹\ 2{,}80{,}000$

$$\text{Claim} = \frac{\text{Insured value}}{\text{Value of property}} \times \text{Loss}$$

$$= \frac{560000}{700000} \times 280000$$

$$= ₹\ 2{,}24{,}000$$

$\therefore$ loss that the owner of the property bears

$$= \text{Loss} - \text{Claim earned} + \text{Premium paid at } \frac{1}{2}\%$$

$$= ₹\ (280000 - 224000 + 2800)$$

$$= ₹\ (56000 + 2800)$$

$$= ₹\ 58800.$$

The owner bears a loss of ₹ 58,800.

11. Stocks in a shop and godown worth ₹ 75,000 and ₹ 1,30,000 respectively were insured through an agent who receives 15% of premium as commission. If the shop was insured for 80% and godown for 60% of the value, find the amount of agent's commission, when the premium was 0.80% less 20%. If the entire stock in the shop and 20% stock in the godown is destroyed by fire, find the amount that can be claimed under the policy.

Solution :

Stocks in the shop $= ₹\ 75{,}000$.

The worth of godown $= ₹\ 1{,}30{,}000$.

$$\text{Insured value of stock} = ₹ \left(75000 \times \frac{80}{100}\right)$$

$$= ₹\ 60{,}000$$

$$\text{Insured value of godown} = ₹ \left(130000 \times \frac{60}{100}\right)$$

$$= ₹\ 78{,}000$$

The rate of premium

$$= 0.80\% - (20\%\text{ of } 0.80)$$

$$= ₹\ (0.80 - 0.16)\% = 0.64\%$$

Total insured value

$$= ₹\ (60000 + 78000) = ₹\ 138000$$

Premium on insured value

$$= ₹\ 138000 \times \frac{64}{100} \times \frac{1}{100} = ₹\ 883.20$$

Agent's commission at 15% of the premium

$$= ₹\ 883.20 \times \frac{15}{100} = ₹\ 132.48$$

Damage to the stock in the shop $= ₹\ 75{,}000$

Damage to the stock in the godown

$= ₹\ 130000 \times \frac{20}{100} = ₹\ 26,000$

Claim for the stock in the shop :

$$\text{Claim} = \frac{\text{Policy value of stock}}{\text{Total value of stock}} \times \text{Loss}$$

$$= \frac{60000}{75000} \times 75000 = ₹\ 60,000$$

Claim for godown :

$$\text{Claim} = \frac{\text{Policy value of godown}}{\text{Total value of the godown}} \times \text{Loss}$$

$$= \frac{78000}{130000} \times 26000 \qquad = ₹\ 15,600$$

Total claim = ₹ (60000 + 15600) = ₹ 75,600.*

Agents' commission is ₹ 132.48.*

[* **Note :** *Answers given in the textbook are incorrect.*]

12. A person holding a life policy of ₹ 1,20,000 for a term of 25 years wants to discontinue after paying premium for 8 years at the rate of ₹ 58 per thousand per annum. Find the amount of paid up value he will receive on the policy. Find the amount he will receive if the surrender value granted is 35% of the premiums paid, excluding the first year's premium.

Solution :

The policy value = ₹ 1,20,000.

The rate of premium = ₹ 58 per thousand.

∴ amount of premium for 8 years

$$= 8 \times \frac{58}{1000} \times 120000 = ₹\ 55,680$$

∴ amount of premium for first year

$$= \frac{55680}{8} = ₹\ 6960$$

Paid up value of the policy

$$= \frac{\text{Number of premium paid}}{\text{Term of the policy}} \times \text{Sum assured}$$

$$= \frac{8}{25} \times 120000 = ₹\ 38400$$

Surrender value of the policy

= 35% of (the total premium paid − First year premium)

= 35% of [55680 − 6960]

$= 35\%$ of ₹ 48720 $\qquad = \frac{35}{100} \times 48720 = ₹\ 17052$

Paid up value of the policy = ₹ 38,400.

Surrender value of the policy = ₹ 17,052.

13. A godown valued at ₹ 80,000 contained stock worth ₹ 4,80,000. Both were insured against fire. Godown for ₹ 50,000 and stock for 80% of its value. A part of stock worth ₹ 60,000 was completely destroyed and the rest was reduced to 60% of its value. The amount of damage to the godown is ₹ 40,000. Find the amount that can be claimed under the policy.

Solution :

Value of the godown = ₹ 80,000.

Value of the stock = ₹ 4,80,000.

∴ total value of the property = ₹ 5,60,000.

Insured value of the godown = ₹ 50,000

Insured value of the stock $= ₹ \left(480000 \times \frac{80}{100}\right)$

$= ₹\ 3,84,000$

Damage :

Stock worth ₹ 60,000 completely destroyed.

Rest stock of ₹ (480000 − 60000)

= ₹ 4,20,000 was reduced to 60%.

∴ stock worth $₹ \left(420000 \times \frac{60}{100}\right)$

= ₹ 2,52,000 reduced in value.

∴ stock worth ₹ (420000 − 252000)

= ₹ 1,68,000 reduced in value.

Damages to the stock = ₹ (60000 + 168000)

= ₹ 2,28,000

Damage to the godown = ₹ 40,000

Claim for stock :

$$\text{Claim} = \frac{\text{Policy value of stock}}{\text{Value of stock}} \times \text{Loss}$$

$$= \frac{384000}{480000} \times 228000$$

= ₹ 1,82,400.

Claim for godown :

$$\text{Claim} = \frac{\text{Policy value of godown}}{\text{Property value}} \times \text{Loss}$$

$$= \frac{50000}{80000} \times 40000$$

= ₹ 25,000.

Total claim = ₹ (182400 + 25000) = ₹ 2,07,400.

14. Find the amount of an ordinary annuity, if payment of ₹ 500 is made at the end of every quarter for 5 years at the rate of 12% p.a. compounded quarterly.

Solution :

$C = ₹\ 500,\ n = 5,\ r = 12\%.$

The period of payment is every quarter.

$\therefore\ n = 5 \times 4 = 20,\ r = \frac{12}{4} = 3\%,\ A = ?$

$i = \frac{r}{100} = \frac{3}{100} = 0.03$

$A = \frac{C}{i}[(1+i)^n - 1]$

$= \frac{500}{0.03}[(1+0.03)^{20} - 1]$

$= \frac{500}{0.03}[1.8061 - 1]$

$= \frac{500 \times 0.8061}{0.03}$

$= 13435$

The amount of an ordinary annuity is ₹ 13,435.

15. Find the amount a company should set aside at the end of every year, if it wants to buy a machine expected to cost ₹ 1,00,000 at the end of 4 years and interest rate is 5% p.a. compounded annually.

Solution :

Here, $A = ₹\ 1{,}00{,}000,\ n = 4,\ r = 5\%,\ C = ?$

$i = \frac{r}{100} = \frac{5}{100} = 0.05$

Now, $A = \frac{C}{i}[(1+i)^n - 1]$

$\therefore\ 100000 = \frac{C}{0.05}[(1+0.05)^4 - 1]$

$\therefore\ 100000 \times 0.05 = C[(1.05)^4 - 1]$

$\therefore\ 5000 = C[1.2155 - 1]$

$\therefore\ 5000 = C \times 0.2155$

$\therefore\ C = \frac{5000}{0.2155}$

$\therefore\ C = 23201.85$

The company should set aside a sum of ₹ 23,201.85 at the end of every year.

16. Find the least number of years for which an annuity of ₹ 3000 per annum must run in order that its amount just exceeds ₹ 60,000 at 10% compounded annually. $[(1.1)^{11} = 2.8531,\ (1.1)^{12} = 3.1384]$

Solution :

$A = ₹\ 60{,}000,\ C = ₹\ 3000,\ r = 10\%,\ n = ?$

$i = \frac{r}{100} = \frac{10}{100} = 0.1$

$A = \frac{C}{i}[(1+i)^n - 1]$

$\therefore\ 60000 = \frac{3000}{0.1}[(1+0.1)^n - 1]$

$\therefore\ 60000 = 30000[(1.1)^n - 1]$

$\therefore\ \frac{60000}{30000} = (1.1)^n - 1$

$\therefore\ 2 = (1.1)^n - 1$

$\therefore\ 3 = (1.1)^n$

$\therefore\ \log 3 = n \log 1.1$

$\therefore\ n = \frac{\log 3}{\log 1.1} = \frac{0.4771}{0.0414} = 11.52$

$\therefore\ n = 11.52 \approx 12$ years.

$3 = (1.1)^n$

Now, $(1.1)^{11} = 2.8531$ and $(1.1)^{12} = 3.1384$ ... (Given)

3 is nearer to 3.1384 than 2.8531

$\therefore\ 3 = (1.1)^n = (1.1)^{12}$

$\therefore\ n = 12$ years.

17. Find the rate of interest compounded annually, if an ordinary annuity of ₹ 20,000 per year amounts to ₹ 41,000 in 2 years.

Solution :

$C = ₹\ 20{,}000,\ A = ₹\ 41{,}000,\ n = 2,\ r = ?$

$A = \frac{C}{i}[(1+i)^n - 1]$

$\therefore\ 41000 = \frac{20000}{i}[(1+i)^2 - 1]$

$\therefore\ \frac{41000}{20000} = \frac{(1+i)^2 - 1}{i}$

$\therefore\ 2.05 = \frac{1 + 2i + i^2 - 1}{i}$

$\therefore\ 2.05 = \frac{i(2+i)}{i}$

$\therefore\ 2.05 = 2 + i$

$\therefore\ 2.05 - 2 = i$

$\therefore\ i = 0.05$

Now, $i = \frac{r}{100}$

$\therefore\ 0.05 = \frac{r}{100}$

$\therefore\ r = 0.05 \times 100$

$\therefore\ r = 5\%$

The rate of interest is 5%.

18. A person purchases a television paying ₹ 20,000 in cash and promising to pay ₹ 1000 at the end of every month for the next 2 years. If money is worth 12% p.a., converted monthly, what is the cash price of the television?

Solution :

Initial payment for a television is ₹ 20,000.

At the end of every month the amount paid is ₹ 1000, i.e. $C = 1000$, $n = 2 \times 12 = 24$,

$r = \frac{12}{12} = 1\%$,

$i = \frac{r}{100} = \frac{1}{100} = 0.01.$

We have to find the present value of all the instalments paid at the end of every month, i.e. we have to find P.

$$P = \frac{C}{i}[1-(1+i)^{-n}]$$

$$= \frac{1000}{0.01}[1-(1+0.01)^{-24}]$$

$$= 100000\,[1-(1.01)^{-24}]$$

$$= 100000\,[1-0.7875]$$

$$= 100000\,[0.2125]$$

$$= 21250$$

$\therefore$ the present value of all paid instalment is ₹ 21250.

Hence, the cash price of the television = Present value of all paid instalment + Amount paid at the time of buying

= ₹ (21250 + 20000)

= ₹ 41,250.

19. Find the present value of an annuity immediate of ₹ 20,000 per annum for 3 years at 10% per annum compounded annually.

Solution :

$C =$ ₹ 20,000, $n = 3$, $r = 10\%$, $P = ?$

$i = \frac{r}{100} = \frac{10}{100} = 0.1$

$$P = \frac{C}{i}[1-(1+i)^{-n}]$$

$$= \frac{2160}{0.2}[1-(1-(1.02)^{-3}]$$

$$= 10800\left[1-\frac{1}{1.728}\right]$$

$$= 10800\,[1-0.5787]$$

$$= 10800 \times 0.4213$$

$= 4250.04$ i.e. $p \approx$ ₹ 4550.

The present value of an annuity is ₹ 49,740.

20. A man borrowed some money and paid back in 3 equal instalments of ₹ 2160 each. What amount did he borrow, if the rate of interest was 20% per annum compounded annually? Also find the total interest charged.

Solution :

$C =$ ₹ 2160, $n = 3$, $r = 20\%$,

$i = \frac{20}{100} = 0.20$

We have to find the present value of the money borrowed, i.e. the present value of annuity due.

$$P = \frac{C}{i}[1-(1+i)^{-n}]$$

$$= \frac{2160}{0.2}[1-(1-0.2)^{-3}]$$

$$= 10800\left[1-\frac{1}{1.728}\right]$$

$$= 10800\,(1-0.5787)$$

$$= 10800 \times 0.4213$$

$$= 4550.04.$$

i.e. $P \approx$ ₹ 4550.

The man paid 3 equal instalments of ₹ 2160 each, i.e. ₹ 6480

$\therefore$ interest charged = ₹ (6480 − 4550)

= ₹ 1930.

The man borrowed ₹ 4550.

Total interest charged is ₹ 1930.

21. A company decides to set aside a certain amount at the end of every year to create a sinking fund, that should amount to ₹ 9,28,200 in 4 years at 10% p.a. Find the amount to be set aside every year.

Solution :

$A = ₹\ 9{,}28{,}200,\ n = 4,\ r = 10\%,\ C = ?$

$$i = \frac{r}{100} = \frac{10}{100} = 0.1$$

$$A = \frac{C}{i}[(1+i)^n - 1]$$

$$\therefore\ 928200 = \frac{C}{0.1}[(1+0.1)^4 - 1]$$

$$\therefore\ 928200 \times 0.1 = C[(1.1)^4 - 1]$$

$$\therefore\ 92820 = C[1.4641 - 1]$$

$$\therefore\ 92820 = C \times 0.4641$$

$$\therefore\ C = \frac{92820}{0.4641} = 200000$$

The company has to set aside ₹ 2,00,000 at the end of every year.

22. Find the future value after 2 years if an amount of ₹ 12,000 is invested at the end of every half year at 12% p.a., compounded half yearly.

Solution :

$C = ₹\ 12{,}000,\ n = 2,\ r = 12\%,\ A = ?$

Investment is half yearly.

$$\therefore\ n = 2 \times 2 = 4,\ r = \frac{12\%}{2} = 6\%$$

$$i = \frac{r}{100} = \frac{6}{100} = 0.06$$

$$A = \frac{C}{i}[(1+i)^n - 1]$$

$$= \frac{12000}{0.06}[(1+0.06)^4 - 1]$$

$$= 200000[(1.06)^4 - 1]$$

$$= 200000(1.2625 - 1)$$

$$= 200000 \times 0.2625 = 52500$$

The future value is ₹ 52,500.

[**Note :** *Answer given in the textbook is incorrect.*]

23. After how many years would an annuity due of ₹ 3000 p.a. accumulated ₹ 19,324.80 at 20% p.a. compounded yearly? [Given : $(1.2)^4 = 2.0736$]

Solution :

Here, $A' = ₹\ 19324.80,\ C = ₹\ 3000,\ r = 20\%,\ n = ?$

$$i = \frac{r}{100} = \frac{20}{100} = 0.2$$

$$\text{Now, } A' = \frac{C(1+i)}{i}[(1+i)^n - 1]$$

$$\therefore\ 19324.80 = \frac{3000(1+0.2)}{0.2}[(1+0.2)^n - 1]$$

$$\therefore\ 19324.80 = \frac{3000 \times 1.2}{0.2}[(1.2)^n - 1]$$

$$\therefore\ 19324.80 = 18000[(1.2)^n - 1]$$

$$\therefore\ \frac{19324.80}{18000} = (1.2)^n - 1$$

$$\therefore\ 1.0736 = (1.2)^n - 1$$

$$\therefore\ 1.0736 + 1 = (1.2)^n$$

$$\therefore\ 2.0736 = (1.2)^n$$

$$\therefore\ (1.2)^4 = (1.2)^n$$

$$\therefore\ n = 4 \qquad \ldots (\because \text{ base is same})$$

An annuity due is to be accumulated for 4 years.

24. Some machinery is expected to cost 25% more over its present cost of ₹ 6,96,000 after 20 years. The scrap value of the machinery will realize ₹ 1,50,000. What amount should be set aside at the end of every year at 5% per annum, compound interest for 20 years to replace the machinery? [Given : $(1.05)^{20} = 2.655$]

Solution :

Present cost of the machinery = ₹ 6,96,000.

$$25\% \text{ of } ₹\ 6{,}96{,}000 = ₹\ 696000 \times \frac{25}{100}$$

$$= ₹\ 1{,}74{,}000$$

$\therefore$ expected cost of the machinery will be

$$= ₹\ (696000 + 174000)$$

$$= ₹\ 8{,}70{,}000$$

Scrap value realize is ₹ 1,50,000

$$\therefore \text{ sinking fund} = ₹\ (870000 - 150000)$$

$$= ₹\ 7{,}20{,}000$$

$A = ₹\ 720000,\ n = 20,\ r = 5\%,\ C = ?$

$$i = \frac{r}{100} = \frac{5}{100} = 0.05$$

$$A = \frac{C}{i}[(1+i)^n - 1]$$

$$\therefore\ 720000 = \frac{C}{0.05}[(1+0.05)^{20} - 1]$$

$$\therefore\ 720000 \times 0.05 = C[(1.05)^{20} - 1]$$

$$\therefore\ 36000 = C[2.655 - 1]$$

$$\therefore\ 36000 = C \times 1.655$$

$$\therefore\ C = \frac{36000}{1.655} = 21752.27$$

$\therefore\ C = 21752.27 \approx 21752.30$

A sum of ₹ 21752.30 should be set aside at the end of every year.

ACTIVITIES Textbook pages 32 and 33

(Answers are given directly.)

1. Property Value = ₹ 1,00,000.

Policy value = 70% of property value

$= \boxed{₹\ 70{,}000}$

Rate of premium = 0.4 %

$$\text{Amount of premium} = \frac{0.4}{100} \times \boxed{70000}$$

$= ₹\ 280$

Property worth ₹ 60,000 is destroyed.

$$\therefore\ \text{claim} = \text{Loss} \times \frac{\text{Policy Value}}{\text{Property Value}}$$

$$= \boxed{₹\ 60000} \times \frac{7}{\boxed{10}}$$

$= ₹\ 42{,}000$

Now, the property worth ₹ 60,000 is totally destroyed and in addition the remaining property is so damaged as to reduce its value by 40%

$$\therefore\ \text{loss} = ₹\left(60000 + \frac{40}{100} \times \boxed{40000}\right)$$

$= ₹\ (60000 + 16000)$

$= ₹\ \boxed{76000}$

$$\therefore\ \text{claim} = \boxed{76000} \times \frac{7}{10}$$

$= ₹\ 53{,}200$

2. Policy value = ₹ 70,000

Period of policy = 15 years

Rate of premium = ₹ 56.50 per thousand per annum

$$\therefore\ \text{amount of premium} = \frac{56.50}{1000} \times \boxed{70000}$$

$= ₹\ 3955$

total premium paid = ₹ 3955 × $\boxed{15}$

$= ₹\ 59{,}325$

Rate of bonus = ₹ 6 per thousand per annum

$$\therefore\ \text{amount of bonus} = \left(₹\ 6 \times \boxed{\frac{70000}{1000}}\right)$$

$= ₹\ 420$

$\therefore$ bonus for 15 years = (₹ 420 × $\boxed{15}$)

$= ₹\ 6300$

$\therefore$ the person gets ₹ { $\boxed{70000}$ + 6300 }

$= ₹\ 76{,}300$

$\therefore$ benefit = ₹ ($\boxed{76300}$ − 59325)

$= ₹\ 16{,}975.$

3. For an immediate annuity,

$P = ₹\ 2000,\ \ A = ₹\ 4000,\ \ r = 10\%$ per annum

$$\therefore\ i = \frac{r}{100} = \frac{\boxed{10}}{100} = 0.1$$

$$\frac{1}{P} - \frac{1}{A} = \frac{i}{C}$$

$$\therefore\ \frac{1}{\boxed{2000}} - \frac{1}{\boxed{4000}} = \frac{0.1}{C}$$

$$\therefore\ \frac{\boxed{1}}{4000} = \frac{0.1}{C}$$

$\therefore\ C = ₹\ \boxed{400}$

4. For an annuity due, $C = ₹\ 2000$, rate = 16% per annum compounded quarterly for 1 year

$$\therefore\ \text{rate of interest per quarter} = \frac{\boxed{16}}{4} = 4$$

$\therefore\ r = 4\%$

$$\therefore\ i = \frac{r}{100} = \frac{\boxed{4}}{100} = 0.04$$

n = Number of quarters

$= 4 \times 1$

$= \boxed{4}$

$$P' = \frac{C(1+i)}{i}\ [1-(1+i)^{-n}]$$

$$\therefore\ P' = \frac{\boxed{2000}\left(1+\boxed{0.04}\right)}{0.04}\left[1-\left(\boxed{1}+0.04\right)^{\boxed{-4}}\right]$$

$$= \frac{2000\left(\boxed{1.04}\right)}{\boxed{0.04}}\left[1-\left(\boxed{1.04}\right)^{-4}\right]$$

$$= 50000\left(\boxed{1.04}\right)(1-0.8548)$$

$$= 50000\,(1.04)\left(\boxed{0.1452}\right)$$

$$= \text{₹}\ 7550.40$$

5. The cost of machinery $=$ ₹ 10,00,000

Effective life of machinery $= 12$ years

Scrap value of machinery $=$ ₹ 50,000

$r = 5\%$ p.a.

$$\therefore\ i = \frac{r}{100} = \frac{\boxed{5}}{\boxed{100}} = 0.05$$

$$A = \text{₹}\ (1000000 - 50000)$$

$$= \boxed{\text{₹}\ 950000}$$

For an immediate annuity

$$A = \frac{C}{i}\left[(1+i)^n - 1\right]$$

$$\therefore\ \boxed{950000} = \frac{C}{\boxed{0.05}}\left[(1+0.05)^{\boxed{12}} - 1\right]$$

$$\therefore\ 950000 = \frac{C}{0.05}\left[1.797 - \boxed{1}\right]$$

$$\therefore\ C = \frac{950000 \times \boxed{0.05}}{0.797}$$

$$= \text{₹}\ 59{,}598.50.$$

* * *

3 LINEAR REGRESSION

CHAPTER OUTLINE

INTRODUCTION

The study of correlation indicates the degree and direction of linear relationship between two random variables. But this much information is insufficient to study the perfect relationship between the variables. Hence, it is necessary to know any functional (or algebric) relationship between two variables. So we can determine or predict the value of one variable when the value of the other variable is given. This concept is studied with the help of Regression Analysis. It measures the effect of change in one variable on the other and estimates the value of one variable given the value of another variable.

Carl Friedrich Gauss developed the Least Squares Method for finding the linear equation that best describes the relationship between two or more variables. Such relationship is called Linear regression.

Notes :

- If an unknown quality is a parameter or a random variable, then the method used to determine them is called estimation.
- If an unknown quantity is a variable, then the method used to determine it is called predication.

IMPORTANT FORMULAE

1. Equation of Linear Regression :

$y = a + bx$, where y = value of Y, x = value of X.

a, b are constants.

2. Equation of best fitted Linear Regression :

$\hat{y} = a + bx$, where $\hat{y}$ = predicted value of y

$$a = \bar{y} - b\bar{x}$$

$$b = \frac{\text{Cov}(x,\ y)}{\sigma_x^2}$$

σ_x^2 = Variance of x

3. Error :

$$e = y - \hat{y}$$

4. Lines of Regression :

- **Line of Regression of Y on X :**

$y - \bar{y} = b_{yx}(x - \bar{x})$ *OR*

$y = a + bx$, where $b = b_{yx}$ and $a = \bar{y} - b\bar{x}$

The value of Y can be estimated for the given value of X.

- **Line of Regression of X on Y :**

$x - \bar{x} = b_{xy}(y - \bar{y})$ *OR*

$x = a' + b'y$, where $b' = b_{xy}$ and $a' = \bar{x} - b'\bar{y}$

The value of X can be estimated for the given value of Y.

Point of Intersection of Two Regression Lines :

The point of intersection of two regression lines is $(\bar{x}, \bar{y})$. To find $\bar{x}$, $\bar{y}$ solve two regression equations in x and y. These values of x and y are values of $\bar{x}$ and $\bar{y}$.

5. Regression Coefficient :

- **Regression Coefficient of Y on X :**

$$b_{yx} = \frac{\text{Cov}(x,\ y)}{\sigma_x^2} = \frac{\Sigma(x - \bar{x})(y - \bar{y})}{\Sigma(x - \bar{x})^2}$$

$$= \frac{\frac{\Sigma xy}{n} - \bar{x}\bar{y}}{\frac{\Sigma x^2}{n} - \bar{x}^2}$$

$$= \frac{\Sigma xy - n\bar{x}\bar{y}}{\Sigma x^2 - n\bar{x}^2}$$

$$= r \cdot \frac{\sigma_y}{\sigma_x}$$

[Note : $b_{yx} = b$]

- **Regression Coefficient of X on Y :**

$$b_{xy} = \frac{\text{Cov}(x, y)}{\sigma_y^2} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(y-\bar{y})^2}$$

$$= \frac{\frac{\Sigma xy}{n} - \bar{x}\bar{y}}{\frac{\Sigma y^2}{n} - \bar{y}^2}$$

$$= \frac{\Sigma xy - n\bar{x}\bar{y}}{\Sigma y^2 - n\bar{y}^2}$$

$$= r \cdot \frac{\sigma_x}{\sigma_y}$$

[Note : $b_{xy} = b'$]

Properties of Regression Coefficients :

(i) $r^2 = b_{yx} \cdot b_{xy} = b \times b' \leqslant (\pm 1)^2$

(ii) $r = \pm\sqrt{b_{yx} \cdot b_{xy}}$

Sign of r is same as the signs of b_{yx} and b_{xy}

(iii) b_{yx} and b_{xy} always have the similar sign.

(iv) If $b_{yx} > 1$, then $b_{xy} < 1$

(v) $\left|\frac{b_{yx} + b_{xy}}{2}\right| \geqslant |r|$

(vi) b and b' are independent of change of origin but not of scale.

- **If $u = \frac{x-a}{h}$ and $v = \frac{y-b}{k}$ then**

$b_{yx} = \frac{k}{h} b_{vu}$ and $b_{xy} = \frac{h}{k} b_{uv}$.

3.1 : MEANING AND TYPES OF REGRESSION

(1) **Meaning of Regression :** Linear Regression is the statistical method which helps to formulate a functional relationship between two or more variables in the form of linear equations and predicts the value of one variable given the value of the other variable.

- The variable being predicted is called the response or dependent variable. Usually it is denoted by Y.
- The variables used for predicting the response or dependent variable are called predictors or independent variables. Usually, they are denoted by X.
- **Linear Regression Model :** A Linear equation with unknown coefficients is called a linear regression model. The unknown coefficients are called parameters of the linear regression model and they are estimated by the observed values of independent variables. The equation of linear regression model is of the form $Y = a + bX$ where Y = Dependent variable, X = Independent variable, a and b are parameters of the linear regression model.

The linear regression model develops a formula for predicting the value of dependent variable when the values of independent variables are known.

Linear regression model will be useful for prediction only if there is strong correlation between two variables.

(2) **Types of Linear Regression :**

(1) **Simple Linear Regression Model :** When the linear regression model represents the relationship between the dependent variable and one independent variable, then it is called a simple regression model.

Examples :

- The relationship between advertising (X) and sales of a product (Y).
- The relationship between the ages (X) and heights (Y) of seeding in a experiment.

(2) **Multiple Regression Model :** When the linear regression model represents the relationship between one dependent variable and two or more independent variables, then it is called a multiple linear regression model.

Examples :

- The blood pressure of a person (Y) is associated with several independent variables like age, weight, level of blood sugar and the level of blood cholesterol.

- Annual savings of a family (Y) are associated with several independent variables like family size, annual income, number of children in school or college and health conditions of family members.

3.2 : FITTING SIMPLE LINEAR REGRESSION

3.2.1 Least Square Method

From a scatter diagram several lines can be drawn through the points. The best line is one which is close to all points. Such a line is called 'the line of best fit'. The principle involved in obtaining 'the line of best fit' is called the principle of least squares.

The method of least squares was developed by Adrien-Maire Lagendre and Carl Friedrich Gauss independently.

Suppose the equation of linear regression is $y = a + bx$, where a and b are constants and $\hat{y} = a + bx$ is the equation of the line that best fits the data consisting of n pairs of observations (x_i, y_i), $i = 1, 2, \ldots, n$.

For the given value x_i of independent variable X, if we can get the predicted value, $\hat{y}_i$ (read as y_i cap) corresponding to observed value y_i of the dependent variable Y by using the line of best fit, then $(y_i - \hat{y}_i)$ is called the residual or error.

According to principle of least squares, we find the constants a and b so that the sum of squares of error part is minimum, i.e. symbolically.

$$\sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2, \text{ put } \hat{y}_i = a + bx_i$$

$$\therefore \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} [y_i - (a + bx_i)]^2 \text{ is minimum}$$

$$\text{i.e. } S^2 = \sum_{i=1}^{n} [y_i - (a + bx_i)]^2 \text{ is minimum}$$

In order to minimise S^2, we differentiate S^2 with respect to 'a' and 'b' and equate both these derivatives to zero. As the result, we get the following two linear equations :

$$\sum_{i=1}^{n} y_i = na + b\sum_{i=1}^{n} x_i \quad \ldots (1)$$

$$\sum_{i=1}^{n} x_i y_i = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^2 \quad \ldots (2)$$

Solving the equations (1) and (2), we get

$$a = \bar{y} - b\bar{x},\ b = \frac{\text{Cov}(x, y)}{\sigma_x^2},$$

$$\text{where Cov}(x, y) = \frac{1}{n}\sum_{i=1}^{n} x_i y_i - \bar{x}\bar{y}$$

$$\sigma_x^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2$$

Substituting 'a' and 'b' in the regression equation $y = a + bx$, we get the equation

$$y - \bar{y} = b(x - \bar{x})$$

Notes :

- The constant 'b' is called the regression coefficient (or the slope of the regression line) and the constant 'a' is called the y-intercept.
- When observations on variables X and Y are available, it is possible to fit (i) a linear regression of Y on X and (ii) a linear regression of X on Y.

3.2.2 Regression of Y on X

Linear regression of Y on X assumes that the variable X is the independent variable and the variable Y is the dependent variable.

The linear regression model of Y on X is expressed as follows :

$$y - \bar{y} = b_{yx}(x - \bar{x}) \quad OR$$

$$Y = b_{yx} \cdot X, \text{ where } Y = (y - \bar{y}) \text{ and } X = (x - \bar{x})$$

Replacing b by b_{yx},

$$y - \bar{y} = b(x - \bar{x}) \quad OR$$

$$Y = b \cdot x$$

$$\text{Here, } b_{yx} = \frac{\text{Cov}(x, y)}{\text{Var}(x)}$$

$$= \frac{\Sigma(x_1 - \bar{x})(y_1 - \bar{y})}{\Sigma(x_1 - \bar{x})^2}$$

$$= \frac{\Sigma x_1 y_1 - n\bar{x}\bar{y}}{\Sigma x_i^2 - n \cdot \bar{x}^2}$$

3.2.3 Regression of X on Y

Linear regression of X on Y assumes that the varible Y is the independent variable and the variable X is the dependent variable.

The linear regression model is different from that of Y on X. It is expressed as follows :

$x = a' + b'y$, where $a' = \bar{x} - b'\bar{y}$

$b' = \frac{\text{Cov}(x,\ y)}{\text{Var}(x)}$ = The regression coefficient of X on Y.

Substituting the values of a' and b' we get

$$x = \bar{x} - b'\bar{y} + b'y$$

$$\Rightarrow (x - \bar{x}) = b'(y - \bar{y})$$

Replacing b' by b_{xy}, the linear regression model of X on Y will be written as follows

$$(x - \bar{x}) = b_{xy}(y - \bar{y})$$

$$\text{Here, } b_{xy} = \frac{\text{Cov}\ (x,\ y)}{\text{Var}\ (y)} = \frac{\Sigma(x_i - \bar{x})(y_i - \bar{y})}{\Sigma(y_i - \bar{y})^2} = \frac{\Sigma x_i y_i - n\bar{x}\bar{y}}{\Sigma y_i^2 - n \cdot \bar{y}^2}$$

[Notes :

- $\frac{1}{b_{xy}}$ is the slope of the line of regression of X on Y.
- The point $(\bar{x},\ \bar{y})$ satisfies equation of both the lines of regression. Therefore the point $(\bar{x},\ \bar{y})$ is the point of intersection of two lines of regression.]

EXERCISE 3.1 | **Textbook pages 41 and 42**

1. The HRD manager of a company wants to find a measure which he can use to fix the monthly income of persons applying for the job in the production department. As an experimental project, he collected data of 7 persons from that department referring to years of service and their monthly incomes.

Years of service (X)	11	7	9	5	8	6	10
Monthly Income (₹ 1000's) (Y)	10	8	6	5	9	7	11

(i) Find the regression equation of income on years of service.

(ii) What initial start would you recommend for a person applying for the job after having served in similar capacity in another company for 13 years?

Solution :

Let X = Years of service. Y = Monthly income (in ₹ 1000's)

	x	y	xy	x^2
	11	10	110	121
	7	8	56	49
	9	6	54	81
	5	5	25	25
	8	9	72	64
	6	7	42	36
	10	11	110	100
$n = 7$	$\Sigma x = 56$	$\Sigma y = 56$	$\Sigma xy = 469$	$\Sigma x^2 = 476$

$$\bar{x} = \frac{\Sigma x}{n} = \frac{56}{7} = 8,\quad \bar{y} = \frac{\Sigma y}{n} = \frac{56}{7} = 8$$

Regression coefficient of Y on X :

$$b_{yx} = \frac{\frac{\Sigma xy}{n} - \bar{x} \cdot \bar{y}}{\frac{\Sigma x^2}{n} - \bar{x}^2}$$

$$= \frac{\frac{469}{7} - 8 \times 8}{\frac{476}{7} - (8)^2}$$

$$= \frac{67 - 64}{68 - 64} = \frac{3}{4} = 0.75$$

(i) Regression equation of income (Y) on years of service (X) :

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

$$\therefore\ y - 8 = 0.75\ (x - 8)$$

$$\therefore\ y - 8 = 0.75x - 6$$

$$\therefore\ y = 0.75x - 6 + 8$$

$$\therefore\ y = 2 + 0.75x$$

(ii) Estimate of initial start (Y) when years of service X = 13 :

$y = 2 + 0.75x$

Putting $x = 13$ we get

$$y = 2 + 0.75 \times 13$$

$\therefore\ y = 2 + 9.75$

$\therefore\ y = 11.75$

Hence, the initial start, the person will get is ₹ 11.75 × 1000 = ₹ 11,750.

2. Calculate the regression equations of X on Y and Y on X from the following data :

X	**10**	**12**	**13**	**17**	**18**
Y	**5**	**6**	**7**	**9**	**13**

Solution :

We prepare the following table showing the calculations :

X = x_i	Y = y_i	$(x_i - \bar{x})$ $\bar{x} = 14$	$(y_i - \bar{y})$ $\bar{y} = 8$	$(x_i - \bar{x})(y_i - \bar{y})$	$(x_i - \bar{x})^2$	$(y_i - \bar{y})^2$
10	5	−4	−3	12	16	9
12	6	−2	−2	4	4	4
13	7	−1	−1	1	1	1
17	9	3	1	3	9	1
18	13	4	5	20	16	25
$\Sigma x = 70$	$\Sigma y = 40$	$\Sigma(x_i - \bar{x}) = 0$	$\Sigma(y_i - \bar{y})^2 = 0$	$\Sigma(x_i - \bar{x})(y_i - \bar{y}) = 40$	$\Sigma(x_i - \bar{x})^2 = 46$	$\Sigma(y_i - \bar{y})^2 = 40$

Here, $n = 5$

$\bar{x} = \frac{\Sigma x}{n} = \frac{70}{5} = 14;\ \bar{y} = \frac{\Sigma y}{n} = \frac{40}{5} = 8$

Regression equation of X on Y :

$x = a' + b_{xy} \cdot y$

$$b_{xy} = \frac{\Sigma(x_i - \bar{x})(y_i - \bar{y})}{\Sigma(y_i - \bar{y})^2}$$

Putting the values, we get

$$b_{xy} = \frac{40}{40} = 1$$

$a' = \bar{x} - b_{xy} \cdot \bar{y}$

Putting $\bar{x} = 14$, $b_{xy} = 1$, $\bar{y} = 8$, we get

$a' = 14 - 1(8) = 14 - 8 = 6.$

Putting $a' = 6$ and $b_{xy} = 1$, we get the regression equation of X on Y as

$x = 6 + 1(y) \Rightarrow x = y + 6$

Regression equation of Y on X :

$y = a + b_{yx} \cdot x$

$$b_{xy} = \frac{\Sigma(x_i - \bar{x})(y_i - \bar{y})}{\Sigma(x_i - \bar{x})^2}$$

$$= \frac{40}{46} = 0.87$$

$\therefore\ a = \bar{y} - b_{yx} \cdot \bar{x}$

Putting $\bar{y} = 8$, $b_{yx} = 0.87$ and $\bar{x} = 14$, we get

$a = 8 - 0.87(14)$

$= 8 - 12.18$

$= -4.18$

Putting $a = -4.18$ and $b_{yx} = 0.87$, we get the regression equation of Y on X as

$y = -4.18 + 0.87x \Rightarrow y = 0.87x - 4.18$

Hence, the regression equation of X on Y is $x = y + 6$ and the regression equation of Y on X is $y = 0.87x - 4.18$.

3. For a certain bivariate data on 5 pairs of observations given

$\Sigma x = 20$, $\Sigma y = 20$, $\Sigma x^2 = 90$, $\Sigma y^2 = 90$, $\Sigma xy = 76$

Calculate (i) cov(x, y) (ii) b_{yx} and b_{xy} (iii) r.

Solution :

Given : $n = 5$, $\Sigma x = 20$, $\Sigma y = 20$, $\Sigma x^2 = 90$, $\Sigma y^2 = 90$, $\Sigma xy = 76$

$$\therefore\ \bar{x} = \frac{\Sigma x}{n} = \frac{20}{5} = 4;\ \bar{y} = \frac{\Sigma y}{n} = \frac{20}{5} = 4$$

(i) Cov (x, y) :

$$\text{Cov}(x, y) = \frac{1}{n}\Sigma xy - \bar{x}\bar{y}$$

$$= \frac{76}{5} - (4 \times 4) = 15.2 - 16$$

$$\therefore\ \text{Cov}(x, y) = -0.8$$

(ii) b_{yx} and b_{xy} :

$$b_{yx} = \frac{\Sigma xy - n\bar{x}\bar{y}}{\Sigma x^2 - n\bar{x}^2}$$

$$= \frac{76 - 5(4 \times 4)}{90 - 5(4)^2} = \frac{76 - 80}{90 - 80}$$

$$= \frac{-4}{10} = -0.4$$

$$\therefore\ b_{yx} = -0.4$$

$$b_{xy} = = \frac{\Sigma xy - n\bar{x}\bar{y}}{\Sigma y^2 - n\bar{y}^2}$$

$$= \frac{76 - 5(4 \times 4)}{90 - 5(4)^2} = \frac{76 - 80}{90 - 80}$$

$$= \frac{-4}{10} = -0.4$$

$$\therefore\ b_{xy} = -0.4$$

(iii) r :

$$r = \pm\sqrt{b_{yx} \cdot b_{xy}}$$

$$= -\sqrt{b_{yx} \cdot b_{xy}} \quad \dots (\because b_{yx} \text{ and } b_{xy} \text{ are negative})$$

$$= -\sqrt{(-0.4)(-0.4)}$$

$$= -\sqrt{0.16}$$

$$= -0.4$$

$$\therefore\ r = -0.4$$

4. From the following data estimate y when $x = 125$

X	120	115	120	125	126	123
Y	13	15	14	13	12	14

Solution :

We have to estimate Y when X = 125. So we obtain the regression equation of Y on X.

The values of regression coefficients are independent of change of origin. Therefore we take the new variables $u = (x - a)$ and $v = (y - b)$, taking $a = 120$ and $b = 14$

We prepare the following table for calculation :

X = x	Y = y	$u = (x - a)$ $a = 120$	$v = (y - b)$ $b = 14$	uv	u^2
120	13	0	−1	0	0
115	15	−5	1	−5	25
120	14	0	0	0	0
125	13	5	−1	−5	25
126	12	6	−2	−12	36
123	14	3	0	0	9
Σx = 729	Σy = 81	14 −5 $\Sigma u = 9$	1 −4 $\Sigma v = -3$	Σuv = −22	Σu^2 = 95

Here, $n = 6$

$$\bar{x} = \frac{\Sigma x}{n} = \frac{729}{6} = 121.5;\ \bar{y} = \frac{\Sigma y}{n} = \frac{81}{6} = 13.5$$

$$\bar{u} = \frac{\Sigma u}{n} = \frac{9}{6} = 1.5;\ \bar{v} = \frac{\Sigma v}{n} = \frac{-3}{6} = -0.5.$$

Regression equation of Y on X :

$$y = a + b_{yx} \cdot x$$

$$b_{yx} = b_{uv} = \frac{\Sigma uv - n\bar{u}\bar{v}}{\Sigma u^2 - n\bar{u}^2}$$

$$= \frac{-22 - 6[1.5 \times (-0.5)]}{95 - 6(1.5)^2}$$

$$= \frac{-22 + 4.5}{95 - 13.5}$$

$$= \frac{-17.5}{81.5}$$

$$= -0.21$$

$$a = \bar{y} - b_{yx} \cdot \bar{x}$$

Putting $\bar{y} = 13.5$, $b_{yx} = -0.21$, $\bar{x} = 121.5$, we get

$$a = 13.5 - (-0.21 \times 121.5)$$

$$= 13.5 + 25.515 = 39.015$$

Putting $a = 39.015$ and $b_{yx} = -0.21$, we get the regression equation of Y on X as follows :

$$y = 39.015 - 0.21x$$

Estimation of Y when X = 125

Putting X = 125 in the equation $y = 39.015 - 0.21x$, we get

$$y = 39.015 - 0.21\,(125)$$

$$= 39.015 - 26.25$$

$$= 12.765^*$$

[* **Note :** *Answer given in the textbook is incorrect.*]

5. The following table gives the aptitude test scores and productivity indices of 10 workers selected at random :

Aptitude score (X)	60	62	65	70	72	48	53	73	65	82
Productivity index (Y)	68	60	62	80	85	40	52	62	60	81

Obtain the two regression equations and estimate :

(i) The productivity index of a worker whose test score is 95.

(ii) The test score when productivity index is 75.

Solution :

We prepare the following table for calculation :

	Aptitude score $\bar{x}$	Productivity index $\bar{y}$	$(x-\bar{x})$ $\bar{x}=65$	$(y-\bar{y})$ $\bar{y}=65$	$(x-\bar{x})$ × $(y-\bar{y})$	$(x-\bar{x})^2$	$(y-\bar{y})^2$
	60	68	−5	3	−15	25	9
	62	60	−3	−5	15	9	25
	65	62	0	−3	0	0	9
	70	80	5	15	75	25	225
	72	85	7	20	140	49	400
	48	40	−17	−25	425	289	625
	53	52	−12	−13	156	144	169
	73	62	8	−3	−24	64	9
	65	60	0	−5	0	0	25
	82	81	17	16	272	289	256
$n=10$	$\Sigma x=650$	$\Sigma y=650$	$\Sigma(x-\bar{x})=0$	$\Sigma(y-\bar{y})=0$	1083 −39 $\Sigma(x-\bar{x})(y-\bar{y})=1044$	$\Sigma(x-\bar{x})^2=894$	$\Sigma(y-\bar{y})^2=1752$

$$\bar{x}=\frac{\Sigma x}{n}=\frac{650}{10}=65,\quad \bar{y}=\frac{\Sigma y}{n}=\frac{650}{10}=65$$

Regression equation of Y on X :

$y=a+b_{yx}\cdot x$

$$b_{yx}=\frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(x-\bar{x})^2}=\frac{1044}{894}=1.1678$$

$a=\bar{y}-b_{yx}\cdot x$

Putting $\bar{y}=65$, $b_{yx}=1.1678$, we get

$a=65-1.1678(65)=65-75.907=-10.907$

Therefore the regression equation of Y on X is

$y=-10.907+1.1678x$ … (1)

Regression equation of X on Y :

$y=a'+b_{xy}\cdot y$

$$b_{xy}=\frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(y-\bar{y})^2}=\frac{1044}{1752}=0.5959$$

$a'=\bar{x}-b_{xy}\cdot\bar{y}$

Putting $\bar{x}=65$, $b_{xy}=0.5959$, we get

$a'=65-0.5959\times 65=65-38.73=26.27$

Therefore, the regression equation of X on Y is

$x=26.27+0.5959y$ … (2)

(i) Estimation of productivity index (Y) when $X=95$

Putting $x=95$ in equation (1), we get

$y=-10.907+1.1678(95)$

$\therefore\ y=-10.907+110.941$

$\therefore\ y=100.034$

(ii) Estimation of aptitude score (X) when $Y=75$

Putting $y=75$ in the equation (2), we get

$x=26.27+0.5959(75)$

$\therefore\ x=26.27+44.6925$

$\therefore\ x=70.9625$

[**Note :** *Answers given in the textbook are incorrect.*]

6. Compute the appropriate regression equation for the following data :

X [Independent Variable]	2	4	5	6	8	11
Y [Dependent Variable]	18	12	10	8	7	5

Solution :

The appropriate regression equation for the given data is of Y on X.

We prepare the following table for calculation :

$X_i = x$	$Y_i = y$	xy	x^2
2	18	36	4
4	12	48	16
5	10	50	25
6	8	48	36
8	7	56	64
11	5	55	121
$\Sigma x = 36$	$\Sigma y = 60$	$\Sigma xy = 293$	$\Sigma x^2 = 266$

Here, $n = 6$

$$\bar{x} = \frac{\Sigma x}{n} = \frac{36}{6} = 6; \ \bar{y} = \frac{\Sigma y}{n} = \frac{60}{6} = 10$$

Regression equation of Y on X :

$y = a + b_{yx} \cdot x$

$$\text{Now, } b_{yx} = \frac{\Sigma xy - n\bar{x}\bar{y}}{\Sigma x^2 - n\bar{x}^2}$$

$$= \frac{293 - 6(6 \times 10)}{266 - 6(6)^2} = \frac{293 - 360}{266 - 216}$$

$$= \frac{-67}{50} = -1.34$$

$\therefore \ b_{yx} = -1.34$

$a = \bar{y} - b_{yx} \cdot \bar{x}$

Putting $\bar{y} = 10$, $b_{yx} = -1.34$ and $\bar{x} = 6$, we get

$a = 10 - (-1.34)(6)$

$= 10 + 8.04$

$= 18.04$

Putting $a = 18.04$ and $b_{yx} = -1.34$ in $y = a + b_{yx} \cdot x$, we get the regression equation of Y on X as follows :

$y = 18.04 - 1.34x$

$\Rightarrow y = -1.34x + 18.04$

7. The following are the marks obtained by the students in Economics (X) and Mathematics (Y) :

X	59	60	61	62	63
Y	78	82	82	79	81

Find the regression equation of Y on X.

Solution :

X = marks in Economics, Y = Marks in Mathematics.

We prepare the following table for calculation :

x	y	$(x - \bar{x})$ $\bar{x} = 61$	$(y - \bar{y})$ $\bar{y} = 80.4$	$(x - \bar{x})$ $(y - \bar{y})$	$(x - \bar{x})^2$
59	78	−2	−2.4	4.8	4
60	82	−1	1.6	−1.6	1
61	82	0	1.6	0	0
62	79	1	−1.4	−1.4	1
63	81	2	0.6	1.2	4
Σx $= 305$	Σy $= 402$	$\Sigma(x - \bar{x})$ $= 0$	$\Sigma(y - \bar{y})$ $= 0$	6.0 −3.0 $\Sigma(x - \bar{x})(y - \bar{y}) = 3$	$\Sigma(x - \bar{x})^2$ $= 10$

Here, $n = 5$

$$\bar{x} = \frac{\Sigma x}{n} = \frac{305}{5} = 61; \ \bar{y} = \frac{\Sigma y}{n} = \frac{402}{5} = 80.4$$

Regression equation of Y on X :

$y = a + b_{yx} \cdot x$

$$b_{yx} = \frac{\Sigma(x - \bar{x})(y - \bar{y})}{\Sigma(x - \bar{x})^2} = \frac{3}{10} = 0.3$$

$a = \bar{y} - b_{yx} \cdot \bar{x}$

Putting $\bar{y} = 80.4$, $b_{yx} = 0.3$ and $\bar{x} = 61$, we get

$a = 80.4 - 0.3(61)$

$= 80.4 - 18.3 = 62.1$

$\therefore \ a = 62.1.$

Putting $a = 62.1$ and $b_{yx} = 0.3$ in $y = a + b_{yx} \cdot x$, we get the regression equation of Y on X as follows.

$y = 62.1 + 0.3x$

$\Rightarrow y = 0.3x + 62.1.$

8. For the following bivariate data obtain the equations of two regression lines :

X	1	2	3	4	5
Y	5	7	9	11	13

Solution : We prepare the following table for calculation :

x	y	xy	x^2	y^2
1	5	5	1	25
2	7	14	4	49
3	9	27	9	81
4	11	44	16	121
5	13	65	25	169
$\Sigma x = 15$	$\Sigma y = 45$	$\Sigma xy = 155$	$\Sigma x^2 = 55$	$\Sigma y^2 = 445$

Here, $n = 5$

$$\bar{x} = \frac{\Sigma x}{n} = \frac{15}{5} = 3;\ \bar{y} = \frac{\Sigma y}{n} = \frac{45}{5} = 9$$

Regression equation of Y on X :

$$y = a + b_{yx} \cdot x$$

$$b_{yx} = \frac{\Sigma xy - n\bar{x}\bar{y}}{\Sigma x^2 - n\bar{x}^2}$$

$$= \frac{155 - 5(3 \times 9)}{55 - 5(3)^2} = \frac{155 - 135}{55 - 45}$$

$$= \frac{20}{10} = 2$$

$$\therefore\ b_{yx} = 2$$

$$a = \bar{y} - b_{yx} \cdot \bar{x}$$

Putting $\bar{y} = 9$, $b_{yx} = 2$, $\bar{x} = 3$, we get

$$a = 9 - 2(3) = 9 - 6 = 3$$

Putting $a = 3$ and $b_{yx} = 2$ in $y = a + b_{yx} \cdot x$, we get the regression equation Y on X as follows :

$$y = 3 + 2x$$

$$\Rightarrow y = 2x + 3$$

Regression equation of X on Y :

$$x = a' + b_{xy} \cdot y$$

$$b_{xy} = \frac{\Sigma xy - n\bar{x}\bar{y}}{\Sigma y^2 - n\bar{y}^2}$$

$$= \frac{155 - 5(3 \times 9)}{445 - 5(9)^2} = \frac{155 - 135}{445 - 405}$$

$$= \frac{20}{40} = \frac{1}{2} = 0.5$$

$$\therefore\ b_{xy} = 0.5$$

$$a' = \bar{x} - b_{xy} \cdot \bar{y}$$

Putting $\bar{x} = 3$, $b_{xy} = 0.5$ and $\bar{y} = 9$, we get

$$a' = 3 - 0.5(9) = 3 - 4.5 = -1.5$$

$$\therefore\ a' = -1.5$$

Putting $a' = -1.5$ and b_{xy} in $x = a' + b_{xy} \cdot y$. we get the regression equation of X on Y as follows :

$$x = -1.5 + 0.5y$$

$$\Rightarrow x = 0.5y - 1.5$$

9. From the following data obtain the equation of two regression lines :

X	6	2	10	4	8
Y	9	11	5	8	7

Solution :

We prepare the following table for calculation :

x	y	$(x - \bar{x})$ $\bar{x} = 6$	$(y - \bar{y})$ $\bar{y} = 8$	$(x - \bar{x})(y - \bar{y})$	$(x - \bar{x})^2$	$(y - \bar{y})^2$
6	9	0	1	0	0	1
2	11	-4	3	-12	16	9
10	5	4	-3	-12	16	9
4	8	-2	0	0	4	0
8	7	2	-1	-2	4	1
$\Sigma x = 30$	$\Sigma y = 40$	$\Sigma (x - \bar{x}) = 0$	$\Sigma (y - \bar{y}) = 0$	$\Sigma (x - \bar{x})(y - \bar{y}) = -26$	$\Sigma (x - \bar{x})^2 = 40$	$\Sigma (y - \bar{y})^2 = 20$

Here, $n = 5$

$$\bar{x} = \frac{\Sigma x}{n} = \frac{30}{5} = 6;\ \bar{y} = \frac{\Sigma y}{n} = \frac{40}{5} = 8$$

Regression equation of Y on X :

$y = a + b_{yx} \cdot x$

$$b_{yx} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(x-\bar{x})^2} = \frac{-26}{40} = -0.65$$

$\therefore\ b_{yx} = -0.65$

$a = \bar{y} - b_{yx} \cdot \bar{x}$

Putting $\bar{y} = 8$, $b_{yx} = -0.65$ and $\bar{x} = 6$, we get

$a = 8 - (-0.65)6 = 8 + 3.9 = 11.9$

$\therefore\ a = 11.9$

Hence, regression equation of Y on X is

$y = 11.9 - 0.65x$

$\therefore\ y = -0.65x + 11.9$

Regression equation of X on Y :

$x = a' + b_{xy} \cdot y$

$$b_{xy} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(y-\bar{y})^2} = \frac{-26}{20} = -1.3$$

$\therefore\ b_{xy} = -1.3$

$a' = \bar{x} - b_{xy} \cdot \bar{y}$

Putting $\bar{x} = 6$, $b_{xy} = -1.3$ and $\bar{y} = 8$, we get

$a' = 6 - (-1.3)8$

$\therefore\ a' = 6 + 10.4 \qquad \therefore\ a' = 16.4$

Hence, the regression equation of X on Y is

$x = 16.4 - 1.3y$

$\therefore\ x = -1.3y + 16.4$

10. For the following data, find the regression line of Y on X :

X	1	2	3
Y	2	1	6

Hence find the most likely value of y when $x = 4$.

Solution :

We prepare the following table for calculation :

x	y	$(x-\bar{x})$ $\bar{x}=2$	$(y-\bar{y})$ $\bar{y}=3$	$(x-\bar{x})(y-\bar{y})$	$(x-\bar{x})^2$
1	2	-1	-1	1	1
2	1	0	-2	0	0
3	6	1	3	3	1
Σx $=6$	Σy $=9$	$\Sigma(x-\bar{x})$ $=0$	$\Sigma(y-\bar{y})$ $=0$	$\Sigma(x-\bar{x})(y-\bar{y})$ $=4$	$\Sigma(x-\bar{x})^2$ $=2$

Here, $n = 3$

$$\bar{x} = \frac{\Sigma x}{n} = \frac{6}{3} = 2;\ \bar{y} = \frac{\Sigma y}{n} = \frac{9}{3} = 3$$

Regression line of Y on X :

$y = a + b_{yx} \cdot x$

$$b_{yx} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(x-\bar{x})^2} = \frac{4}{2} = 2$$

$\therefore\ b_{yx} = 2$

$a = \bar{y} - b_{yx} \cdot \bar{x}$

Putting $\bar{y} = 3$, $b_{yx} = 2$, $\bar{x} = 2$, we get

$a = 3 - 2(2) = 3 - 4 = -1$

$\therefore\ a = -1$

Hence, regression line of Y on X is

$y = -1 + 2x$

$\therefore\ y = 2x - 1$

Most likely value of Y when X = 4 :

Putting $x = 4$ in $y = 2x - 1$, we get

$y = 2(4) - 1$

$\therefore\ y = 8 - 1 \qquad \therefore\ y = 7.$

11. From the following data, find the regression equation of Y on X and estimate Y when X = 10 :

X	1	2	3	4	5	6
Y	2	4	7	6	5	6

Solution :

We prepare the following table for calculation :

x	y	xy	x^2
1	2	2	1
2	4	8	4
3	7	21	9
4	6	24	16
5	5	25	25
6	6	36	36
$\Sigma x = 21$	$\Sigma y = 30$	$\Sigma xy = 116$	$\Sigma x^2 = 91$

Here, $n = 6$

$$\bar{x} = \frac{\Sigma x}{n} = \frac{21}{6} = 3.5;\ \bar{y} = \frac{\Sigma y}{n} = \frac{30}{6} = 5$$

Regression equation of Y on X :

$y = a + b_{yx} \cdot x$

$$b_{yx} = \frac{\Sigma xy - n\bar{x}\bar{y}}{\Sigma x^2 - n\bar{x}^2} = \frac{116 - 6(3.5 \times 5)}{91 - 6(3.5)^2} = \frac{116 - 105}{91 - 73.5} = \frac{11}{17.5}$$

$= 0.63$

$\therefore\ b_{yx} = 0.63 \qquad a = \bar{y} - b_{yx} \cdot \bar{x}$

Putting $\bar{y} = 5$, $b_{yx} = 0.63$, $\bar{x} = 3.5$, we get

$a = 5 - 0.63(3.5)$

$= 5 - 2.205 = 2.795$

$\therefore \; a = 2.795 \approx 2.8$

Hence, the regression equation of Y on X is

$y = 2.8 + 0.63x$

$\Rightarrow y = 0.63x + 2.8^*$

Estimate of Y when X = 10 :

Put $x = 10$ in $y = 0.63x + 2.8$

$\therefore \; y = 0.63(10) + 2.8 \qquad \therefore \; y = 6.3 + 2.8$

$\therefore \; y = 9.1^*$

[* **Note :** *Answers given in the textbook are incorrect.*]

12. The following sample gives the number of hours of study (X) per day for an examination and marks (Y) obtained by 12 students :

X	3	3	3	4	4	5	5	5	6	6	7	8
Y	45	60	55	60	75	70	80	75	90	80	75	85

Obtain the line of regression of marks on hours of study.

Solution :

We take new variables $u = (x - a)$, $a = 5$ and $v = (y - b)$, $b = 70$ to make the calculation easier.

We prepare the following table for calculation :

No. of hours of study x	Grades obtained y	$u = (x - a)$ $a = 5$	$v = (y - b)$ $b = 70$	uv	u^2
3	45	-2	-25	50	4
3	60	-2	-10	20	4
3	55	-2	-15	30	4
4	60	-1	-10	10	1
4	75	-1	5	-5	1
5	70	0	0	0	0
5	80	0	10	0	0
5	75	0	5	0	0
6	90	1	20	20	1
6	80	1	10	10	1
7	75	2	5	10	4
8	85	3	15	45	9
$\Sigma x = 59$	$\Sigma y = 850$	7 -8 $\Sigma u = -1$	70 -60 $\Sigma v = 10$	195 $-\;5$ $\Sigma uv = 190$	$\Sigma u^2 = 29$

Here, $n = 12$

$$\bar{x} = \frac{\Sigma x}{n} = \frac{59}{12} = 4.92, \quad \bar{y} = \frac{\Sigma y}{n} = \frac{850}{12} = 70.83$$

$$\bar{u} = \frac{\Sigma u}{n} = \frac{-1}{12} = -0.083; \; \bar{v} = \frac{\Sigma v}{n} = \frac{10}{12} = 0.83$$

Regression line of marks (Y) on hours of study (X) :

$y = a + b_{yx} \cdot x$

[**Note :** *The value of b_{yx} is independent of change of origin.*]

$$\therefore \; b_{yx} = b_{vu} = \frac{\Sigma uv - n(\bar{u}\bar{v})}{\Sigma u^2 - n\bar{u}^2}$$

$$= \frac{190 - 12(-0.083 \times 0.83)}{29 - 12(-0.083)^2}$$

$$= \frac{190 + 0.069}{29 - 0.083} = \frac{190.069}{28.917} = 6.57$$

$\therefore \; b_{yx} = 6.57 \approx 6.6$

$a = \bar{y} - b_{yx} \cdot \bar{x}$

Putting $\bar{y} = 70.83$, $b_{yx} = 6.6$ and $\bar{x} = 4.92$, we get

$a = 70.83$, $b_{yx} = 6.6$ and $\bar{x} = 4.92$, we get

$a = 70.83 - 6.6(4.92)$

$\therefore \; a = 70.83 - 32.47$

$\therefore \; a = 38.36$

Hence, the regression line of marks (Y) on hours of study (X) is

$y = 38.36 + 6.6x$

$\Rightarrow y = 6.6x + 38.36.$

EXAMPLES FOR PRACTICE 3.1

1. Find the equation of regression line of X on Y from the following data :

X	6	2	10	4	8
Y	9	11	5	8	7

2. Find the equation of the regression line of Y on X, hence find Y when X = 10; given that :

X	1	2	3	4	5	6
Y	2	4	7	6	5	6

3. From the following data, estimate X when Y = 200 :

X	250	248	297	338	463	393
Y	137	147	184	196	276	260

4. In a bivariate data, $n=10$, $\Sigma x=10$, $\Sigma y=210$, $\Sigma x^2=14$, $\Sigma y^2=5340$, $\Sigma xy=180$. Estimate x when $y=15$.

5. Determine the two regression lines from the following data :

x	7	6	10	14	13
y	22	18	20	26	24

6. You are given the following information about two variables X and Y :

$n=10$, $\Sigma x^2=385$, $\Sigma y^2=192$, $\bar{x}=5.5$, $\bar{y}=4$, $\Sigma xy=185$.

Find (i) Regression line of Y on X (ii) Regression line of X on Y.

7. From 10 observations of price (X) and supply (Y) of a certain commodity, the following data was obtained : $\Sigma x=130$, $\Sigma y=220$, $\Sigma x^2=2288$, $\Sigma y^2=5506$, $\Sigma xy=3467$. Estimate the line of regression of Y on X and find the estimate of supply when the price is 16.

8. The following table shows the ages (X) and blood pressure (Y) of 8 patients :

X	52	63	45	36	72	65	47	25
Y	62	53	51	25	79	43	60	33

Obtain the regression equation of Y on X and find the expected blood pressure of a person 49 years old.

9. Following table gives the marks of 10 students of standard 12 in Commercial Mathematics and Language. Obtain the best fitted line of regression of marks in Language (Y) on marks in Commercial Mathematics (X) :

Marks in Commercial Mathematics	45	55	56	58	60	65	68	70	80	85
Marks in Language	40	60	48	52	50	55	45	60	50	70

10. Two judges A and B have given marks independently to 5 plays as below. When the sixth play was enacted, judge B was absent and judge A gave 37 marks to that play. Obtain the best fitted line of regression of the marks given by judge B on the marks given by judge A and estimate the marks that would be given to the sixth play by judge B :

Serial no. of play	1	2	3	4	5
Marks given by judge A (X)	46	44	43	42	40
Marks given by judge B (Y)	42	36	39	38	35

11. Let X be the number of matches played by the player and Y be the number of matches in which he scored more than 50 runs. The following data shown is obtained for 5 players :

No. of Matches played (X)	Data of matches of 5 players				
	21	25	26	24	19
Scored more than 50 in a match (Y)	19	20	24	21	16

Find the regression line of X on Y.

12. A departmental store gives training to the salesmen in service followed by a test. It is experienced that the performance regarding sales of any salesman is linearly related to the scores secured by him. The following data gives the test scores and sales made by nine (9) salesmen during a fixed period :

Text scores (X)	16	22	28	24	29	25	16	23	24
Sales (Y) (₹ in hundreds)	35	42	57	40	54	51	34	47	45

(a) Obtain the line of regression of Y on X.

(b) Estimate Y when X = 17.

Answers

1. $10x=-13y+164$ *OR* $x=-1.3y+16.4$

2. $y=0.63x+2.795$; $y=9.095$

3. $x=331.5$

4. $x=1.190\approx1.2$

5. (i) $y=0.76x+14.4$ (ii) $x=0.95y-10.9$

6. (i) $y=-0.42x+6.31$

(ii) $x=-1.094y+9.876$

7. $y=1.015x+8.805$, $y=25.045$

8. $y=11.87+0.768x$, $y=49.502$

9. $y=25.39+0.43x$

10. $y=-2.85+0.95x$: 32 marks **11.** $x=4.76+0.91y$.

12. $y=7.4525+1.6325x$, ₹ 34.905 hundred.

3.3 : PROPERTIES OF REGRESSION COEFFICIENTS

(1) The line of regression of Y on X is given by

$y = a + b_{yx} \cdot x$

Here, b_{yx} a constant in the equation is called the regression coefficient of Y on X. It represents the appropriate increase (or decrease) in the value of Y corresponding to unit increase (or decrease) in the value of X.

(2) The line of regression of X on Y is given by

$x = a' + b_{xy} \cdot y$

Here, b_{xy} a constant in the equation is called the regression coefficient of X on Y. It represents the appropriate increase (or decrease) in the value of X corresponding to unit increase (or decrease) in the value of Y.

Properties :

(1) $r^2 = b_{yx} \cdot b_{xy}$.

Proof : $b_{yx} = \dfrac{\text{Cov }(x,\ y)}{\sigma_x^2}$ and $b_{xy} = \dfrac{\text{Cov }(x,\ y)}{\sigma_y^2}$

$$\therefore\ b_{yx} . b_{xy} = \frac{\text{Cov }(x,\ y)}{\sigma_x^2} \times \frac{\text{Cov }(x,\ y)}{\sigma_y^2}$$

$$= \left[\frac{\text{Cov }(x,\ y)}{\sigma_x \cdot \sigma_y}\right]^2$$

$$= r^2$$

$\therefore\ r^2 = b_{yx} \cdot b_{xy}$

Hence, $r = \pm\sqrt{b_{yx} \cdot b_{xy}}$

(2) If $b_{yx} > 1$, then $b_{xy} < 1$.

Proof : Let us assume that $b_{yx} > 1$ and $b_{xy} > 1$

Hence, $b_{yx} \cdot b_{xy} > 1$

$\therefore\ r^2 > 1$

which is not possible. Hence our assumption is incorrect and it can be said that b_{yx} and b_{xy} both cannot exceed unity simultaneously.

(3) r, b_{yx} and b_{xy} are of similar sign :

$$r = \frac{\text{Cov }(x,\ y)}{\sigma_x \cdot \sigma_y}.$$

$\therefore\ \text{Cov }(x,\ y) = r \cdot \sigma_x \cdot \sigma_y$

Now, $b_{yx} = \dfrac{\text{Cov }(x,\ y)}{\sigma_x^2}$ | $b_{xy} = \dfrac{\text{Cov }(x,\ y)}{\sigma_y^2}$

$\therefore\ b_{yx} = \dfrac{r \cdot \sigma_x \cdot \sigma_y}{\sigma_x^2}$ | $\therefore\ b_{xy} = \dfrac{r \cdot \sigma_x \cdot \sigma_y}{\sigma_y^2}$

$\therefore\ b_{yx} = r \cdot \dfrac{\sigma_y}{\sigma_x}$ | $\therefore\ b_{xy} = r \cdot \dfrac{\sigma_x}{\sigma_y}$

σ_x and σ_y are always positive, hence r, b_{yx} and b_{xy} are of similar sign.

(4) $\left|\dfrac{b_{yx} + b_{xy}}{2}\right| \geqslant |r|$.

Proof : $b_{yx} = r \cdot \dfrac{\sigma_y}{\sigma_x}$ and $b_{xy} = r \cdot \dfrac{\sigma_x}{\sigma_y}$

$$\therefore\ b_{yx} + b_{xy} = r \cdot \frac{\sigma_y}{\sigma_x} + r \cdot \frac{\sigma_x}{\sigma_y}$$

$$= r\left[\frac{\sigma_y^2 + \sigma_x^2}{\sigma_x \cdot \sigma_y}\right] \qquad \dots (1)$$

Now, $(\sigma_x - \sigma_y)^2 \geqslant 0$

$\therefore\ \sigma_x^2 + \sigma_y^2 \geqslant 2\sigma_x\sigma_y$

$\therefore\ \dfrac{\sigma_x^2 + \sigma_y^2}{\sigma_x \cdot \sigma_y} \geqslant 2$

$$\therefore\ r\left[\frac{\sigma_x^2 + \sigma_y^2}{\sigma_x \cdot \sigma_y}\right] \geqslant 2r \qquad \dots (2)$$

From (1) and (2), we have

$b_{yx} + b_{xy} \geqslant 2r$

$\therefore\ \dfrac{b_{yx} + b_{xy}}{2} \geqslant r$

This results hold only for positive values of b_{yx} and b_{xy}.

$\therefore\ \left|\dfrac{b_{yx} + b_{xy}}{2}\right| \geqslant |r|$

(5) It is interesting to note that :

$$b_{yx} = \frac{\text{Cov}\,(x, y)}{\sigma_x^2},\ b_{xy} = \frac{\text{Cov}\,(x, y)}{\sigma_y^2} \text{ and } r = \frac{\text{Cov}\,(x, y)}{\sigma_x \cdot \sigma_y}.$$

Clearly, denominator of each coefficient is positive and numerator is same. Hence, the numerator decides algebraic sign of the coefficient. Thus all coefficients have same algebraic sign.

If $r > 0$, $b_{yx} > 0$ and $b_{xy} > 0$

$r < 0$, $b_{yx} < 0$ and $b_{xy} < 0$

$r = 0$, $b_{yx} = 0$ and $b_{xy} = 0$

(6) The values of b_{yx} and b_{xy} are independent of change of origin but not of scale.

Proof : Let $u = \frac{x-a}{h}$ and $v = \frac{y-b}{k}$

where a, b, h and k are constants; $h, k \neq 0$

$\therefore\ \sigma_x^2 = h^2\sigma_u^2$ and $\sigma_y^2 = k^2\sigma_v^2$

Also, $\text{Cov}(x, y) = hk\,\text{Cov}(u, v)$

$$\text{Therefore, } b_{yx} = \frac{\text{Cov}(x,\ y)}{\sigma_x^2} = \frac{hk\,\text{Cov}(u,\ v)}{h^2\sigma_u^2}$$

$$= \frac{k}{h}\cdot\frac{\text{Cov}(u,\ v)}{\sigma_u^2} = \frac{k}{h}b_{vu}$$

$$\text{Similarly, } b_{xy} = \frac{\text{Cov}(x,\ y)}{\sigma_y^2} = \frac{hk\,\text{Cov}(u,\ v)}{k^2\sigma_v^2}$$

$$= \frac{h}{k}\cdot\frac{\text{Cov}(u,\ v)}{\sigma_v^2} = \frac{h}{k}b_{uv}$$

This means, regression coefficients are independent of change of origin but not of scale.

EXERCISE 3.2	Textbook pages 47 and 48

1. For a bivariate data :

$\bar{x} = 53$, $\bar{y} = 28$, $b_{yx} = -1.2$ and $b_{xy} = -0.3$. Find

(i) Correlation coefficient between X and Y.

(ii) Estimate of Y for X = 50.

(iii) Estimate of X for Y = 25.

Solution :

Given : $\bar{x} = 53$, $\bar{y} = 28$, $b_{yx} = -1.2$, $b_{xy} = -0.3$

(i) Correlation coefficient between X and Y :

$$r = \pm\sqrt{b_{yx}\cdot b_{xy}}$$

$$= \pm\sqrt{(-1.2)(-0.3)}$$

$$= \pm\sqrt{0.36}$$

$\therefore\ r = -0.6$ ... ($\because$ b_{yx} and b_{xy} are negative.)

(ii) Estimation of Y for X = 50 :

Regression equation of Y on X is,

$y = a + b_{yx}\cdot x$

$b_{yx} = -1.2$

$a = \bar{y} - b_{yx}\cdot\bar{x}$

$= 28 - (-1.2)53$

$= 28 + 63.6$

$= 91.6$

$\therefore\ y = 91.6 - 1.2x$

$\Rightarrow y = -1.2x + 91.6$

Put $x = 50$

$\therefore\ y = -1.2(50) + 91.6$

$\therefore\ y = -60 + 91.6$ $\quad\therefore\ y = 31.6$

(iii) Estimation of X for Y = 25 :

Regression equation of X on Y is,

$x = a' + b_{xy}\cdot y$

$b_{xy} = -0.3$

$a' = \bar{x} - b_{xy}(\bar{y})$

$= 53 - (-0.3)(28)$

$= 53 + 8.4$

$= 61.4$

$\therefore\ x = 61.4 - 0.3y$

$\Rightarrow x = -0.3y + 61.4$

Put $y = 25$,

$\therefore\ x = -0.3(25) + 61.4$

$\therefore\ x = -7.5 + 61.4$ $\quad\therefore\ x = 53.9$

2. From the data of 20 pairs of observations on X and Y, following results are obtained :

$\bar{x} = 199$, $\bar{y} = 94$, $\Sigma(x_i - \bar{x})^2 = 1200$, $\Sigma(y_i - \bar{y})^2 = 300$, $\Sigma(x_i - \bar{x})(y_i - \bar{y}) = -250$. Find

(i) The line of regression of Y on X.

(ii) The line of regression of X on Y.

(iii) Correlation coefficient between X and Y.

Solution :

Given : $\bar{x} = 199$, $\bar{y} = 94$, $\Sigma(x_i - \bar{x})^2 = 1200$, $\Sigma(y_i - \bar{y})^2 = 300$,

$\Sigma(x_i - \bar{x})(y_i - \bar{y}) = -250$

(i) The line of regression of Y on X :

$$b_{yx} = \frac{\Sigma(x_i - \bar{x})(y_i - \bar{y})}{\Sigma(x_i - \bar{x})^2}$$

$$= \frac{-250}{1200} = -\frac{5}{24}$$

$y = a + b_{yx}\cdot x$

$a = \bar{y} - b_{yx}\cdot\bar{x}$

$$= 94 - \left(-\frac{5}{24}\right)199$$

$$= 94 + \frac{995}{24} = \frac{2256 + 995}{24}$$

$$= \frac{3251}{24}$$

$\therefore$ line of regression of Y on X is

$$y = \frac{3251}{24} - \frac{5}{24}x$$

$\therefore\ 24y = 3251 - 5x$

$\therefore\ 5x + 24y = 3251$

(ii) The line of regression of X on Y :

$$b_{xy} = \frac{\Sigma(x_i - \bar{x})(y_i - \bar{y})}{\Sigma(y_i - \bar{y})^2}$$

$$= \frac{-250}{300} = -\frac{5}{6}$$

$x = a' + b_{xy} \cdot y$

$a' = \bar{x} - b_{xy} \cdot \bar{y}$

$$= 199 - \left(-\frac{5}{6}\right)(94)$$

$$= 199 + \frac{470}{6} = \frac{1194 + 470}{6} = \frac{1664}{6}$$

$\therefore$ line of regression of X on Y is

$$x = \frac{1664}{6} - \frac{5}{6}y$$

$\therefore\ 6x = 1664 - 5y$

$\therefore\ 6x + 5y = 1664$

(iii) Correlation coefficient between X and Y :

$$b_{yx} = -\frac{5}{24},\ b_{xy} = -\frac{5}{6}$$

Now, $r = \pm\sqrt{b_{yx} \cdot b_{xy}}$

$$= \pm\sqrt{\left(-\frac{5}{24}\right)\left(-\frac{5}{6}\right)}$$

$$= \pm\sqrt{\frac{25}{144}}$$

$\therefore\ r = -\frac{5}{12}$ ... ($\because$ b_{yx} and b_{xy} are negative)

3. From the data of 7 pairs of observations on X and Y, following results are obtained :

$\Sigma(x_i - 70) = -35$, $\Sigma(y_i - 60) = -7$,

$\Sigma(x_i - 70)^2 = 2989$, $\Sigma(y_i - 60)^2 = 476$,

$\Sigma(x_i - 70)(y_i - 60) = 1064$.

[Given : $\sqrt{0.7884} = 0.8879$]

Obtain

(i) The line of regression of Y on X.

(ii) The line of regression of X on Y.

(iii) The correlation coefficient between X and Y.

Solution :

Given : $n = 7$, $\Sigma(x_i - 70) = -35$,

$\Sigma(y_i - 60) = -7$, $\Sigma(x_i - 70)^2 = 2989$,

$\Sigma(y_i - 60)^2 = 476$,

$\Sigma(x_i - 70)(y_i - 60) = 1064$.

Regression coefficients are independent of change of origin.

Therefore, let $u_i = x_i - 70$, $v_i = y_i - 60$

$\therefore\ \Sigma u_i = -35$, $\Sigma v_i = -7$, $\Sigma u_i^2 = 2989$, $\Sigma v_i^2 = 476$,

$\Sigma u_i v_i = 1064$

Now, $\bar{u} = \frac{\Sigma u_i}{n} = \frac{-35}{7} = -5$

$\bar{v} = \frac{\Sigma v_i}{n} = \frac{-7}{7} = -1$

Now $\bar{x} = \bar{u} + 70$: $\bar{y} = \bar{v} + 60$

$= -5 + 70$ $\quad$ $= -1 + 60$

$= 65$ $\quad$ $= 59$

(i) The line of regression of Y on X :

$y = a + b_{yx} \cdot x$

$$b_{yx} = b_{vu} = \frac{\Sigma u_i v_i - n\bar{u}\bar{v}}{\Sigma u_i^2 - n\bar{u}^2}$$

$$= \frac{1064 - 7(-5)(-1)}{2989 - 7(-5)^2} = \frac{1064 - 35}{2989 - 175}$$

$$= \frac{1029}{2814} = 0.3657$$

$\therefore\ b_{yx} = 0.3657 \approx 0.37$

$a = \bar{y} - b_{yx} \cdot \bar{x}$

$= 59 - 0.37(65)$

$= 59 - 24.05$

$= 34.95$

≈ 35

$\therefore$ line of regression of Y on X is

$y = 35 + 0.37x$

$\Rightarrow y = 0.37x + 35$*

(ii) The line of regression of X on Y :

$x = a' + b_{xy} \cdot y$

$$b_{xy} = b_{uv} = \frac{\Sigma u_i v_i - n\bar{u}\bar{v}}{\Sigma v_i^2 - n\bar{v}^2}$$

$$= \frac{1064 - 7(-5)(-1)}{476 - 7(-1)^2} = \frac{1064 - 35}{476 - 7}$$

$$= \frac{1029}{469} = 2.19$$

$\therefore\ b_{xy} = 2.19$

$a' = \bar{x} - b_{xy} \cdot \bar{y}$

$= 65 - 2.19(59)$

$= 65 - 129.21 = -64.21$

$\therefore$ line of regression of X on Y is

$x = -64.21 + 2.19y$

$\Rightarrow x = 2.19y - 64.21^*$

(iii) The correlation coefficient between X and Y :

$b_{yx} = 0.37,\ b_{xy} = 2.19$

Now, $r = \pm\sqrt{b_{yx} \cdot b_{xy}}$

$\therefore\ r = \pm\sqrt{0.37 \times 2.19}$

$\therefore\ r = \pm\sqrt{0.8103}$

$\therefore\ r = 0.90^*$ ... ($\because\ b_{yx}$ and b_{xy} are > 0)

[***Note :** *Answers given in the textbook are incorrect.*]

4. You are given the following information about advertising expenditure and sales :

	Advertisement expenditure (₹ in lakh) (X)	**Sales (₹ in lakh) (Y)**
Arithmetic mean	**10**	**90**
Standard deviation	**3**	**12**

Correlation coefficient between X and Y is 0.8

(i) Obtain the two regression equations.

(ii) What is the likely sales when the advertising budget is ₹ 15 lakh ?

(iii) What should be the advertising budget if the company wants to attain sales target of ₹ 120 lakh ?

Solution :

Given : $\bar{x} = 10,\ \bar{y} = 90,\ \sigma_x = 3,\ \sigma_y = 12,\ r = 0.8$

(i) Regression equation of Y on X :

$y = a + b_{yx} \cdot x$

$$b_{yx} = r \cdot \frac{\sigma_y}{\sigma_x}$$

$$\therefore\ b_{yx} = 0.8 \times \frac{12}{3} = 3.2$$

$a = \bar{y} - b_{yx}(\bar{x})$

$= 90 - 3.2(10)$

$= 90 - 32$

$= 58$

Now, $y = a + b_{yx} \cdot x$

$\therefore\ y = 58 + 3.2x$

$\Rightarrow y = 3.2x + 58$

Regression equation of X on Y :

$x = a' + b_{xy} \cdot y$

$$b_{xy} = r \cdot \frac{\sigma_x}{\sigma_y}$$

$$\therefore\ b_{xy} = 0.8 \times \frac{3}{12} = 0.2$$

$a' = \bar{x} - b_{xy} \cdot \bar{y}$

$= 10 - 0.2(90)$

$= 10 - 18$

$= -8$

Now, $x = a' + b_{xy} \cdot y$

$\therefore\ x = -8 + 0.2y$

$\Rightarrow x = 0.2y - 8$

(ii) Likely sales (Y) when X = 15 :

Putting $x = 15$ in $y = 58 + 3.2x$, we get

$y = 58 + 3.2\ (15)$

$\therefore\ y = 58 + 48.0 \quad \therefore\ y = 106$

Hence, likely sales is ₹ 106 lakh when advertising budget is ₹ 15 lakh.

(iii) Estimation of advertising budget (X) when Y = 120 :

Putting $y = 120$ in $x = -8 + 0.2y$, we get

$x = -8 + 0.2\ (120)$

$\therefore\ x = -8 + 24$

$\therefore\ x = 16$

Hence, the advertising budget is ₹ 16 lakh to attain sales target of ₹ 120 lakh.

5. Bring out inconsistency if any, in the following :

(i) $b_{yx} + b_{xy} = 1.30$ and $r = 0.75$.

(ii) $b_{yx} = b_{xy} = 1.50$ and $r = -0.9$.

(iii) $b_{yx} = 1.9$ and $b_{xy} = -0.25$.

(iv) $b_{yx} = 2.6$ and $b_{xy} = \frac{1}{2.6}$.

Solution :

(i) $b_{yx} + b_{xy} = 1.30,\ r = 0.75$

$$\frac{b_{yx} + b_{xy}}{2} = \frac{1.30}{2} = 0.65 \text{ and } r = 0.75$$

$$\therefore\ \frac{b_{yx} + b_{xy}}{2} < r$$

Hence, the data is inconsistent.

(ii) $b_{yx} = b_{xy} = 1.50$, $r = -0.9$

The sign of r is not similar to the signs of b_{yx} and b_{xy}.

Hence, the data is inconsistent.

(iii) $b_{yx} = 1.9$, $b_{xy} = -0.25$

The signs of b_{yx} and b_{xy} are not similar.

Hence, the data is inconsistent.

(iv) $b_{yx} = 2.6$ and $b_{xy} = \frac{1}{2.6}$

$b_{yx} > 1$ and $b_{xy} < 1$ and both are of similar signs.

Hence, the data is consistent.

6. Two samples from bivariate populations have 15 observations each. The sample means of X and Y are 25 and 18 respectively. The corresponding sum of squares of deviations from respective means are 136 and 150. The sum of product of deviations from respective means is 123. Obtain the equation of line of regression of X on Y.

Solution :

Given : $n = 15$, $\bar{x} = 25$, $\bar{y} = 18$, $\Sigma(x-\bar{x})^2 = 136$,

$\Sigma(y-\bar{y})^2 = 148$, $\Sigma(x-\bar{x})(y-\bar{y}) = 122$.

Regression coefficient of X on Y :

Equation of line of regression of X on Y :

$x = a' + b_{xy} \cdot y$

$$b_{xy} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(y-\bar{y})^2}$$

$$= \frac{122}{148} = 0.8243 \approx 0.82$$

$a' = \bar{x} - b_{xy} \cdot \bar{y}$

$= 25 - 0.82(18)$

$= 25 - 14.76$

$= 10.24$

$\therefore$ equation of line of regression of X on Y is

$x = 10.24 + 0.82y$

$\Rightarrow x = 0.82y + 10.24$

7. For a certain bivariate data

	X	Y
Mean	25	20
S.D.	4	3

And $r = 0.5$, estimate y when $x = 10$ and estimate x when $y = 16$.

Solution :

Given : $\bar{x} = 25$, $\bar{y} = 20$, $\sigma_x = 4$, $\sigma_y = 3$, $r = 0.5$

Estimation of Y when X = 10 :

Regression equation of Y on X is

$y = a + b_{yx} \cdot x$

Now, $b_{yx} = r \cdot \frac{\sigma_y}{\sigma_x}$

$$= 0.5 \times \frac{3}{4} = \frac{3}{8}$$

$a = \bar{y} - b_{yx} \cdot \bar{x}$

$$= 20 - \frac{3}{8} \times 25$$

$$= 20 - \frac{75}{8}$$

$$= \frac{160-75}{8} = \frac{85}{8}$$

$$\therefore\ y = \frac{85}{8} + \frac{3}{8}x$$

Putting $x = 10$,

$$y = \frac{85}{8} + \frac{3}{8} \times 10$$

$$= \frac{85+30}{8} = \frac{115}{8} = 14.375$$

Estimation of X when Y = 16 :

Regression equation of X on Y is

$x = a' + b_{xy} \cdot y$

$b_{xy} = r \cdot \frac{\sigma_x}{\sigma_y}$

$$= 0.5 \times \frac{4}{3}$$

$$= \frac{2}{3}$$

$a' = \bar{x} - b_{xy} \cdot \bar{y}$

$$= 25 - \frac{2}{3}(20)$$

$$= 25 - \frac{40}{3} = \frac{75-40}{3} = \frac{35}{3}$$

$$\therefore\ x = \frac{35}{3} + \frac{2}{3}y$$

Putting $y = 16$

$$x = \frac{35}{3} + \frac{2}{3} \times 16$$

$= \frac{35}{3} + \frac{32}{3} = \frac{35+32}{3}$

$= \frac{67}{3}$

$= 22.33$

[**Note :** *Answers given in the textbook are incorrect.*]

8. Given the following information about the production and demand of a commodity obtain the two regression lines :

	Production X	**Demand Y**
Mean	85	90
S.D.	5	6

Coefficient of correlation between X and Y is 0.6. Also, estimate the production when demand is 100.

Solution :

Given : $\bar{x} = 85$, $\bar{y} = 90$, $\sigma_x = 5$, $\sigma_y = 6$, $r = 0.6$

Estimation of production (X) when demand (Y) = 100 :

Regression equation of X on Y :

$x = a' + b_{xy} \cdot y$

Now, $b_{xy} = r \cdot \frac{\sigma_x}{\sigma_y}$

$= 0.6 \times \frac{5}{6} = 0.5$

$\therefore \; b_{xy} = 0.5$

$a' = \bar{x} - b_{xy} \cdot \bar{y}$

$\therefore \; a' = 85 - 0.5(90)$

$= 85 - 45$

$= 40$

Hence, regression equation of X on Y is

$x = 40 + 0.5y$

$\Rightarrow x = 0.5y + 40$

Put $y = 100$, we get

$x = 0.5(100) + 40$

$\therefore \; x = 50 + 40$

$\therefore \; x = 90$

Hence, production (X) will be 90 when demand (Y) is 100.

Regression line of Y on X :

$y = a + b_{yx} \cdot x$

Now, $b_{yx} = r \cdot \frac{\sigma_y}{\sigma_x}$

$= 0.6 \times \frac{6}{5}$

$= 0.72$

$a = \bar{y} - b_{yx} \cdot (\bar{x})$

$\therefore \; a = 90 - 0.72(85)$

$= 90 - 61.2$

$= 28.8$

Hence, the regression line of Y on X is

$y = 28.8 + 0.72x$

$\Rightarrow y = 0.72x + 28.8$

9. Given the following data, obtain linear regression estimate of X for Y = 10

$\bar{x} = 7.6$, $\bar{y} = 14.8$, $\sigma_x = 3.2$, $\sigma_y = 16$ and $r = 0.7$.

Solution :

Given : $\bar{x} = 7.6$, $\bar{y} = 14.8$, $\sigma_x = 3.2$, $\sigma_y = 16$, $r = 0.7$

Linear regression estimate of X for Y = 10 :

Regession equation of X on Y :

$x = a' + b_{xy} \cdot y$

Now, $b_{xy} = r \cdot \frac{\sigma_x}{\sigma_y}$

$= 0.7 \times \frac{3.2}{16}$

$= 0.7 \times 0.2 = 0.14$

$\therefore \; b_{xy} = 0.14$

$a' = \bar{x} - b_{xy} \cdot \bar{y}$

$\therefore \; a = 7.6 - 0.14(14.8)$

$= 7.6 - 2.072$

$= 5.528$

Hence, regression equation of X on Y is

$x = 5.528 + 0.14y$

$\Rightarrow x = 0.14y + 5.528$

Putting $y = 10$, we get

$x = 0.14(10) + 5.528$

$= 1.4 + 5.528 = 6.928$

$\therefore \; x = 6.928$

Hence, linear regression estimate of x is 6.928 for $y = 10$.

10. An inquiry of 50 families to study the relationship between expenditure on accommodation (₹ x) and

expenditure on food and entertainment (₹ y) gave the following results :

$\Sigma x = 8500$, $\Sigma y = 9600$, $\sigma_x = 60$, $\sigma_y = 20$, $r = 0.6$

Estimate the expenditure on food and entertainment when expenditure on accommodation is ₹ 200.

Solution :

Here, X = Expenditure on accommodation (in ₹)

Y = Expenditure on food and entertainment (in ₹)

Given : $\Sigma x = 8500$, $\Sigma y = 9600$, $\sigma_x = 60$, $\sigma_y = 20$, $r = 0.6$

$n = 50$

$$\therefore\ \bar{x} = \frac{\Sigma x}{n} = \frac{8500}{50} = ₹\ 170$$

$$\bar{y} = \frac{\Sigma y}{n} = \frac{9600}{50} = ₹\ 192$$

Estimation of Y when X = 200 :

Regression equation of Y on X :

$y = a + b_{yx} \cdot x$

Now, $b_{yx} = r \cdot \dfrac{\sigma_y}{\sigma_x}$

$= 0.6 \times \dfrac{20}{60} = 0.2$

$\therefore\ b_{yx} = 0.2$

$a = \bar{y} - b_{yx} \cdot (\bar{x})$

$= 192 - 0.2(170)$

$= 192 - 34$

$= 158$

$\therefore\ a = 158$

Hence, regression equation of Y on X is

$y = 158 + 0.2x$

$\Rightarrow y = 0.2x + 158$

Putting $x = 200$, we get

$y = 0.2 \times 200 + 158$

$= 40 + 158$

$= 198$

$\therefore\ y = ₹\ 198$*

Hence, the estimated expenditure on food and entertainment will be ₹ 198 when expenditure on accommodation is ₹ 200.

[* **Note :** *Answer given in the textbook is incorrect.*]

11. The following data about the sales and advertisement expenditure of a firms is given below (in ₹ crores).

	Sales	Adv. Exp.
Mean	40	6
S.D.	10	1.5

***Correlation of coefficient between sales and expenditure = 0.9**

(i) Estimate the likely sales for a proposed advertisement expenditure of ₹ 10 crores.

(ii) What should be the advertisement expenditure if the firm proposes a sales target ₹ 60 crores.

[* *Correction in text.*]

Solution :

Here, X = Sales, Y = advertisement expenditure.

Given : $\bar{x} = 40$, $\bar{y} = 6$, $\sigma_x = 10$, $\sigma_y = 1.5$, $r = 0.9$

(i) Estimation of likely sales (X) for Y = 10 :

Regression equation of X on Y :

$x = a' + b_{xy} \cdot y$

Now, $b_{xy} = r \cdot \dfrac{\sigma_x}{\sigma_y}$

$= 0.9 \times \dfrac{10}{1.5}$

$= 6$

$\therefore\ b_{xy} = 6$

$a' = \bar{x} - b_{xy} \cdot \bar{y}$

$= 40 - 6(6)$

$= 40 - 36 = 4$

$\therefore\ a = 4$

Hence, the regression equation of X on Y is

$x = 4 + 6y$

$\Rightarrow x = 6y + 4$

Putting $y = 10$, we get

$x = 6 \times 10 + 4$

$\therefore\ x = 60 + 4$

$\therefore\ x = 64$

Hence, the estimated sales is ₹ 64 crores when advertisement expenditure (Y) is ₹ 10 crores.

(ii) Estimate of advertisement expenditure (Y) when Sales (X) = 60 :

Regression equation of Y on X :

$y = a + b_{yx} \cdot x$

Now, $b_{yx} = r \cdot \frac{\sigma_y}{\sigma_x}$

$= 0.9 \times \frac{1.5}{10}$

$= 0.135$

$\therefore\ b_{yx} = 0.135$

$a = \bar{y} - b_{yx} \cdot (\bar{x})$

$= 6 - 0.135(40)$

$= 6 - 5.4$

$= 0.6$

$\therefore\ a = 0.6$

Hence, the regression equation of Y on X is

$y = 0.6 + 0.135x$

$\Rightarrow y = 0.135x + 0.6$

Putting $x = 60$, we get

$y = 0.135(60) + 0.6$

$\therefore\ y = 8.1 + 0.6$

$\therefore\ y = 8.7$

Hence, the estimated expenditure is ₹ 8.7 crores when sales is of ₹ 60 crores.

12. For a certain bivariate data the following information are available :

	X	Y
A.M.	13	17
S.D.	3	2

Correlation coefficient between x and y is 0.6. Estimate x when $y = 15$ and estimate y when $x = 10$.

Solution :

Given : $\bar{x} = 13$, $\bar{y} = 17$, $\sigma_x = 3$, $\sigma_y = 2$, $r = 0.6$

Estimation of X when Y = 15 :

Regression equation of X on Y is

$x = a' + b_{xy} \cdot y$

Now, $b_{xy} = r \cdot \frac{\sigma_x}{\sigma_y}$

$= 0.6 \times \frac{3}{2} = 0.9$

$\therefore\ b_{xy} = 0.9$

$a' = \bar{x} - b_{xy} \cdot \bar{y}$

$= 13 - 0.9(17)$

$= 13 - 15.3$

$\therefore\ a' = -2.3$

Hence, the regression equation of X on Y is

$x = -2.3 + 0.9y$

$\Rightarrow x = 0.9y - 2.3$

Putting $y = 15$, we get

$x = 0.9 \times 15 - 2.3$

$= 13.5 - 2.3 = 11.2$

$\therefore\ x = 11.2$

Hence, the estimated value of x is 11.2, when $y = 15$.

Estimation of Y when X = 10 :

Regression equation of Y on X is :

$y = a + b_{yx} \cdot x$

Now, $b_{yx} = r \cdot \frac{\sigma_y}{\sigma_x}$

$= 0.6 \times \frac{2}{3} = 0.4$

$\therefore\ b_{yx} = 0.4$

$a = \bar{y} - b_{yx} \cdot \bar{x}$

$= 17 - 0.4(13)$

$= 17 - 5.2 = 11.8$

$\therefore\ a = 11.8$

Hence, the regression equation of Y on X is

$y = 11.8 + 0.4x$

$\Rightarrow y = 0.4x + 11.8$

Putting $x = 10$, we get

$y = 0.4(10) + 11.8$

$= 4 + 11.8 = 15.8$

$\therefore\ y = 15.8$

Hence, the estimated value of y is 15.8, when $x = 10$.

EXAMPLES FOR PRACTICE 3.2

1. For a sample of firms the coefficient of correlation between the profit rate (X) and growth rate (Y) is found to be 0.9. Given the following additional

information, (a) obtain the equation of the regression line of Y on X (b) estimate the growth rate when profit rate is 16.

	X	Y
Mean	20	25
Variance	9	25

2. If for a bivariate data, $\bar{x} = 10$, $\bar{y} = 12$, variance $v_x = 9$, $\sigma_y = 4$ and $r = 0.6$. Estimate y when $x = 5$.

3. In a bivariate data, $\bar{x} = 27.6$, $\bar{y} = 14.8$, $\sigma_x = 4$, $\sigma_y = 2$ and the coefficient of correlations is $r = 0.8$. Find the most probable value of Y when X = 20.

4. The following data relate to the age of husbands and wives :

	Husbands	Wives
Mean	27 years	23 years
Standard deviation	3 years	2 years

The coefficient of correlation is $r = 0.93$. Estimate the age of a woman whose husband is aged 23.

5. Given the data : $\bar{x} = 6$, $\bar{y} = 40$, $\sigma_x = 1.6$, $\sigma_y = 10$, $r = 0.9$. Estimate Y for X = 10. What would be the estimated value of X if Y = 60 ?

6. If $\Sigma x_i = 56$, $\Sigma y_i = 56$, $\Sigma x_i^2 = 476$. $\Sigma y_i^2 = 476$, $\Sigma x_i y_i = 469$ and $n = 7$. Find (a) The regression equation of Y on X (b) y, if $x = 12$.

7. The following results were obtained from records of age (X) and systolic blood pressure (Y), of a group of 10 women.

	X	Y
Mean	53	142
Variance	130	165

$\Sigma(x_i - \bar{x})(y_i - \bar{y}) = 1220$

Find the appropriate regression equation and use it to estimate the blood pressure of a woman with age 47 years.

8. Compute regression coefficients from the following data on the variables weight (X) and height (Y) of 8 individuals :

$n = 8$, $\Sigma(x_i - 45) = 45$, $\Sigma(x_i - 45)^2 = 4400$,
$\Sigma(y_i - 150) = 280$, $\Sigma(y_i - 150)^2 = 167432$,
$\Sigma(x_i - 45)(y_i - 150) = 21680$.

Answers

1. $y = -5 + 1.5x$, $y = 19$
2. $y = 8$
3. $y = 11.76$
4. Age of woman is 20.52 years
5. $y = 62.5$, $x = 8.88$
6. (a) $y = \frac{3}{4}x + 2$, (b) $y = 11$
7. $y = 0.94x + 92.18$, $y = 136.36$
8. $b_{yx} = 4.86$, $b_{xy} = 0.13$.

EXERCISE 3.3 | Textbook pages 49 and 50

1. From the two regression equations find r, $\bar{x}$ and $\bar{y}$: $4y = 9x + 15$ and $25x = 4y + 17$.

Solution :

To find r :

Let $4y = 9x + 15$ be the equation of line of regression of Y on X.

$$\therefore\ y = \frac{9}{4}x + \frac{15}{4}$$

$$\therefore\ b_{yx} = \frac{9}{4} \qquad \ldots (\because \text{ it is coefficient of } x)$$

Then the other equation $25x = 4y + 17$ is the equation of line of regression of X on Y.

$$\therefore\ x = \frac{4}{25}y + \frac{17}{25}$$

$$\therefore\ b_{xy} = \frac{4}{25} \qquad \ldots (\because \text{ it is coefficient of } y)$$

Now, $r^2 = b_{yx} \cdot b_{xy}$

$$\therefore\ r = \pm\sqrt{b_{yx} \cdot b_{xy}} = \pm\sqrt{\frac{9}{4} \times \frac{4}{25}}$$

$$= \pm\sqrt{\frac{9}{25}} = \pm\sqrt{0.36} = \pm 0.6$$

$$\therefore\ r = 0.6 \qquad \ldots (\because\ b_{yx} \text{ and } b_{xy} \text{ are positive})$$

To find $\bar{x}$, $\bar{y}$:

$$9x - 4y + 15 = 0 \qquad \ldots (1)$$
$$25x - 4y - 17 = 0 \qquad \ldots (2)$$

Subtracting equation (2) from equation (1)

$$\begin{array}{r} 9x - 4y + 15 = 0 \\ 25x - 4y - 17 = 0 \\ - \quad + \quad + \quad \\ \hline -16x \qquad + 32 = 0 \end{array}$$

$$\therefore\ 32 = 16x$$

$$\therefore\ x = \frac{32}{16} = 2$$

$\therefore\ \bar{x}=2$

Putting $x=2$ in the equation (1),

$9(2)-4y+15=0$

$\therefore\ 18-4y+15=0$

$\therefore\ 33=4y$

$\therefore\ y=\frac{33}{4}=8.25$

$\therefore\ \bar{y}=8.25$

Hence, $r=0.6$, $\bar{x}=2$, $\bar{y}=8.25$.

2. In a partially destroyed laboratory record of an analysis of regression data, the following data are legible :

Variance of X = 9.

Regression equations : $8x-10y+66=0$ and $40x-18y=214$.

Find on the basis of above information :

(i) The mean values of X and Y.

(ii) Correlation coefficient between X and Y.

(iii) Standard Deviation of Y.

Solution :

Given : $8x-10y+66=0$, $40x-18y=214$,

$\sigma_x^2=9\quad\therefore\ \sigma_x=3$

(i) The mean values of X and Y :

$$8x-10y=-66 \qquad \dots(1)$$

$$40x-18y=214 \qquad \dots(2)$$

Multiplying equation (1) by 5 and subtracting equation (2) from it, we get

$$\begin{array}{rcr} 40x-50y & = & -330 \\ 40x-18y & = & 214 \\ -\quad\ + & & - \\ \hline -32y & = & -544 \end{array}$$

$\therefore\ y=\frac{544}{32}=17$

Put $y=17$ in equation (1)

$\therefore\ 8x-10(17)=-66$

$\therefore\ 8x=-66+170$

$\therefore\ 8x=104$

$\therefore\ x=\frac{104}{8}=13$

Hence, $\bar{x}=13$ and $\bar{y}=17$.

(ii) Correlation coefficient between X and Y :

Let $8x-10y+66=0$ be the regression equation of Y on X.

$\therefore\ 10y=8x+66$

$\therefore\ y=\frac{8}{10}x+\frac{66}{10}$

$\therefore\ b_{yx}=\frac{8}{10}$ ... ($\because$ it is coefficient of x)

and the other equation $40x-18y=214$ be the regression equation of X and Y.

$\therefore\ 40x=18y+214$

$\therefore\ x=\frac{18}{40}y+\frac{214}{40}$

$\therefore\ b_{xy}=\frac{18}{40}$ ... ($\because$ it is coefficient of y)

Now, $r=\pm\sqrt{b_{yx}\cdot b_{xy}}$

$$=\pm\sqrt{\frac{8}{10}\times\frac{18}{40}}=\pm\sqrt{\frac{9}{25}}=\pm\frac{3}{5}$$

$\therefore\ r=\frac{3}{5}=0.6$ ... ($\because$ b_{yx} and b_{xy} are positive)

(iii) Standard deviation of y :

$b_{yx}=r\cdot\frac{\sigma_y}{\sigma_x}$

$\therefore\ \frac{8}{10}=\frac{3}{5}\times\frac{\sigma_y}{3}$ ... ($\because\ \sigma_x^2=9\ \therefore\ \sigma_x=3$)

$\therefore\ \frac{8\times5\times3}{10\times3}=\sigma_y$

$\therefore\ \sigma_y=4$

3. For 50 students of a class, the regression equation of marks in Statistics (X) on the marks in Accountancy (Y) is $3y-5x+180=0$. The mean marks in Accountancy is 44 and the variance of marks in Statistics is $\left(\frac{9}{16}\right)^{\text{th}}$ of the variance of marks in Accountancy. Find the mean marks in Statistics and the correlation coefficient between marks in the two subjects.

Solution :

Let, X = Marks in Statistics, Y = Marks in Accountancy.

Given : Regression equation of X on Y is,

$3y-5x+180=0$

$\therefore\ 5x=3y+180$

$$\therefore\ x = \frac{3}{5}y + 36$$

$$\therefore\ b_{xy} = \frac{3}{5} \qquad \ldots (\because \text{ it is coefficient of } y)$$

$$\bar{y} = 44,\ \sigma_x^2 = \frac{9}{16}\sigma_y^2 \qquad \therefore\ \sigma_x = \frac{3}{4}\sigma_y$$

Mean marks in Statistics ($\bar{x}$) :

$\bar{y} = 44$

$\therefore$ put $y = 44$ in $3y - 5x + 180 = 0$

$$\therefore\ 3(44) - 5x + 180 = 0$$

$$\therefore\ 5x = 180 + 132$$

$$\therefore\ x = \frac{312}{5} = 62.4$$

$\therefore\ \bar{x} = 62.4$ marks.*

Correlation coefficient between X and Y :

We have, $b_{xy} = \dfrac{3}{5}$

$$\text{Now, } b_{xy} = r\cdot\frac{\sigma_x}{\sigma_y} \qquad \therefore\ \frac{3}{5} = r\cdot\frac{\frac{3}{4}\sigma_y}{\sigma_y} \qquad \therefore\ \frac{3}{5} = r\cdot\frac{3}{4}$$

$$\therefore\ r = \frac{3}{5}\times\frac{4}{3} = \frac{4}{5} = 0.8.$$

[***Note :** *Answer given in the textbook is incorrect.*]

4. For a bivariate data, the regression coefficient of Y on X is 0.4 and the regression coefficient of X on Y is 0.9. Find the value of variance of Y if variance of X is 9.

Solution :

Given : $b_{yx} = 0.4,\ b_{xy} = 0.9,\ r = ?,\ \sigma_x^2 = 9,\ \sigma_y^2 = ?$

$$r = \pm\sqrt{b_{yx}\cdot b_{xy}} = \pm\sqrt{0.4\times 0.9} = \pm\sqrt{0.36}$$

$$= 0.6 \qquad \ldots (\because\ b_{yx} \text{ and } b_{xy} \text{ are positive})$$

Variance of Y :

Now, $b_{yx} = 0.4$

$$\therefore\ r\cdot\frac{\sigma_y}{\sigma_x} = 0.4$$

$$\therefore\ 0.6\times\frac{\sigma_y}{3} = 0.4 \qquad \ldots (\because\ \sigma_x^2 = 9 \quad \therefore\ \sigma_x = 3)$$

$$\therefore\ 0.2\sigma_y = 0.4$$

$$\therefore\ \sigma_y = \frac{0.4}{0.2} = 2$$

$$\therefore\ \sigma_y^2 = (2)^2 = 4$$

Hence, the variance of Y is 4.

5. The equations of two regression lines are $2x + 3y - 6 = 0$ and $2x + 2y - 12 = 0$.

Find (i) Correlation coefficient (ii) $\dfrac{\sigma_x}{\sigma_y}$.

Solution :

Let the equation $2x + 3y - 6 = 0$ be the regression equation of Y on X.

$$\therefore\ 3y = -2x + 6$$

$$\therefore\ y = -\frac{2}{3}x + 2$$

$$\therefore\ b_{yx} = -\frac{2}{3}$$

Another equation $2x + 2y - 12 = 0$ be the regression equation of X on Y.

$$\therefore\ 2x = -2y + 12$$

$$\therefore\ x = -\frac{2}{2}y + \frac{12}{2} \qquad \therefore\ x = -y + 6$$

$$\therefore\ b_{xy} = -1$$

(i) Correlation coefficient :

$$r = \pm\sqrt{b_{yx}\cdot b_{xy}} = \pm\sqrt{\left(-\frac{2}{3}\right)\times(-1)}$$

$$= \pm\sqrt{\frac{2}{3}}$$

$$= \pm 0.82$$

$$\therefore\ r = -0.82^{*} \qquad \ldots (\because\ b_{yx} \text{ and } b_{xy} \text{ are negative})$$

(ii) $\dfrac{\sigma_x}{\sigma_y}$:

We know, $b_{xy} = r\cdot\dfrac{\sigma_x}{\sigma_y}$

$$\therefore\ -1 = -0.82\times\frac{\sigma_x}{\sigma_y}$$

$$\therefore\ (-1)\times\left(\frac{1}{-0.82}\right) = \frac{\sigma_x}{\sigma_y}$$

$$\therefore\ \frac{\sigma_x}{\sigma_y} = \frac{1}{0.82}$$

$$\therefore\ \frac{\sigma_x}{\sigma_y} = 1.22.^{*}$$

[***Note :** *Answers given in the textbook are incorrect.*]

6. For a bivariate data : $\bar{x} = 53,\ \bar{y} = 28,\ b_{yx} = -1.5$ and $b_{xy} = -0.2$. Estimate Y when X = 50.

Solution :

Given : $\bar{x} = 53,\ \bar{y} = 28,\ b_{yx} = -1.5,\ b_{xy} = -0.2$

Using regression equation of Y on X, we estimate Y when X = 50.

Now, $y-\bar{y}=b_{yx}(x-\bar{x})$

$\therefore\ y-28=-1.5(x-53)$

$\therefore\ y=-1.5x+79.5+28$

$\therefore\ y=-1.5x+107.5$

Estimate of Y when X = 50 :

Put $x=50$ in $y=-1.5x+107.5$

$\therefore\ y=-1.5\times 50+107.5$

$\therefore\ y=-75+107.5 \qquad \therefore\ y=32.5.$

7. The equations of two regression lines are $x-4y=5$ and $16y-x=64$. Find means of X and Y. Also, find correlation coefficient between X and Y.

Solution :

Given : Two regression lines : $x-4y=5$, $16y-x=64$

Means of X and Y :

$$\begin{array}{r} x-\ 4y=\ 5 \\ -x+16y=64 \\ \hline 12y=69 \end{array}$$

$\therefore\ y=\dfrac{69}{12}$

$=5.75$

Put $y=5.75$ in $x-4y=5$

$\therefore\ x-4(5.75)=5$

$\therefore\ x=5+23$

$\therefore\ x=28$

Hence, $\bar{x}=28$, $\bar{y}=5.75$

Coefficient of correlation :

$x-4y=5$

$\therefore\ x=4y+5$

$\therefore\ b_{xy}=4$

$16y-x=64$

$\therefore\ 16y=x+64$

$\therefore\ y=\dfrac{1}{16}x+4$

$\therefore\ b_{yx}=\dfrac{1}{16}$

Now, $r=\pm\sqrt{b_{yx}\cdot b_{xy}}$

$\therefore\ r=\pm\sqrt{\dfrac{1}{16}\times 4}$

$\therefore\ r=\pm\sqrt{\dfrac{1}{4}}$

$\therefore\ r=\dfrac{1}{2}=0.5 \qquad \ldots(\because\ b_{yx}$ and b_{xy} are positive)

8. In a partially destroyed record, the following data are available variance of X = 25. Regression equation of Y on X is $5y-x=22$ and Regression equation of X on Y is $64x-45y=22$. Find

(i) Mean values of X and Y.

(ii) Coefficient of correlation between X and Y.

(iii) Standard deviation of Y if $\sigma_x=5$.

[*Correction in text.*]

Solution :

Given : Var$(x)=25$

Regression equation of Y on X : $5y-x=22$

Regression equation of X on Y : $64x-45y=22$

(i) Mean values of X and Y :

$5y-x=22\Rightarrow -x+5y=22 \qquad \ldots(1)$

$64x-45y=22 \qquad \ldots(2)$

From equation (1) $x=5y-22$

Putting $x=5y-22$ in equation (2), we get

$64(5y-22)-45y=22$

$\therefore\ 320y-1480-45y=22$

$\therefore\ 275y==22+1408$

$\therefore\ 275y=1430$

$\therefore\ y=\dfrac{1430}{275} \qquad \therefore\ y=5.2$

$\therefore\ \bar{y}=5.2$

Putting $y=5.2$ in $x=5y-22$, we get

$x=5(5.2)-22$

$\therefore\ x=26-22$

$\therefore\ x=4 \qquad \therefore\ \bar{x}=4$

Hence, $\bar{x}=4$. $\bar{y}=5.2$

(ii) Regression equation of Y on X is $5y-x=22$

$\therefore\ 5y=x+22$

$\therefore\ y=\dfrac{x}{5}+\dfrac{22}{5}$

$\therefore\ b_{yx}=$ coefficient of $x=\dfrac{1}{5}$

Regression equation of X on Y is $64x-45y=22$

$\therefore\ 64x=45y+22$

$\therefore\ x=\dfrac{45y}{64}+\dfrac{22}{64}$

$\therefore\ b_{xy}=$ coefficient of $Y=\dfrac{45}{64}$

Now, $r=\pm\sqrt{b_{yx}\cdot b_{xy}}$

$$= \pm\sqrt{\frac{1}{5}\times\frac{45}{64}} = \pm\sqrt{\frac{9}{64}}$$

$$= \pm\frac{3}{8}$$

$$\therefore\ r = \frac{3}{8} \qquad \ldots (\because b_{yx} \text{ and } b_{xy} \text{ are positive})$$

(iii) $b_{yx} = \frac{1}{5},\ r = \frac{3}{8},\ \sigma_x = 5$

Now, $b_{yx} = r\cdot\frac{\sigma_y}{\sigma_x}$

$$\therefore\ \frac{1}{5} = \frac{3}{8}\cdot\frac{\sigma_y}{\sigma_x}$$

$$\therefore\ \frac{1}{5}\times\frac{8}{3} = \frac{\sigma_y}{\sigma_x}$$

$$\therefore\ \frac{1}{5}\times\frac{8}{3} = \frac{\sigma_y}{5}$$

$$\therefore\ \sigma_y = \frac{5\times 8}{5\times 3} = \frac{8}{3}$$

Hence, standard deviation of Y is $\frac{8}{3}$.

9. If the two regression lines for a bivariate data are $2x = y + 15$ (x on y) and $4y = 3x + 25$ (y on x), find

(i) $\bar{x}$, (ii) $\bar{y}$, (iii) b_{yx}, (iv) b_{xy},

(v) r [Given : $\sqrt{0.375} = 0.61$]

Solution :

Given : Regression equation of X on Y : $2x = y + 15$

Regression equation of Y on X : $4y = 3x + 25$

(i) and (ii) :

$2x = y + 15$

$\therefore\ 2x - y = 15$ … (1)

$4y = 3x + 25$

$\therefore\ -3x + 4y = 25$ … (2)

From equation (1), $y = 2x - 15$

Putting $y = 2x - 15$ in equation (2), we get

$-3x + 4(2x - 15) = 25$

$\therefore\ -3x + 8x - 60 = 25$

$\therefore\ 5x = 25 + 60$

$\therefore\ x = \frac{85}{5} = 17$

$\therefore\ \bar{x} = 17$

Putting $x = 17$ in $y = 2x - 15$, we get

$y = 2(17) - 15$

$= 34 - 15 = 19$

$\therefore\ \bar{y} = 19$

Hence, (i) $\bar{x} = 17$, (ii) $\bar{y} = 19$.

(iii) Regression equation of Y on X is $4y = 3x + 25$

$\therefore\ y = \frac{3}{4}x + \frac{25}{4}$

$\therefore\ b_{yx}$ = coefficient of x

$\therefore\ b_{yx} = \frac{3}{4}$

(iv) Regression equation of X on Y is $2x = y + 15$

$\therefore\ x = \frac{1}{2}y + \frac{15}{2}$

$\therefore\ b_{xy}$ = coefficient of y

$\therefore\ b_{xy} = \frac{1}{2}$

(v) $r = \pm\sqrt{b_{yx}\cdot b_{xy}}$

Putting, $b_{yx} = \frac{3}{4}$ and $b_{xy} = \frac{1}{2}$

$r = \pm\sqrt{\frac{3}{4}\times\frac{1}{2}}$

$\therefore\ r = \pm\sqrt{\frac{3}{8}}$

$\therefore\ r = \pm\sqrt{0.375}$

$\therefore\ r = 0.61$ … ($\because b_{yx}$ and b_{xy} are positive)

10. The two regression equations are $5x - 6y + 90 = 0$ and $15x - 8y - 130 = 0$.

Find $\bar{x}$, $\bar{y}$, r.

Solution :

Given : Two regression equations :

$5x - 6y + 90 = 0$ … (1)

$15x - 8y - 130 = 0$ … (2)

Multiply equation (1) by 3 and subtracting equation (2) from it, we get

$$\begin{array}{rl} 15x - 18y + 270 &= 0 \\ 15x - 8y - 130 &= 0 \\ - \quad + \quad + & \\ \hline -10y + 400 &= 0 \end{array}$$

$\therefore\ 400 = 10y$

$\therefore\ y = \frac{400}{10} = 40 \qquad \therefore\ \bar{y} = 40$

Putting $y = 40$ in equation (1), we get

$5x - 6(40) + 90 = 0$

$\therefore\ 5x - 240 + 90 = 0$

$\therefore\ 5x - 150 = 0$

$\therefore\ 5x = 150$

$\therefore\ x = \dfrac{150}{5} = 30 \qquad \therefore\ \bar{x} = 30$

Let the equation of Y on X be $5x - 6y + 90 = 0$

$\therefore\ 6y = 5x + 90$

$\therefore\ y = \dfrac{5}{6}x + 15$

$\therefore\ b_{yx} = \dfrac{5}{6}$

and the equation of regression of X on Y be

$15x - 8y - 130 = 0$

$\therefore\ 15x = 8y + 130$

$\therefore\ x = \dfrac{8}{15}y + \dfrac{130}{15}$

$\therefore\ b_{xy} = \dfrac{8}{15}$

Now, $r = \pm\sqrt{b_{yx} \cdot b_{xy}}$

$= \pm\sqrt{\dfrac{5}{6} \times \dfrac{8}{15}}$

$= \pm\sqrt{\dfrac{8}{18}} = \pm\sqrt{\dfrac{4}{9}}$

$= \pm\dfrac{2}{3}$

$\therefore\ r = \dfrac{2}{3}$ ... ($\because$ b_{yx} and b_{xy} are positive)

---

11. Two lines of regression are $10x + 3y - 62 = 0$ and $6x + 5y - 50 = 0$. Identify the regression of x on y. Hence find $\bar{x}$, $\bar{y}$ and r.

Solution :

Given : Two lines of regression

$10x + 3y - 62 = 0$; $6x + 5y - 50 = 0$

Identify the line of regression of X on Y :

Let the regression equation of X on Y be

$10x + 3y - 62 = 0$

$\therefore\ 10x = -3y + 62$

$\therefore\ x = -\dfrac{3}{10}y + 6.2$

$\therefore\ b_{xy} = -\dfrac{3}{10}$

and the regression equation of Y on X be

$6x + 5y - 50 = 0$

$\therefore\ 5y = -6x + 50$

$\therefore\ y = -\dfrac{6}{5}x + 10$

$\therefore\ b_{yx} = -\dfrac{6}{5}$

Now, $b_{xy} \cdot b_{yx} = \left(-\dfrac{3}{10}\right) \times \left(-\dfrac{6}{5}\right) = \dfrac{18}{50}$ which is less than 1.

Hence, assumption regarding regression equation holds true.

$\therefore$ regression equation of X on Y is $10x + 3y - 62 = 0$.

To find $\bar{x}$, $\bar{y}$:

$10x + 3y - 62 = 0$ ... (1)

$6x + 5y - 50 = 0$ ... (2)

Multiply equation (1) by 5 and equation (2) by 3.

Then subtracting equation (2) from equation (1), we get

$$\begin{array}{rrr} 50x & +\,15y & -\,310 = 0 \\ 18x & +\,15y & -\,150 = 0 \\ - & - & + \\ \hline 32x & & -\,160 = 0 \end{array}$$

$\therefore\ 32x = 160$

$\therefore\ x = \dfrac{160}{32} = 5$

$\therefore\ \bar{x} = 5$

Putting $x = 5$ in equation (1), we get

$10 \times 5 + 3y - 62 = 0$

$\therefore\ 50 + 3y - 62 = 0$

$\therefore\ 3y - 12 = 0$

$\therefore\ 3y = 12$

$\therefore\ y = \dfrac{12}{3} = 4$

$\therefore\ \bar{y} = 4$

Hence, $\bar{x} = 5$, $\bar{y} = 4$

To find r :

$b_{xy} = -\dfrac{3}{10}$, $b_{yx} = -\dfrac{6}{5}$

Now, $r = \pm\sqrt{b_{yx} \cdot b_{xy}}$

$= \pm\sqrt{\left(-\dfrac{3}{10}\right)\left(-\dfrac{6}{5}\right)}$

$= \pm\sqrt{\frac{18}{50}} = \pm\sqrt{0.36}$

$\therefore\ r = -0.6$ ... ($\because$ b_{yx} and b_{xy} are negative)

12. For certain X and Y series, which are correlated the two lines of regression are $10y = 3x + 170$ and $5x + 70 = 6y$. Find the correlation coefficient between them. Find the mean values of X and Y.

Solution :

Given : Two lines of regression

$10y = 3x + 170$; $5x + 70 = 6y$

Correlation coefficient :

Let the regression equation of Y on X be

$10y = 3x + 170$

$\therefore\ y = \frac{3}{10}x + 17$

$\therefore\ b_{yx} = \frac{3}{10}$

and the regression equation of X on Y be

$5x + 70 = 6y$

$\therefore\ 5x = 6y - 70$

$\therefore\ x = \frac{6}{5}y - 14$

$\therefore\ b_{xy} = \frac{6}{5}$

Now, $r = \pm\sqrt{b_{yx} \cdot b_{xy}}$

$= \pm\sqrt{\frac{3}{10} \times \frac{6}{5}}$

$= \pm\sqrt{\frac{18}{50}}$

$= \pm\sqrt{0.36}$

$\therefore\ r = 0.6$ ... ($\because$ b_{yx} and b_{xy} are positive)

Mean values of X and Y :

$10y = 3x + 170$

$\therefore\ 3x - 10y + 170 = 0$ ... (1)

$5x + 70 = 6y$

$\therefore\ 5x - 6y + 70 = 0$ ... (2)

Multiply equation (1) by 5 and multiply equation (2) by 3. Then subtracting equation (2) from equation (1) we get

$$\begin{array}{l} 15x - 50y + 850 = 0 \\ 15x - 18y + 210 = 0 \\ -\quad +\quad - \\ \hline -32y + 640 = 0 \end{array}$$

$\therefore\ 640 = 32y$

$\therefore\ y = \frac{640}{32} = 20$

$\therefore\ \bar{y} = 20$

Putting $y = 20$ in equation (1) we get

$3x - 10(20) + 170 = 0$

$\therefore\ 3x - 200 + 170 = 0$

$\therefore\ 3x - 30 = 0$ $\quad\therefore\ 3x = 30$

$\therefore\ x = \frac{30}{3} = 10$ $\quad\therefore\ \bar{x} = 10$

Hence, $\bar{x} = 10$, $\bar{y} = 20$.

13. Regression equations of two series are $2x - y - 15 = 0$ and $3x - 4y + 25 = 0$. Find $\bar{x}$, $\bar{y}$ and regression coefficients. Also, find coefficients of correlation. [Given : $\sqrt{0.375} = 0.61$]

Solution :

Given : Regression equations $2x - y - 15 = 0$ and $3x - 4y + 25 = 0$

To find $\bar{x}$, $\bar{y}$:

$2x - y - 15 = 0$ ... (1)

$3x - 4y + 25 = 0$ ... (2)

From equation (1) $y = 2x - 15$

Putting $y = 2x - 15$ in the equation (2), we get

$3x - 4(2x - 15) + 25 = 0$

$\therefore\ 3x - 8x + 60 + 25 = 0$

$\therefore\ -5x + 85 = 0$

$\therefore\ 85 = 5x$

$\therefore\ x = \frac{85}{5} = 17$

$\therefore\ \bar{x} = 17$

Now, putting $x = 17$ in $y = 2x - 15$, we get

$y = 2(17) - 15$

$= 34 - 15 = 19$ $\quad\therefore\ \bar{y} = 19$

Hence, $\bar{x} = 17$, $\bar{y} = 19$.

Regression coefficients :

Let the regression equation of X on Y be

$2x - y - 15 = 0$

$\therefore\ 2x = y + 15$

$\therefore\ x = \frac{1}{2}y + \frac{15}{2}$

$\therefore\ b_{xy}=\frac{1}{2}$

and the regression equation of Y on X be

$3x-4y+25=0$

$\therefore\ 4y=3x+25$

$\therefore\ y=\frac{3}{4}x+\frac{25}{4}$

$\therefore\ b_{yx}=\frac{3}{4}$

Now, $b_{yx}\cdot b_{xy}=\frac{3}{4}\times\frac{1}{2}=\frac{3}{8}$, which is less than 1. Hence the assumption regarding regression equations holds true.

$\therefore\ b_{xy}=\frac{1}{2}$ and $b_{yx}=\frac{3}{4}$.

Coefficient correlation :

$b_{xy}=\frac{1}{2},\ b_{yx}=\frac{3}{4}$

Now, $r=\pm\sqrt{b_{yx}\cdot b_{xy}}$

$=\pm\sqrt{\frac{3}{4}\times\frac{1}{2}}=\pm\sqrt{\frac{3}{8}}$

$=\pm 0.375$

$\therefore\ r=0.61$ ... ($\because$ b_{yx} and b_{xy} are positive)

14. The two regression lines between height (X) in inches and weight (Y) in kgs of girls are
$4y-15x+500=0$ and $20x-3y-900=0$.
Find mean height and weight of the group. Also, estimate weight of a girl whose height is 70 inches.

Solution :

Given : $4y-15x+500=0,\quad 20x-3y-900=0$

Mean height ($\bar{x}$) and mean weight ($\bar{y}$) :

$-15x+4y+500=0$ ... (1)

$20x-3y-900=0$ ... (2)

Multiplying equation (1) by 3 and equation (2) by 4 and then adding them,

$$\begin{array}{r} -45x+12y+1500=0 \\ 80x-12y-3600=0 \\ \hline \therefore\ 35x\qquad -2100=0 \end{array}$$

$\therefore\ 35x=2100$

$\therefore\ \bar{x}=\frac{2100}{35}=60$

Put $x=60$ in equation (1), we get

$4y-15(60)+500=0$

$\therefore\ 4y-900+500=0$

$\therefore\ 4y=400$

$\therefore\ \bar{y}=\frac{400}{4}=100$

Hence, mean height $\bar{x}=60$ inches and mean weight $\bar{y}=100$ kg.

Estimate of weight (y) of a girl when $X=70$:

We use the regression equation of Y on X.

From the given equations let $4y-15x+500=0$ be the regression equation of Y on X.

$\therefore\ 4y=15x-500$

$\therefore\ y=\frac{15}{4}x-\frac{500}{4}$

$\therefore\ b_{yx}=\frac{15}{4}$

Now, regression equation of X on Y is

$20x-3y-900=0$

$\therefore\ 20x=3y+900$

$\therefore\ x=\frac{3}{20}y+45$

$\therefore\ b_{xy}=\frac{3}{20}$

Now, $b_{yx}\cdot b_{xy}=\frac{15}{4}\times\frac{3}{20}=\frac{45}{80}$ which is less than 1. Hence, assumption regarding regression equations is true.

$\therefore$ regression equation of Y on X is

$y=\frac{15}{4}x-\frac{500}{4}$

Put $x=70$ in the equation

$\therefore\ y=\frac{15}{4}(70)-\frac{500}{4}$

$\therefore\ y=\frac{1050-500}{4}=\frac{550}{4}=137.5$

Hence, the estimated weight of a girl is 137.5 kg.

EXAMPLES FOR PRACTICE 3.3

1. Given the two regression lines of a bivariate data. Find both the coefficients of regression, coefficient correlation.

   (a) $2x-y+10=0$ and $3x-4y+100=0$

   (b) $2x-3y=4$ and $3x-5y=18$

   (c) $x+y+5=0$ and $16x+9y+1=0$.

2. If for a bivariate data $b_{yx} = 1.2$, $\sigma_x = 4$, $r = 0.6$, find σ_y, b_{xy} and the measure of the acute angle between the two regression lines.

3. If $\Sigma x = 40$, $\Sigma y = 32$, $\Sigma (x - \bar{x})^2 = 30$, $\Sigma (y - \bar{y})^2 = 36$, $\Sigma (x - \bar{x})(y - \bar{y}) = 6$, find the regression coefficients and the measure of the acute angle between the regression line.

4. Find both the regression coefficients and the measure of the acute angle between the regression lines from the following data :
$n = 10$, $\Sigma x = 250$, $\Sigma y = 300$, $\Sigma x^2 = 6500$, $\Sigma y^2 = 10{,}000$ and $\Sigma xy = 7900$.

5. Given two regression equations $4x + 5y - 155 = 0$ and $16x + 15y - 545 = 0$ find (i) mean values of X and Y (ii) r_{xy} and (iii) σ_y if $\sigma_x = 3$.

6. The equations of two regression lines are
$10x + 3y - 62 = 0$ and $6x + 5y - 50 = 0$.
Find (a) b_{yx} (b) b_{xy} (c) σ_x if $\sigma_y = 2$.

7. The equations of lines of regression are
$5x - 6y + 90 = 0$ and $15x - 8y - 130 = 0$.
Find (i) Coefficient of correlation r (ii) $\bar{x}$ and $\bar{y}$.

8. If $5x + y = 21$ and $4x + 5y = 42$ are the two regression equations, find
$\bar{x}$, $\bar{y}$ and r (the coefficient of correlation).

9. The two regression equations are $10x + 3y - 62 = 0$ and $6x + 5y - 50 = 0$. Identify the regression of X on Y. Hence, find $\bar{x}$, $\bar{y}$ and r. Also find σ_x if $\sigma_y = 2$.

10. Find the values of $\bar{x}$, $\bar{y}$, $\sigma_x : \sigma_y$ and r if the equations of regression lines are $3x - 2y - 10 = 0$ and $24x - 25y + 145 = 0$.

11. The regression equations are $3x - y - 5 = 0$ and $4x - 3y = 0$, find
(i) Arithmetic mean of X and Y.
(ii) Coefficient of variation of X and Y if $\sigma_x = 2$.
(iii) Correlation coefficient between X and Y.

12. Values of two regression coefficients between the variables X and Y are $b_{yx} = -0.4$ and $b_{xy} = -2.025$ respectively. Obtain the value of correlatoin coefficient.

13. For a bivariate data $b_{yx} = -1.2$ and $b_{xy} = -0.3$, find the correlation coefficient between x and y.

14. From two regression equations $y = 4x - 5$ and $3x = 2y + 5$, find $\bar{x}$ and $\bar{y}$.

15. The regression equation of Y on X is $y = \frac{2}{9}x$ and the regression equation of X on Y is $x = \frac{y}{2} + \frac{7}{6}$.
Find : (i) Correlation coefficient between X and Y.
(ii) σ_y^2 if $\sigma_x^2 = 4$.

16. Identify the regression equations of X on Y and Y on X from the following equations :
$2x + 3y = 6$ and $5x + 7y - 12 = 0$.

17. The regression equation of Y on X is given by $3x + 2y - 26 = 0$. Find b_{yx}.

Answers

1. (a) $b_{yx} = \frac{3}{4}$, $b_{xy} = \frac{1}{2}$, $r = 0.612$

(b) $b_{yx} = \frac{3}{5}$, $b_{xy} = \frac{3}{2}$, $r = 0.9487$

(c) $b_{yx} = -1$, $b_{xy} = -\frac{9}{16}$, $r = -0.75$

2. $\sigma_y = 8$, $b_{xy} = 0.3$

3. $b_{yx} = \frac{1}{5}$, $b_{xy} = \frac{1}{6}$

4. $b_{yx} = \frac{8}{5}$, $b_{xy} = \frac{2}{5}$

5. (i) $\bar{x} = 20$, $\bar{y} = 15$ (ii) $r_{xy} = -0.866$ (iii) $\sigma_y = 2.771$

6. (a) $b_{yx} = -1.2$, (b) $b_{xy} = -0.3$, (c) $\sigma_x = 1$

7. (i) $r = \frac{2}{3}$ (ii) $\bar{x} = 30$, $\bar{y} = 40$

8. $\bar{x} = 3$, $\bar{y} = 6$, $r = -0.4$

9. $\bar{x} = 5$, $\bar{y} = 4$, $r = -0.6$, $\sigma_x = 1$

10. $\bar{x} = 20$, $\bar{y} = 25$, $\sigma_x : \sigma_y = 5 : 6$, $r = 0.8$

11. (i) $\bar{x} = 3$, $\bar{y} = 4$ (ii) C.V. of $x = 66.67\%$
C.V. of $y = 100\%$ (iii) $r = 0.6667$

12. -0.9 **13.** -0.6 **14.** $\bar{x} = 1$, $\bar{y} = 1$

15. (i) $\frac{1}{3} = 0.33$ (ii) $\frac{16}{9}$

16. X on Y : $5x + 7y - 12 = 0$; Y on X : $2x + 3y = 6$

17. $-\frac{3}{2}$.

MISCELLANEOUS EXERCISE – 3

(Textbook pages 51 to 54)

I. Choose the correct alternative :

1. Regression analysis is the theory of

(a) Estimation (b) Prediction

(c) Both (a) and (b) (d) Calculation

2. We can estimate the value of one variable with the help of other known variable only if they are

(a) Correlated (b) Positively correlated

(c) Negatively correlated (d) Uncorrelated.

3. There are ……… types of regression equations.

(a) 4 (b) 2 (c) 3 (d) 1.

4. In the regression equation of Y on X

(a) X is independent and Y is dependent

(b) Y is independent and X is dependent

(c) Both X and Y are independent

(d) Both X and Y are dependent.

5. In the regression equation of X on Y

(a) X is independent and Y is dependent

(b) Y is independent and X is dependent

(c) Both X and Y are independent

(d) Both X and Y are dependent.

6. b_{xy} is ………

(a) Regression coefficient of Y on X

(b) Regression coefficient of X on Y

(c) Correlation coefficient between X and Y

(d) Covariance between X and Y.

7. b_{yx} is ………

(a) Regression coefficient of Y on X

(b) Regression coefficient of X on Y

(c) Correlation coefficient between X and Y

(d) Covariance between X and Y.

8. 'r' is ………

(a) Regression coefficient of Y on X

(b) Regression coefficient of X on Y

(c) Correlation coefficient between X and Y

(d) Covariance between X and Y.

9. $b_{xy} \cdot b_{yx}$ ………

(a) $v(x)$ (b) σ_x (c) r^2 (d) $(\sigma_y)^2$.

10. If $b_{yx} > 1$, then b_{xy} is ………

(a) >1 (b) <1

(c) >0 (d) <0.

11. $|b_{xy} + b_{yx}| \geqslant$ ………

(a) $|r|$ (b) $2|r|$

(c) r (d) $2r$.

12. b_{xy} and b_{yx} are ………

(a) Independent of change of origin and scale

(b) Independent of change of origin but not of scale

(c) Independent of change of scale but not of origin

(d) Affected by change of origin and scale.

13. If $u = \frac{x-a}{c}$ and $v = \frac{y-b}{d}$ then $b_{yx} =$ ………

(a) $\frac{d}{c} b_{vu}$ (b) $\frac{c}{d} b_{vu}$

(c) $\frac{a}{b} b_{vu}$ (d) $\frac{b}{a} b_{vu}$.

14. If $u = \frac{x-a}{c}$ and $v = \frac{y-b}{d}$ then $b_{xy} =$ ………

(a) $\frac{d}{c} b_{uv}$ (b) $\frac{c}{d} b_{uv}$

(c) $\frac{a}{b} b_{uv}$ (d) $\frac{b}{a} b_{uv}$.

15. Corr $(x, x) =$ ………

(a) 0 (b) 1 (c) -1 (d) can't be found.

16. Corr $(x, y) =$ ………

(a) corr (x, x) (b) corr (y, y)

(c) corr (y, x) (d) cov (y, x).

17. Corr $\left(\frac{x-a}{c}, \frac{y-b}{d}\right) = -$ corr (x, y) if,

(a) c and d are opposite in sign

(b) c and d are same in sign

(c) a and b are opposite in sign

(d) a and b are same in sign.

18. Regression equation of X on Y is ………

(a) $y - \bar{y} = b_{yx}(x - \bar{x})$ (b) $x - \bar{x} = b_{xy}(y - \bar{y})$

(c) $y - \bar{y} = b_{xy}(x - \bar{x})$ (d) $x - \bar{x} = b_{yx}(y - \bar{y})$.

19. Regression equation of Y on X is ………

(a) $y - \bar{y} = b_{yx}(x - \bar{x})$ (b) $x - \bar{x} = b_{xy}(y - \bar{y})$

(c) $y - \bar{y} = b_{xy}(x - \bar{x})$ (d) $x - \bar{x} = b_{yx}(y - \bar{y})$.

20. $b_{yx} = \dots\dots\dots$

(a) $r\frac{\sigma_x}{\sigma_y}$ (b) $r\frac{\sigma_y}{\sigma_x}$ (c) $\frac{1}{r}\frac{\sigma_y}{\sigma_x}$ (d) $\frac{1}{r}\frac{\sigma_x}{\sigma_y}$.

21. $b_{xy} = \dots\dots\dots$

(a) $r\frac{\sigma_x}{\sigma_y}$ (b) $r\frac{\sigma_y}{\sigma_x}$ (c) $\frac{1}{r}\frac{\sigma_y}{\sigma_x}$ (d) $\frac{1}{r}\frac{\sigma_x}{\sigma_y}$.

22. $\text{Cov}(x, y) = \dots\dots\dots$

(a) $\Sigma(x-\bar{x})(y-\bar{y})$ (b) $\frac{\Sigma(x-\bar{x})(y-\bar{y})}{n}$

(c) $\frac{\Sigma xy}{n} - \bar{x}\bar{y}$ (d) (b) and (c) both.

23. If $b_{xy} < 0$ and $b_{yx} < 0$ then 'r' is ....

(a) >0 (b) <0 (c) >1 (d) not found.

24. If equations of regression lines are $3x+2y-26=0$ and $6x+y-31=0$ then means of x and y are ....

(a) (7, 4) (b) (4, 7) (c) (2, 9) (d) (−4, 7).

Answers

1. (c) Both (a) and (b) 2. (a) Correlated 3. (b) 2
4. (a) X is independent and Y is dependent
5. (b) Y is independent and X is dependent
6. (b) Regression coefficient of X on Y
7. (a) Regression coefficient of Y on X
8. (c) Correlation coefficient between X and Y
9. (c) r^2 10. (b) <1 11. (b) $2|r|$
12. (b) Independent of change of origin but not of scale
13. (a) $\frac{d}{c}b_{vu}$ 14. (b) $\frac{c}{d}b_{uv}$ 15. (b) 1
16. (c) corr (y, x) 17. (a) c and d are opposite in sign
18. (b) $x-\bar{x}=b_{xy}(y-\bar{y})$ 19. (a) $y-\bar{y}=b_{yx}(x-\bar{x})$
20. (b) $r\frac{\sigma_y}{\sigma_x}$ 21. (a) $r\frac{\sigma_x}{\sigma_y}$ 22. (d) (b) and (c) both
23. (b) <0 24. (b) (4, 7).

II. Fill in the blanks :

1. If $b_{xy} < 0$ and $b_{yx} < 0$ then 'r' is ....
2. Regression equation of Y on X is ....
3. Regression equation of X on Y is ....
4. There are .... types of regression equations.
5. Corr $(x, -x) = \dots\dots\dots\dots$
6. If $u = \frac{x-a}{c}$ and $v = \frac{y-b}{d}$ then $b_{xy} = \dots\dots\dots$
7. If $u = \frac{x-a}{c}$ and $v = \frac{y-b}{d}$ then $b_{yx} = \dots\dots\dots$
8. $|b_{xy} + b_{yx}| \geqslant \dots\dots\dots$
9. If $b_{yx} > 1$ then b_{xy} is ....
10. $b_{xy} \cdot b_{yx} = \dots\dots\dots$

Answers

1. Negative 2. $(y-\bar{y})=b_{yx}(x-\bar{x})$ 3. $(x-\bar{x})=b_{xy}(y-\bar{y})$
4. 2 5. -1 6. $\frac{c}{d}b_{uv}$ 7. $\frac{d}{c}b_{uv}$ 8. $2|r|$
9. <1 10. r^2.

III. State whether each of the following is *True* or *False* :

1. Corr $(x, x) = 1$.
2. Regression equation of X on Y is $y-\bar{y}=b_{yx}(x-\bar{x})$.
3. Regression equation of Y on X is $y-\bar{y}=b_{yx}(x-\bar{x})$.
4. Corr $(x, y) =$ Corr (y, x).
5. b_{xy} and b_{yx} are independent of change of origin and scale.
6. 'r' is regression coefficient of Y on X.
7. b_{yx} is correlation coefficient between X and Y.
8. If $u = x - a$ and $v = y - b$ then $b_{xy} = b_{uv}$.
9. If $u = x - a$ and $v = y - b$ then $r_{xy} = r_{uv}$.
10. In the regression equation of Y on X, b_{yx} represents slope of the line.

Answers

1. True	2. False	3. True	4. True	5. False
6. False	7. False	8. True	9. True	10. True.

IV. Solve the following problems :

1. The data obtained on X, the length of time in weeks that a promotional project has been in progress at a small business, and Y, the percentage increase in weekly sales over the period just prior to the beginning of the campaign.

X	1	2	3	4	1	3	1	2	3	4	2	4
Y	10	10	18	20	11	15	12	15	17	19	13	16

Find the equation of regression line to predict the percentage increase in sales if the campaign has been in progress for 1.5 weeks.

Solution :

X = Length of time, Y = % increase in weekly sales.

	x	y	xy	x^2
	1	10	10	1
	2	10	20	4
	3	18	54	9
	4	20	80	16
	1	11	11	1
	3	15	45	9
	1	12	12	1
	2	15	30	4
	3	17	51	9
	4	19	76	16
	2	13	26	4
	4	16	64	16
$n = 12$	$\Sigma x = 30$	$\Sigma y = 176$	$\Sigma xy = 479$	$\Sigma x^2 = 90$

$$\bar{x} = \frac{\Sigma x}{n} = \frac{30}{12} = 2.5, \quad \bar{y} = \frac{\Sigma y}{n} = \frac{176}{12} = 14.67$$

Regression equation of percentage increase in weekly sales (Y) on the length of time (X) :

$y = a + b_{yx} \cdot x$

$$\text{Now, } b_{xy} = \frac{\Sigma xy - n\bar{x}\bar{y}}{\Sigma x^2 - n\bar{x}^2}$$

$$= \frac{479 - 12(2.5 \times 14.67)}{90 - 12(2.5)^2}$$

$$= \frac{479 - 440.1}{90 - 75} = \frac{38.9}{15} = 2.59^*$$

$\therefore \; b_{yx} = 2.59$

$a = \bar{y} - b_{yx} \cdot \bar{x}$

$= 14.67 - 2.59(2.5)$

$= 14.67 - 6.475$

$= 8.19 \approx 8.2$

$\therefore \; a = 8.2$

Hence, the regression equation of Y on X is

$y = 8.2 + 2.59x$

$\Rightarrow y = 2.59x + 8.2$

Prediction of Y when X = 1.5 :

Putting $x = 1.5$ in $y = 8.2 + 2.59x$

$\therefore \; y = 8.2 + 2.59 \times 1.5$

$\therefore \; y = 8.2 + 3.885 \qquad \therefore \; y = 12.085^*.$

Hence, when campaign has been progressed for 1.5 weeks, the percentage increase in weekly sales is 12.085%.

[* **Note :** *Answers given in the textbook are incorrect.*]

2. The regression equation of y on x is given by $3x + 2y - 26 = 0$. Find b_{yx}.

Solution :

Given : Regression equation of Y on X

$3x + 2y - 26 = 0$

$\therefore \; 2y = -3x + 26$

$$\therefore \; y = -\frac{3}{2}x + 13$$

$$\therefore \; b_{yx} = \text{coefficient of } x = -\frac{3}{2}.$$

3. If for a bivariate data $\bar{x} = 10$, $\bar{y} = 12$, $v(x) = 9$, $\sigma_y = 4$ and $r = 0.6$. Estimate y when $x = 5$.

Solution :

Given : $\bar{x} = 10$, $\bar{y} = 12$, $v(x) = 9 \quad \therefore \; \sigma_x = 3,$

$\sigma_y = 4$, $r = 0.6$

Estimation of Y when X = 5 :

Regression of Y on X is $y = a + b_{yx} \cdot x$

$$\text{Now, } b_{yx} = r \cdot \frac{\sigma_y}{\sigma_x}$$

$$= 0.6 \times \frac{4}{3} = 0.8$$

$\therefore \; b_{yx} = 0.8$

$a = \bar{y} - b_{yx} \cdot \bar{x}$

$= 12 - 0.8(10) = 12 - 8 = 4$

$\therefore \; a = 4$

Hence, the regression equation of Y on X is

$y = 4 + 0.8x$

$\Rightarrow y = 0.8x + 4$

Putting $x = 5$, we get

$y = 0.8(5) + 4$

$\therefore \; y = 4 + 4 \qquad \therefore \; y = 8$

Hence, the estimate of y is 8 when $x = 5$.

4. The equation of the line regression of y on x is $y = \frac{2}{9}x$ and x on y is $x = \frac{y}{2} + \frac{7}{6}$. Find (i) r (ii) $\sigma_y^{\,2}$ if $\sigma_x^{\,2} = 4$.

Solution :

Given : Line of regression of Y on X is $y = \frac{2}{9}x$ and that of X on Y is $x = \frac{y}{2} + \frac{7}{6}$

(i) To find r :

Regression equation of Y on X is $y = \frac{2}{9}x$

$\therefore\ b_{yx} = \frac{2}{9}$

Regression equation of X on Y is

$x = \frac{y}{2} + \frac{7}{6}.\qquad \therefore\ b_{xy} = \frac{1}{2}$

Now, $r = \pm\sqrt{b_{yx} \cdot b_{xy}}$

$= \pm\sqrt{\frac{2}{9} \times \frac{1}{2}} = \pm\sqrt{\frac{1}{9}}$

$\therefore\ r = \frac{1}{3}$ ... ($\because$ b_{yx} and b_{xy} are positive)

(ii) To find $\sigma_y{}^2$:

$\sigma_x{}^2 = 4$

Now, $b_{yx} = r \cdot \frac{\sigma_y}{\sigma_x}$

$\therefore\ (b_{yx})^2 = r^2 \cdot \frac{\sigma_y{}^2}{\sigma_x{}^2}$

$\therefore\ \left(\frac{2}{9}\right)^2 = \left(\frac{1}{3}\right)^2 \times \frac{\sigma_y{}^2}{4}$

$\therefore\ \frac{4}{81} = \frac{1}{9} \times \frac{\sigma_y{}^2}{4}$

$\therefore\ \frac{4}{81} \times 36 = \sigma_y{}^2$

$\therefore\ \sigma_y{}^2 = \frac{16}{9}$

Hence, $\sigma_y{}^2 = \frac{16}{9}$.

5. Identify the regression equations of x on y and y on x from the following equations,
$2x + 3y = 6$ and $5x + 7y - 12 = 0$.

Solution :

Given : $2x + 3y = 6$; $5x + 7y - 12 = 0$

Let the regression equation of Y on X be

$2x + 3y = 6$

$\therefore\ 3y = 6 - 2x$

$\therefore\ y = \frac{6}{3} - \frac{2}{3}x$

$\therefore\ b_{yx} = -\frac{2}{3}$

and the regression equation of X on Y be

$5x + 7y - 12 = 0$

$\therefore\ 5x = 12 - 7y$

$\therefore\ x = \frac{12}{5} - \frac{7}{5}y$

$\therefore\ b_{xy} = -\frac{7}{5}$

Now, $b_{yx} \cdot b_{xy} = \left(-\frac{2}{3}\right) \times \left(-\frac{7}{5}\right) = \frac{14}{15}$ which is less than 1.

Therefore assumption made regarding the regression equations holds true.

Hence, the regression equation of X on Y is $5x + 7y - 12 = 0$ and the regression equation of Y on X is $2x + 3y = 6$.

6. (i) If for a bivariate data $b_{yx} = -1.2$ and $b_{xy} = -0.3$ then find r.

(ii) From the two regression equations $y = 4x - 5$ and $3x = 2y + 5$, find $\bar{x}$ and $\bar{y}$.

Solution :

(i) Given : $b_{yx} = -1.2$, $b_{xy} = -0.3$

Now, $r = \pm\sqrt{b_{yx} \cdot b_{xy}}$

$= \pm\sqrt{(-1.2) \cdot (-0.3)} = \pm\sqrt{0.36}$

$\therefore\ r = -0.6$ ... ($\because$ b_{yx} and b_{xy} are negative)

Hence, $r = -0.6$

(ii) Given : $y = 4x - 5$ ... (1)

$3x = 2y + 5$ ... (2)

Putting $y = 4x - 5$ in equation (2), we get

$3x = 2(4x - 5) + 5$

$\therefore\ 3x = 8x - 10 + 5$

$\therefore\ 3x - 8x = -5$

$\therefore\ -5x = -5$

$\therefore\ x = 1\qquad \therefore\ \bar{x} = 1$

Putting $x = 1$ in equation (1), we get

$y = 4(1) - 5$

$\therefore\ y = 4 - 5$

$\therefore\ y = -1\qquad \therefore\ \bar{y} = -1$

Hence, $\bar{x} = 1$, $\bar{y} = -1$.

7. The equations of the two lines of regression are $3x + 2y - 26 = 0$ and $6x + y - 31 = 0$. Find :

(i) Means of X and Y.

(ii) Correlation coefficient between X and Y.

(iii) Estimate of Y for X = 2.

(iv) Var (X) if Var (Y) = 36.

Solution :

Given : $3x+2y-26=0,\ 6x+y-31=0$

(i) Means of X and Y :

$$3x+2y-26=0 \qquad \dots (1)$$

$$6x+\ y-31=0 \qquad \dots (2)$$

Multiplying equation (2) by 2 and subtracting it from the equation (1), we get

$$\begin{array}{r} 3x+2y-26=0 \\ 12x+2y-62=0 \\ -\quad\ -\quad\ + \\ \hline -9x+36=0 \end{array}$$

$\therefore\ 9x=36 \qquad \therefore\ x=\dfrac{36}{9}=4$

Put $x=4$ in equation (1),

$\therefore\ 3(4)+2y-26=0$

$\therefore\ 2y+12-26=0$

$\therefore\ 2y=14 \qquad \therefore\ y=7$

Hence, $\bar{x}=4,\ \bar{y}=7$

(ii) Correlation coefficient between X and Y :

Let $3x+2y-26=0$ be the equation of Y on X.

$\therefore\ 2y=-3x+26$

$\therefore\ y=-\dfrac{3}{2}x+13$

$\therefore\ b_{yx}=-\dfrac{3}{2}$

Another equation $6x+y-31=0$ be the equation of X on Y.

$\therefore\ 6x=-y+31$

$\therefore\ x=-\dfrac{1}{6}y+\dfrac{31}{6}$

$\therefore\ b_{xy}=-\dfrac{1}{6}$

Now, $r=\pm\sqrt{b_{yx}\cdot b_{xy}}$

$$=\pm\sqrt{\left(-\frac{3}{2}\right)\times\left(-\frac{1}{6}\right)}$$

$$=\pm\sqrt{\frac{1}{4}}$$

$\therefore\ r=-\dfrac{1}{2}=-0.5 \quad \dots (\because\ b_{yx}$ and b_{xy} are negative)

Hence, $r=-0.5$

(iii) Estimate of Y for X = 2 :

Regression line of Y on X is

$$y-\bar{y}=b_{yx}(x-\bar{x})$$

$\therefore\ y-7=-\dfrac{3}{2}(x-4)$

$\therefore\ y=-\dfrac{3}{2}x+6+7 \qquad \therefore\ y=-\dfrac{3}{2}x+13$

Put $x=2$ in $y=-\dfrac{3}{2}x+13$

$\therefore\ y=-\dfrac{3}{2}\times 2+13$

$\therefore\ y=-3+13$

$\therefore\ y=10$

(iv) Var (X) if Var (Y) = 36 :

$\text{Var}\ (Y)=36=\sigma_y^2$

$\therefore\ \sigma_y=6,\ \text{Var}\ (X)=?$

Now, $b_{yx}=r\cdot\dfrac{\sigma_y}{\sigma_x}$

$\therefore\ -\dfrac{3}{2}=-\dfrac{1}{2}\times\dfrac{6}{\sigma_x}$

$\therefore\ 3=\dfrac{6}{\sigma_x}$

$\therefore\ \sigma_x=\dfrac{6}{3}=2$

$\therefore\ \text{Var}\ (X)=\sigma_x{}^2=4.$

8. Find the line of regression of X on Y for the following data :

$n=8,\ \Sigma(x_i-\bar{x})^2=36,\ \Sigma(y_i-\bar{y})^2=44,$

$\Sigma(x_i-\bar{x})(y_i-\bar{y})=24.$

Solution :

Given : $n=8,\ \Sigma(x_i-\bar{x})^2=36,\ \Sigma(y_i-\bar{y})^2=44,$

$$\Sigma(x_i-\bar{x})(y_i-\bar{y})=24.$$

Line of regression of X on Y :

$x-\bar{x}=b_{xy}(y-\bar{y})$

Now, $b_{xy}=\dfrac{\Sigma(x_i-\bar{x})(y_i-\bar{y})}{\Sigma(y_i-\bar{y})^2}$

$$=\frac{24}{44}=\frac{6}{11}$$

Line of regression of X on Y is obtained as follows :

$$x-\bar{x}=b_{xy}(y-\bar{y})$$

$\therefore\ x-\bar{x}=\dfrac{6}{11}(y-\bar{y}).$

9. Find the equation of line of regression of Y on X for the following data :

$n=8$, $\Sigma(x_i-\bar{x})(y_i-\bar{y})=120$, $\bar{x}=20$, $\bar{y}=36$, $\sigma_x=2$, $\sigma_y=3$.

Solution :

Given : $n=8$, $\bar{x}=20$, $\bar{y}=36$,

$\Sigma(x_i-\bar{x})(y_i-\bar{y})=120$, $\sigma_x=2$, $\sigma_y=3$.

$$b_{yx}=\frac{\text{Cov}(x,\ y)}{{\sigma_x}^2}$$

$$\text{Now, Cov}(x,\ y)=\frac{\Sigma(x_i-\bar{x})(y_i-\bar{y})}{n}$$

$$=\frac{120}{8}=15$$

$$\therefore\ b_{yx}=\frac{15}{(2)^2}=\frac{15}{4}$$

$$=3.75$$

Line of regression of Y on X :

$y=a+b_{yx}\cdot x$

$a=\bar{y}-b_{yx}\cdot\bar{x}$

$=36-3.75(20)$

$=36-75=-39$

$\therefore\ a=-39$

$\therefore$ line of regression of Y on X is

$y=-39+3.75x$

$\Rightarrow y=3.75x-39.$

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10. The following results were obtained from records of age (X) and systolic blood pressure (Y) of a group of 10 men :

	X	Y
Mean	50	140
Variance	150	165

and $\Sigma(x_i-\bar{x})(y_i-\bar{y})=1120$.

Find the prediction of blood pressure of a man of age 40 years.

Solution :

X = Age, Y = Systolic blood pressure.

Given : $\bar{x}=50$, $\bar{y}=140$, ${\sigma_x}^2=150$, ${\sigma_y}^2=165$, $n=10$,

$\Sigma(x_i-\bar{x})(y_i-\bar{y})=1120$

$$b_{yx}=\frac{\text{Cov}(x,y)}{{\sigma_x}^2}$$

$$\text{Now, Cov}(x,y)=\frac{\Sigma(x_i-\bar{x})(y_i-\bar{y})}{n}$$

$$=\frac{1120}{10}$$

$$=112$$

$$\therefore\ b_{yx}=\frac{112}{150}$$

$$=0.75$$

Regression line of Y on X :

$y=a+b_{yx}\cdot x$

$b_{yx}=0.75$

$a=\bar{y}-b_{yx}\cdot\bar{x}$

$=140-0.75(50)$

$=140-37.5=102.5$

$\therefore$ regression line of Y on X is

$y=102.5+0.75x$

$\Rightarrow y=0.75x+102.5$

Prediction of blood pressure (Y) for the age X = 40

Putting $x=40$ in $y=0.75x+102.5$, we get

$y=0.75(40)+102.5=30+102.5$

$\therefore\ y=132.5$

[**Note :** *Answers given in the textbook are incorrect.*]

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11. The equations of two regression lines are $10x-4y=80$ and $10y-9x=-40$.

Find :

(i) $\bar{x}$ and $\bar{y}$.

(ii) b_{yx} and b_{xy}.

(iii) If var(Y) = 36, obtain var(X).

(iv) r.

Solution :

Given : $10x-4y=80$, $10y-9x=-40$

(i) $\bar{x}$ and $\bar{y}$:

$$10x-4y=80 \quad \dots(1)$$
$$-9x+10y=-40 \quad \dots(2)$$

Multiplying equation (1) by 9 and equation (2) by 10 and then adding them, we get

$$90x-36y=720 \quad \dots(1)$$
$$-90x+100y=-400 \quad \dots(2)$$
$$\therefore\ 64y=320$$

$$\therefore\ y=\frac{320}{64}=5$$

Put $y=5$ in equation (1), we get

$$10x - 4(5) = 80$$

$$\therefore\ 10x = 80 + 20$$

$$\therefore\ x = \frac{100}{10} = 10$$

Hence, $\bar{x} = 10^{*}$, $\bar{y} = 5$

(ii) b_{yx} and b_{xy} :

Let regression equation of X on Y be

$$10x - 4y = 80$$

$$\therefore\ 10x = 4y + 80$$

$$\therefore\ x = \frac{4}{10}y + 8$$

$$\therefore\ b_{xy} = 0.4$$

And another equation $10y - 9x = -40$ be the regression equation of Y on X.

$$\therefore\ 10y = 9x - 40$$

$$\therefore\ y = \frac{9}{10}x - 4 \qquad \therefore\ b_{yx} = \frac{9}{10} = 0.9$$

Hence, $b_{yx} = 0.9$ and $b_{xy} = 0.4$

(iii) Var(X) if Var(Y) = 36 $\therefore\ \sigma_y = 6$:

Now, $b_{yx} = r \cdot \dfrac{\sigma_y}{\sigma_x}$

$$\therefore\ 0.9 = 0.6 \times \frac{6}{\sigma_x}$$

$$\therefore\ \frac{0.9}{0.6 \times 6} = \frac{1}{\sigma_x} \qquad \therefore\ \frac{1}{4} = \frac{1}{\sigma_x}$$

$$\therefore\ \sigma_x = 4$$

$$\therefore\ \text{Var}(X) = \sigma_x{}^2 = (4)^2 = 16.^{*}$$

(iv) Coefficient of correlation r :

$$r = \pm\sqrt{b_{yx} \cdot b_{xy}}$$

$$= \pm\sqrt{0.9 \times 0.4}$$

$$= \pm\sqrt{0.36}$$

$\therefore\ r = 0.6$ … ($\because$ b_{yx} and b_{xy} are positive)

[* **Note :** *Answers given in the textbook are incorrect.*]

12. If $b_{yx} = -0.6$ and $b_{xy} = -0.216$ then find correlation coefficient between X and Y. Comment on it.

Solution :

Given : $b_{yx} = -0.6$, $b_{xy} = -0.216$

$$r = \pm\sqrt{b_{yx} \cdot b_{xy}}$$

$$= \pm\sqrt{(-0.6)(-0.216)}$$

$$= \pm\sqrt{0.1296}$$

$$\therefore\ r = -0.36$$

Comment :

$r^2 = b_{yx} \cdot b_{xy}$

Hence, $b_{yx} \cdot b_{xy} > 0$ and signs of b_{yx} and b_{xy} should be same.

$-1 \leqslant r^2 \leqslant 1 \Rightarrow r^2 = (\pm 1)^2$. Hence the sign of correlation coefficient is similar as the signs of b_{yx} and b_{xy}. Here, b_{yx} and b_{xy} both are negative. Hence r is also negative and hence X and Y are negatively correlated.

ACTIVITIES Textbook pages 54 to 56

1. Consider a group of 70 students of your class to take their heights in cm (x) and weights in kg (y). Hence find both the regression equations.

Solution :

Let the data of 70 students on height (x) (in cm) and weights (y) in (kg) are summerised in the following results :

$\bar{x} = 150$ cm, $\bar{y} = 45$ kg

$\Sigma(x - \bar{x})^2 = 15000$, $\Sigma(y - \bar{y})^2 = 27000$.

$\Sigma(x - \bar{x})(y - \bar{y}) = 18000$

(i) The regression equation of Y on X :

$$y = a + b_{yx} \cdot \bar{x}$$

Now, $b_{yx} = \dfrac{\Sigma(x - \bar{x})(y - \bar{y})}{\Sigma(x - \bar{x})^2} = \dfrac{18000}{15000} = \dfrac{6}{5} = 1.2$

$$a = \bar{y} - b_{yx} \cdot (\bar{x})$$

$$= 45 - 1.2(150)$$

$$= 45 - 180 = -135$$

$$\therefore\ y = -135 + 1.2x$$

$$\Rightarrow y = 1.2x - 135.$$

(ii) The regression equation of X on Y :

$$x = a' + b_{xy} \cdot y$$

Now, $b_{xy} = \dfrac{\Sigma(x - \bar{x})(y - \bar{y})}{\Sigma(y - \bar{y})^2} = \dfrac{18000}{27000} = \dfrac{2}{3}$

$$a' = \bar{x} - b_{xy} \cdot \bar{y}$$

$$= 150 - \frac{2}{3}(45) = 150 - 30$$

$$= 120$$

$$\therefore\ x = 120 + \frac{2}{3}y$$

$$\Rightarrow x = \frac{2}{3}y + 120.$$

2. The age in years of 7 young couples is given below :

Husband (x)	**21**	**25**	**26**	**24**	**22**	**30**	**20**
Wife (y)	**19**	**20**	**24**	**20**	**22**	**24**	**18**

(i) Find the equation of regression line of age of husband on age of wife.

(ii) Draw the regression line of y on x.

(iii) Predict the age of wife whose husband's age is 27 years.

Solution : (i)

Age of Husband x	**Age of Wife y**	**$(x-\bar{x})$ $\bar{x}=24$**	**$(y-\bar{y})$ $\bar{y}=21$**	**$(x-\bar{x})(y-\bar{y})$**	**$(y-\bar{y})^2$**	**$(x-\bar{x})^2$**
21	19	-3	-2	6	4	9
25	20	1	-1	-1	1	1
26	24	2	3	6	9	4
24	20	0	-1	0	1	0
22	22	-2	1	-2	1	4
30	24	6	3	18	9	36
20	18	-4	-3	12	9	16
$\Sigma x=168$	$\Sigma y=147$	$\Sigma(x-\bar{x})=0$	$\Sigma(y-\bar{y})=0$	42 $-\ 3$ $\Sigma(x-\bar{x})(y-\bar{y})=39$	$\Sigma(y-\bar{y})^2=34$	$\Sigma(x-\bar{x})^2=70$

Regression equation of age of husband (x) on age of wife (y)

$x=a'+b_{xy}\cdot y$

Now, $b_{xy}=\dfrac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(y-\bar{y})^2}=\dfrac{39}{34}$

$a'=\bar{x}-b_{xy}\cdot y$

$=24-\dfrac{39}{34}\times 21$

$=\dfrac{816-819}{34}$

$=-\dfrac{3}{34}$

$\therefore\ x=-\dfrac{3}{34}+\dfrac{39}{34}y$

$\therefore\ 34x=-3+39y$

$\therefore\ 34x-39y+3=0$

(ii) Regression line of y on x :

Here, $n = 7$

$$\bar{x} = \frac{\Sigma x}{n} = \frac{168}{7} = 24, \quad \bar{y} = \frac{\Sigma y}{n} = \frac{147}{7} = 21$$

(iii) From the graph, the estimated age of wife will be 25 years whose husband's age is 27 years.

On the basis of Data given :

Prediction of the age of wife (Y) when the age of husband (X) = 27 years :

Regression equation of Y on X : $y = a + b_{yx} \cdot x$

$$\text{Now, } b_{yx} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(x-\bar{x})^2} = \frac{39}{70}$$

$$a = \bar{y} - b_{yx} \cdot (\bar{x})$$

$$= 21 - \frac{39}{70} \times 24$$

$$= \frac{1470 - 936}{70} = \frac{534}{70}$$

$$\therefore \ a = \frac{534}{70}$$

$$\text{Hence, } y = \frac{534}{70} + \frac{39}{70}x$$

$$\Rightarrow y = \frac{39}{70}x + \frac{534}{70}$$

Put $x = 27$, we get

$$y = \frac{39}{70} \times 27 + \frac{534}{70}$$

$$= \frac{1053 + 534}{70} = \frac{1587}{70}$$

$$= 22.67 \approx 23 \text{ years.}$$

(Answers are given directly.)

3. The equations of two regression lines are

$10x - 4y = 80$ ... (1)

$10y - 9x = -40$ ... (2)

$\therefore$ $(\bar{x}, \bar{y})$ is the point of intersection of both the regression lines.

$\therefore$ solve equations (1) and (2), we get

$\bar{x} = \boxed{10}$ and $\bar{y} = \boxed{5}$

Now, consider $10x - 4y = 80$

$\therefore$ $a = \boxed{10}$, $b = \boxed{-4}$

$\therefore$ slope $(m_1) = -\frac{a}{b} = \frac{\boxed{10}}{\boxed{4}}$

Consider, $10y - 9x = -40$

$\therefore$ $a = \boxed{-9}$, $b = \boxed{10}$

$\therefore$ slope $(m_2) = -\frac{a}{b} = \frac{\boxed{9}}{\boxed{10}}$

$\therefore$ $|m_1| > |m_2|$

$\therefore$ $b_{yx} = \boxed{2.5}$ and $b_{xy} = \frac{1}{\boxed{2.5}}$

$\therefore$ $10x - 4y = 80$ is the regression equations of $\boxed{\text{X}}$ on $\boxed{\text{Y}}$ and

$\therefore$ $10y - 9x = -40$ is the regression equations of $\boxed{\text{Y}}$ on $\boxed{\text{X}}$.

Now, $r = \pm\sqrt{b_{yx} \cdot b_{xy}}$

$\therefore$ $r = \pm\sqrt{\frac{\boxed{9}}{\boxed{10}} \times \frac{4}{10}}$

$r = \boxed{0.6}$

If $\text{Var}(y) = 36$, then $\sigma_y = \sqrt{\boxed{36}} = \boxed{6}$

$\therefore$ $b_{xy} = r\frac{\sigma_y}{\sigma_x}$

$\therefore$ $\frac{\boxed{4}}{\boxed{10}} = \boxed{0.6} \times \frac{\boxed{6}}{\sigma_x}$

$\therefore$ $\sigma_x = (9)$

$\therefore$ $\text{Var}(x) = \sigma_x{}^2 = \boxed{81}$

4. Given : $n = 8$, $\Sigma(x_i - \bar{x})^2 = 36$, $\Sigma(y_i - \bar{y})^2 = 40$,

$\Sigma(x_i - \bar{x})(y_i - \bar{y}) = 24$

$\therefore$ $b_{yx} = \frac{\Sigma(x_i - \bar{x})(y_i - \bar{y})}{\Sigma(x_i - \bar{x})^2}$

$= \frac{\boxed{24}}{\boxed{36}} = \frac{\boxed{2}}{\boxed{3}}$

$\therefore$ $b_{xy} = \frac{\Sigma(x_i - \bar{x})(y_i - \bar{y})}{\Sigma(y_i - \bar{y})^2}$

$= \frac{\boxed{24}}{\boxed{40}} = \frac{\boxed{3}}{\boxed{5}}$

$\therefore$ regression equation of Y on X :

$y - \bar{y} = b_{yx}(x - \bar{x})$

$y - \boxed{\bar{y}} = \frac{\boxed{2}}{\boxed{3}}(x - \boxed{\bar{x}})$

$\therefore$ regression equation of X on Y :

$x - \bar{x} = b_{xy}(y - \bar{y})$

$x - \boxed{\bar{x}} = \frac{\boxed{3}}{\boxed{5}}(y - \boxed{\bar{y}})$

5. Consider, given

$n = 8$, $\Sigma(x_i - \bar{x})(y_i - \bar{y}) = 120$, $\bar{y} = 36$, $\sigma_x = 2$, $\sigma_y = 3$

$\therefore$ $\text{cov}(x, y) = \frac{\Sigma(x - \bar{x})(y - \bar{y})}{n}$

$= \frac{\boxed{120}}{\boxed{8}} = \boxed{15}$

$\therefore$ $b_{yx} = \frac{\boxed{15}}{\sigma_x^{\boxed{2}}} = \frac{150}{\boxed{40}}$

$b_{xy} = \frac{\text{cov}(x, y)}{\boxed{\sigma_y{}^2}} = \frac{\boxed{15}}{9} = \frac{\boxed{5}}{\boxed{3}}$

$\therefore$ regression equation of Y on X :

$y - \boxed{\bar{y}} = b_{yx}(x - \boxed{\bar{x}})$

$y - \boxed{36} = \boxed{3.75}(x - \boxed{\bar{x}})$

$\therefore$ regression equation of Y on X :

$x - \bar{x} = \boxed{b_{xy}}(y - \bar{y})$

$x - \boxed{\bar{x}} = \boxed{1.67}(y - \boxed{36})$

* * *

4 TIME SERIES

CHAPTER OUTLINE

INTRODUCTION

In practice we associate some events with the flow of time. The price of shares and stock of industrial firms, the population of a country, per-capita national income, prices of different commodities, their production, demand and sales, profit of industrial firms, etc. are time based events. By studying the data of such events it can be predicted about the nature of changes in such events or phenomena. The data collected on such time based events form a Time Series. The study and analysis of the time series of the events affecting the growth and development of the country play an important role. This reflects the economic, industrial and social development and growth of the country. In this chapter we will study the basic elements of time series and some statistical methods used in the analysis of time series.

IMPORTANT FORMULAE

Notations :

x = variable quantity

t = time

x_t = value of the variable at time t

T_t = Secular trend or Trend

S_t = Seasonal variation

C_t = Cyclical variation

I_t = Irregular variation

1. Mathematical models of time series :

(1) Additive model :

$$x_t = T_t + S_t + C_t + I_t$$

(2) Multiplicative model :

$$x_t = T_t \times S_t \times C_t \times I_t$$

2. Equation of linear trend :

$x_t = a + bt$; where a, b are constants.

3. Normal equations to fit a linear trend :

$\Sigma x_t = na + b\Sigma t$

$\Sigma tx_t = a\Sigma t + b\Sigma t^2$

4. Short cut method :

We transform the values of t to a new variable u such that $\Sigma u = 0$.

(1) If the number of terms of time series n is an odd number :

$$u = \frac{t - \text{middle } t \text{ value}}{h}$$

where, h = distance between any two consecutive t values.

(2) If the number of terms of time series n is an even number :

$$u = \frac{t - \text{mean of two middle } t \text{ values}}{h/2}$$

where, h = distance between any two consecutive t values.

Two normal equations are :

$\Sigma x_t = na' + b'\Sigma u$

$\Sigma ux_t = a'\Sigma u + b'\Sigma u^2$

The equation of linear trend :

$x_t = a' + b'u$

5. Method of Moving Averages :

k = The moving average period

First moving average $= \dfrac{x_1 + x_2 + ... + x_k}{k}$

Second moving average $= \dfrac{x_2 + x_3 + ... + x_{k+1}}{k}$

Third moving average $= \dfrac{x_3 + x_4 + ... + x_{k+2}}{k}$

and so on ...

When k is odd : Moving averages correspond to the time for which time series is given.

When k is even : Moving averages correspond to midway of two times for which time series is given. Two-unit moving averages correspond to the time period given.

4.1 : TIME SERIES

1. Definition of Time Series :

- Time series is a sequence of observations made on a variable at regular time intervals over a specified period of time.
- The statistical data collected at successive interval of time is referred to as time series.
- 'A time series consists of data arranged chronologically, i.e. according to time.'

– Croxton and Cowden

- 'A set of values depending on time is called time series.'

– Kenny and Keeping

2. Examples of time series :

(i) Yearly production of a commodity

(ii) Weekly price of a commodity

(iii) Monthly sales of an electronic item

(iv) Yearly amount of bank deposits

(v) Daily closing price of a share

(vi) Yearly GDP of a country.

4.2 : TIME SERIES ANALYSIS

- Time series analysis involves the use of statistical methods to extract meaningful statistics from the time series and understand important characteristics of the observed data.
- Time series analysis is useful in understanding the pattern of changes in the variable over time.
- Time series analysis is the method of understanding, interpreting the evaluating chronological variations in the past in time series. It helps us to make reliable predictions in the future.
- **Uses of Time Series Analysis :** The main objective of time series analysis is to understand, interpret and assess chronological change in value of a variable in the past, so that reliable prediction can be made about its future values. For example, an industrialist may be interested in predicting the demand for his product for making the production schedule.

The important uses of time series analysis are as follows :

(1) Past behaviour : Time series analysis enables us to describe the past behaviour of the variable under consideration. This knowledge is useful in understanding various forces which are responsible for the fluctuations in the values of the variable over the passage of time.

(2) Forecasting : The past behaviour of the data is projected into the future behaviour in order to forecast the changes that are likely to occur in future. In economic or business phenomena, such forecasting is of utmost important in decision making regarding inventory, sales and purchases, etc.

(3) Evaluating : Time series analysis is useful in evaluating the performance in comparison with predetermined targets. For example, the achievement of Five Year Plans are evaluated by determining the annual rate of growth in the gross national product. This is made possible by analysis of time series of the relevant variable.

(4) Comparison : It helps in comparing the behaviour of two or more time series. For example, the agricultural production over the last twenty years may be compared with the population for the same sequence of years, prices of gold and silver, prices of shares may be compared over the last ten years period.

4.3 : COMPONENTS OF TIME SERIES

A critical study of time series reveals that the changes in the values of variable under consideration are not always haphazard. A part of the changes accounted for is in the systematic pattern and the remaining is irregular. This systematic part of the time series may be used for forecasting. The systematic part of variations or fluctuations of time series may be referred to as components of a time series. There are four components of time series, which are discussed below :

4.3.1 : Secular Trend (T)

The secular trend of time series is the long-term pattern of the series which can be positive or negative depending on whether the time series exhibits an increasing or a decreasing long-term pattern. It is smooth, regular and long-term movement of the time series. It is also known simply as trend. The concept of trend does not include short-time oscillations but rather the steady changes over a long period of time. The study of trend helps in predicting a value of variable x of time series for some value of time t and in comparing two or more sets of time series data. It may be noted that the trend may be linear or non-linear in nature.

Examples of upward trend :

(i) yearly population
(ii) yearly agricultural production
(iii) yearly price index number
(iv) yearly profit of the company

Examples of downward trend :

(i) cost of electronic goods
(ii) proportion of illiterates
(iii) yearly death rates
(iv) yearly quantity of minerals

Examples of stagnant trend :

(i) yearly rainfall
(ii) monthly electricity consumption of a family
(iii) daily temperature of a place
(iv) daily demand of salt of a family

4.3.2 : Seasonal Variation (S)

In practice, many business and economic time series consist of quarterly or monthly observations. Such series exhibit the phenomenon of seasonality such that patterns are repeated from year to year. These type of patterns of variations found in time series are referred to as seasonal variation. Thus seasonal variation involves patterns of variation within a year that tend to be repeated from year to year. For example, wool and woollens have a market during the winter season only. Fan, coolers, AC and cold-drinks are in great demand in summer. Umbrella and raincoats are demanded more during the rainy season. Certain festivals falling in a particular month create demand of some items in these particular months. For example, in Diwali festival there is a tremendous demand for sweets, crackers, cloth, etc. Thus seasonal variations are a reflection of climate conditions and customers and habits of people.

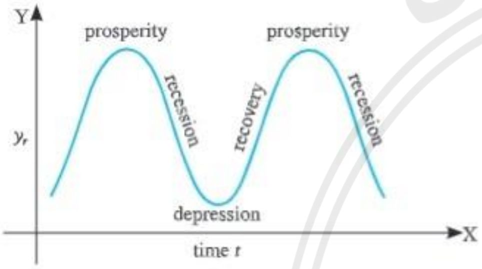
The period of seasonal variation can be taken as year, month, week, day or hour depending on the type of the data available. Seasonal variation may be found in the number of books issued by a library during the days in a week, passenger traffic during the hours of a day.

Seasonal variations are measured by seasonal indices. These types of variations are useful in short-term forecasting. Using such short-term forecasting a departmental store can plan its inventory for various months in a year.

4.3.3 : Cyclical Variation (C)

Cyclical variation is a long-term oscillatory movement that occurs over a long period of time. This time is usually two or more years and is called a cycle. The cyclical variations are not necessarily periodic since the length of a cycle may change from one cycle to the other.

Cyclical variation is found in almost all business and economic time series. In these type of time series it is known as business cycle or trade cycle. The ups and downs in business recurring at intervals of time are the causes of cyclical variation. A business cycle consists of four phases as shown in the following figure :



This figure represents the four phases of business cycle : (i) prosperity (ii) recession (iii) depression and (iv) recovery. Each phase changes gradually in the next phase as shown in the figure.

Usually the period of cyclical variations is 5 to 10 years or even higher in some cases. The period of a cycle often changes from one cycle to the other. Cycles can be affected by the internal factors of an organisation such as inventory policy and by the external factors such as economic conditions, government policies, etc.

4.3.4 : Irregular Variation (I)

Irregular variations are either wholly unaccountable or caused by unforeseen factors such as floods, famines, strikes, wars, etc. These type of variations do not follow any particular pattern and hence they are totally unpredictable. For this reason it is also known as unexplained or unaccounted variation.

4.4 : MATHEMATICAL MODELS OF TIME SERIES

There are two standard mathematical models for time series based on four components of time series, namely, secular trend (T), seasonal variation (S), cyclical variation (C) and irregular variation (I).

4.4.1 : Additive Model

In this model, x_t is assumed to be the sum of the four components of time series.

Thus, $x_t = T_t + S_t + C_t + I_t$

In this model, it is assumed that the components are non-interactive or independent. This assumption is not realistic. Hence this model is not used for most of the economic data.

4.4.2 : Multiplicative Model

Let x_t = Value of the variable at time t.

According to classical approach it is assumed that there is multiplicative relationship among the four components of time series, i.e. x_t is considered as the product of the four components–secular trend (T_t), seasonal variation (S_t), cyclical variation (C_t) and irregular variation (I_t).

Thus, $x_t = T_t \times S_t \times C_t \times I_t$

In this model, only trend is expressed in terms of actual value and the remaining components are expressed in terms of percentage of trend.

4.5 : MEASUREMENT OF SECULAR TREND

4.5.1 : Method of Free hand Curve (Graphical Method)

In this method, we first draw the curve or line-diagram for the given time series by taking time on X-axis and values of x_t on Y-axis. Then we draw a free hand smooth curve which seems to fit the data best.

- This method is simple and does not require any mathematical calculation.
- This method is quite flexible and can be used for linear as well as non-linear trend and involves minimum amount of work.
- In this method, different researchers may draw different trend lines for the same set of data. Thus it is quite subjective and requires sound judgement.
- Forecasting may be risky if free hand curve is not drawn by efficient and experienced person.

4.5.2 : Method of Moving Averages

In this method, firstly the period of moving average is determined. Let k be the moving average period of a time series. This will give us a new series of arithmetic means, each of k successive observations of the time series. In this method, we start with first k observations, then we leave the first observation and include $(k+1)$st observation. This process is repeated until the last k^{th} observation arrived. We find the moving average as follows :

$$\text{First moving average} = \frac{x_1 + x_2 + \ldots + x_k}{k}$$

$$\text{Second moving average} = \frac{x_2 + x_3 + \ldots + x_{k+1}}{k}$$

$$\text{Third moving average} = \frac{x_3 + x_4 + \ldots + x_{k+2}}{k}$$

and so on ...

Each of these averages is centred against the time which is the midpoint of the time interval of the given time series. If k is odd, the moving average is centred against the given time values of the time series. If k is even, the moving averages fall midway between two time values. In this case, we calculate a two-unit moving averages which correspond to the given time period of time series.

If the period of moving average is selected properly, it smooths out cyclical variations from the time series and gives an estimate of the trend. If the trend is linear, then the period of moving average is equal to or a multiple of the period of the cycles.

Merits and Demerits :

The method of moving averages is flexible due to the reason that if we add a few observations to the series, the moving averages remain the same. As this method does not provide a mathematical equation of the trend, it cannot be used for forecasting purpose. Also in this method a number of trend values at each end of the time series remain unestimated.

4.5.3 : Method of Least Squares

This is the best method of measuring trend. Usually, a polynomial of a suitable degree is chosen and its constant are estimated by the method of least squares on the basis of time series data. The choice of an appropriate polynomial can be determined by the graphical representation of the data.

Fitting of Linear Trend : Let the equation of linear trend be $x_t = a + bt$ where a, b are constants. According to the method of least squares the constants 'a' and 'b' are estimated by solving the following two normal equations :

$$\Sigma x_t = na + b\,\Sigma t \qquad \ldots (1)$$

$$\Sigma t x_t = a\,\Sigma t + b\,\Sigma t^2 \qquad \ldots (2)$$

where, $t =$ time, $n =$ number of time periods.

We obtain Σx_t, $\Sigma t x_t$, Σt and Σt^2 from the data and solving the equations we get least squares estimates of a and b. Putting these estimates in $x_t = a + bt$, we get the fitted equation of linear trend.

We can transform 't' to 'u' to make the calculation easier.

When n is odd :

$$u = \frac{t - \text{middle } t \text{ value}}{h}$$

When h is even :

$$u = t - \frac{\text{mean of two middle } t \text{ values}}{h/2}$$

where, $h =$ distance between any two consecutive t values.

The equation of linear trend becomes $y_t = a' + b'u$.

To estimate a' and b' the following two normal equations are solved.

$$\Sigma\, x_t = na' + b'\,\Sigma\, u \qquad \ldots (1)$$

$$\Sigma\, u x_t = a'\,\Sigma\, u + b'\,\Sigma\, u^2 \qquad \ldots (2)$$

Σx_t, Σu, Σu^2 and $\Sigma u x_t$ are obtained from the data, solving the equations we get least squares estimates of a' and b'. Putting these estimates in $x_t = a' + b'u$, we get the fitted equation of linear trend.

EXERCISE 4.1 | **Textbook pages 66 and 67**

1. The following data gives the production of bleaching powder (in '000 tonnes) for the years 1962 to 1972 :

Year	1962	1963	1964	1965	1966	
Production	0	0	1	1	4	
Year	1967	1968	1969	1970	1971	1972
Production	2	4	9	7	10	8

Fit a trend line by graphical method to the above data.

Solution : Taking year on X-axis and production on Y-axis, we plot the points for production corresponding to the years. Joining the points by straight lines we get the graph for the given time series. We draw trend line as shown in the figure 4.1.

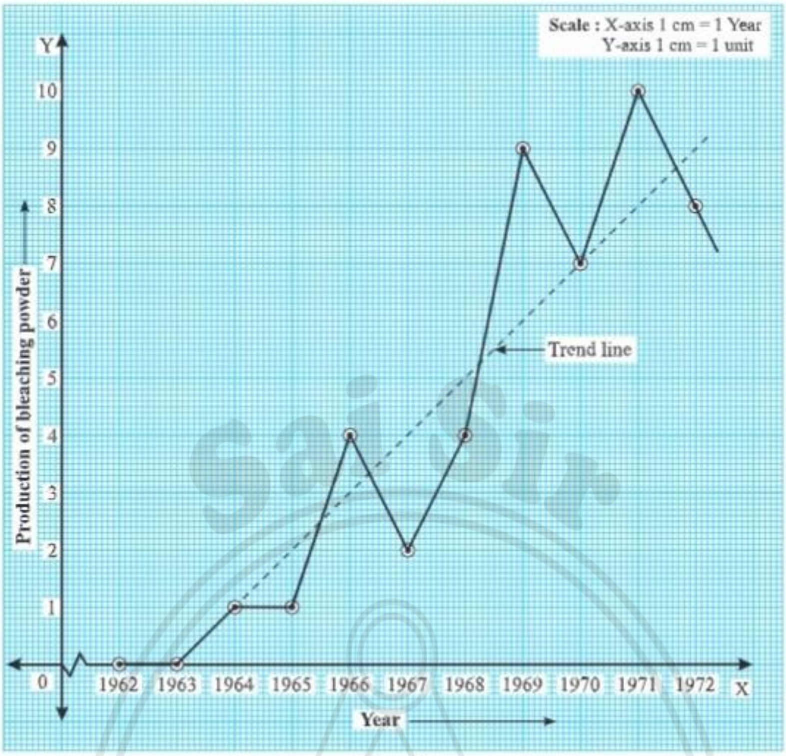


Fig. 4.1

2. Use the method of least squares to fit a trend line to the data in Problem 1 above. Also, obtain the trend value for the year 1975.

Solution : Here, $n = 11$. Hence, we transform year t to 'u' taking origin at 1967.

We construct the following table for calculation of Σu, Σu^2, Σx_t, $\Sigma u x_t$:

Year t	Production x_t ('000 tonnes)	$u = t - '67$	u^2	ux_t
1962	0	-5	25	0
1963	0	-4	16	0
1964	1	-3	9	-3
1965	1	-2	4	-2
1966	4	-1	1	-4
1967	2	0	0	0
1968	4	1	1	4
1969	9	2	4	18
1970	7	3	9	21
1971	10	4	16	40
1972	8	5	25	40
Total	$\Sigma x_t = 46$	$\Sigma u = 0$	$\Sigma u^2 = 110$	123 $-$ 9 $\Sigma u x_t = 114$

The equation of trend line is $x_t = a' + b'u$.

The two normal equations are

$$\Sigma x_t = na' + b'\Sigma u \qquad \ldots (1)$$

$$\Sigma u x_t = a'\Sigma u + b'\Sigma u^2 \qquad \ldots (2)$$

Putting the values $\Sigma x_t = 46$, $n = 11$, $\Sigma u = 0$. $\Sigma u^2 = 110$, $\Sigma u x_t = 114$ in the equations, we get

$$46 = 11a' + b'(0) \qquad \ldots (3)$$

$$114 = a'(0) + b'(110) \qquad \ldots (4)$$

From equation (3), $a' = \frac{46}{11} = 4.182$

From equation (4), $b' = \frac{114}{110} = 1.036$

We get the equation of trend line,

$x_t = 4.182 + 1.036u$, where $u = t - 1967$.

Trend value for the year 1975 :

For $t = 1975$, $u = 1975 - 1967 = 8$

Putting $u = 8$ in $x_t = 4.182 + 1.036\,u$, we get

$$x_{1975} = 4.182 + 1.036(8)$$

$$= 4.182 + 8.288$$

$$\therefore\ x_{1975} = 12.47$$

Hence, the trend value for the year 1975 is 12.47 (in '000 tonnes).

3. Obtain the trend line for the above data using 5-yearly moving averages.

Solution :

We construct the following table to obtain 5-yearly moving averages for the data in problem 1 :

Year t	Production (in '000 tonnes) x_t	5-yearly moving total	5-yearly moving averages Trend value
1962	0	–	–
1963	0	–	–
1964	1	6	1.2
1965	1	8	1.6
1966	4	12	2.4
1967	2	20	4.0
1968	4	26	5.2
1969	9	32	6.4
1970	7	38	7.6
1971	10	–	–
1972	8	–	–

4. The following table shows the index of industrial production for the period from 1976 to 1985, using the year 1976 as the base year :

Year	1976	1977	1978	1979	1980
Index	0	2	3	3	2
Year	1981	1982	1983	1984	1985
Index	4	5	6	7	10

Fit a trend line to above data by graphical method.

Solution :

Taking year on X-axis and index on Y-axis, we plot the points for indices corresponding to the years. We get the graph of the given time series. We draw trend line as shown in the figure 4.2.

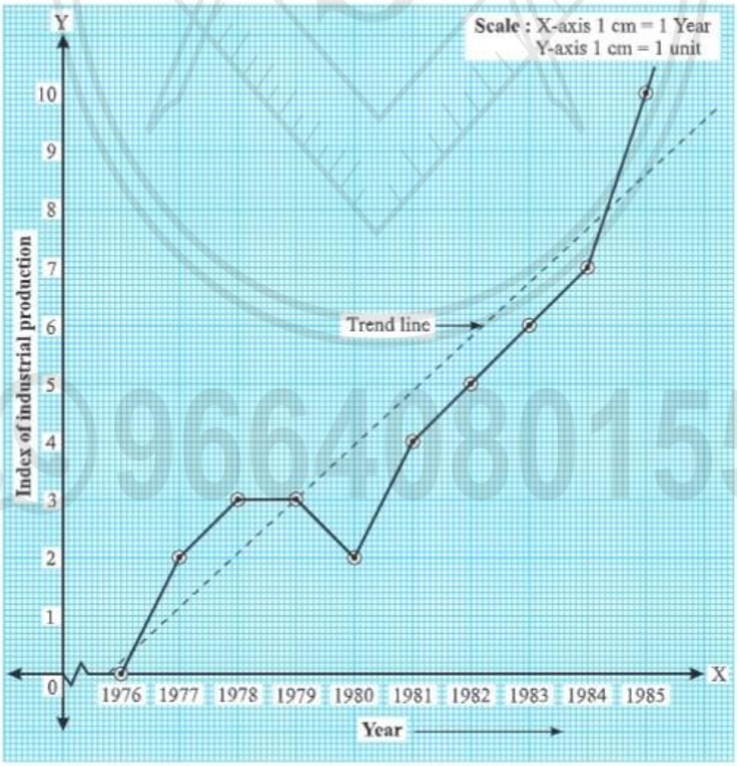


Fig. 4.2

5. Fit a trend line to the data in Problem 4 above by the method of least squares. Also, obtain the trend value for the index of industrial production for the year 1987.

Solution :

Here, $n = 10$, we transform t to u by taking

$u = 2(t - 1980.5)$

We construct the following table for calculation :

Year t	Index of industrial production x_t	$u = 2(t - 1980.5)$	u^2	ux_t
1976	0	−9	81	0
1977	2	−7	49	−14
1978	3	−5	25	−15
1979	3	−3	9	−9
1980	2	−1	1	−2
1981	4	1	1	4
1982	5	3	9	15
1983	6	5	25	30
1984	7	7	49	49
1985	10	9	81	90
Total	$\Sigma x_t = 42$	$\Sigma u = 0$	$\Sigma u^2 = 330$	188 − 40 $\Sigma ux_t = 148$

The equation of trend line, $x_t = a' + b'u$

The normal equations are,

$\Sigma x_t = na' + b' \Sigma u$ ... (1)

$\Sigma ux_t = a' \Sigma u + b' \Sigma u^2$ ... (2)

Here, $n = 10$, $\Sigma x_t = 42$, $\Sigma u = 0$, $\Sigma u^2 = 330$, $\Sigma ux_t = 148$

Putting these values in the normal equations, we get

$42 = 10a' + b'(0)$ ... (3)

$148 = a'(0) + b'(330)$ ... (4)

From the equation (3), we get

$$a' = \frac{42}{10} = 4.2$$

From the equation (4), we get

$$b' = \frac{148}{330} = 0.4485$$

Putting $a' = 4.2$ and $b' = 0.4485$ in $x_t = a' + b'u$, we get the equation of trend line as

$x_t = 4.2 + 0.4485\,u$, where $u = 2(t - 1980.5)$

Trend value for the year 1987 :

For $t = 1987$, $u = 2(1987 - 1980.5)$

$= 2(6.5) = 13$

Putting $u = 13$ in $x_t = 4.2 + 0.4485\,u$, we get

$x_{1987} = 4.2 + 0.4485 \times 13$

$\therefore\ x_{1987} = 4.2 + 5.8305$

$\therefore\ x_{1987} = 10.0305$

Hence, the trend value for the index of industrial production for the year 1987 is 10.0305.

6. Obtain the trend values for the data in problem 4 using 4-yearly centered moving averages.

Solution :

We construct the following table to obtain 4-yearly moving averages for the data in problem 4 :

Year t	Index x_t	4-yearly moving total	4-yearly moving averages	2 unit moving total	4-yearly centred moving averages (Trend value)
1976	0	–	–	–	–
1977	2	8	2.0	–	–
1978	3	10	2.5	4.5	2.25
1979	3	12	3.0	5.5	2.75
1980	2	14	3.5	6.5	3.25
1981	4	17	4.25	7.75	3.875
1982	5	22	5.5	9.75	4.875
1983	6	28	7.0	12.5	6.25
1984	7	–	–	–	–
1985	10			–	–

[**Note :** *Answer given in the textbook is incorrect.*]

7. The following table gives the production of steel (in millions of tonnes) for years 1976 to 1986.

Year	1976	1977	1978	1979	1980	1981
Production	0	4	4	2	6	8
Year	1982	1983	1984	1985	1986	
Production	5	9	4	10	10	

Fit a trend line to the above data by the graphical method.

Solution :

Taking year on X-axis and production on Y-axis, we plot the points for production corresponding to years. Joining these points by staight lines, we get the graph of the given time series. We draw trend line as shown in the figure 4.3.

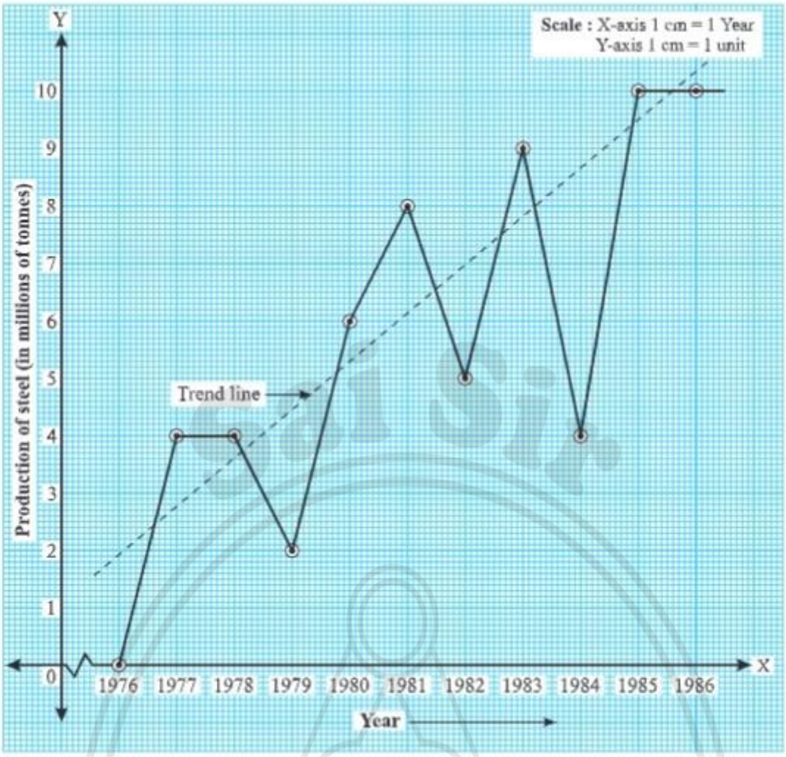


Fig. 4.3

8. Fit a trend line to the data in Problem 7 by the method of least squares. Also, obtain the trend value for the year 1990.

Solution : Here, $n = 11$. We transform year–t to u by taking $u = t - 1981$.

We construct the following table for calculation :

Year t	Production x_t	$u = t - 1981$	u^2	ux_t
1976	0	-5	25	0
1977	4	-4	16	-16
1978	4	-3	9	-12
1979	2	-2	4	-04
1980	6	-1	1	-06
1981	8	0	0	0
1982	5	1	1	5
1983	9	2	4	18
1984	4	3	9	12
1985	10	4	16	40
1986	10	5	25	50
Total	$\Sigma x_t = 62$	$\Sigma u = 0$	$\Sigma u^2 = 110$	125 $-$ 38 $\Sigma ux_t = 87$

The equation of trend line is $x_t = a' + b'u$

The normal equations are

$\Sigma x_t = na' + b' \Sigma u$ ... (1)

$\Sigma u x_t = a' \Sigma u + b' \Sigma u^2$ ... (2)

Here, $n = 11$, $\Sigma x_t = 62$, $\Sigma u = 0$, $\Sigma u^2 = 110$, $\Sigma u x_t = 87$

Putting these values in normal equations, we get

$62 = 11a' + b'(0)$ ... (3)

$87 = a'(0) + b'(110)$ ... (4)

From equation (3), we get

$$a' = \frac{62}{11} = 5.6364$$

From equation (4), we get

$$b' = \frac{87}{110} = 0.7909$$

Putting $a' = 5.6364$ and $b' = 0.7909$ in the equation $x_t = a' + b'u$, we get the equation of trend line as

$x_t = 5.6364 + 0.7909\,u$

Trend for the year 1990 :

For $t = 1990$, $u = 1990 - 1981 = 9$

Putting $u = 9$ in $x_t = 5.6364 + 0.7909\,u$, we get

$x_{1990} = 5.6364 + 0.7909 \times 9$

$\therefore\ x_{1990} = 5.6364 + 7.1181$

$\therefore\ x_{1990} = 12.7545$

Hence, trend value for the year 1990 is 12.7545.

9. Obtain the trend values for the above data using 3-yearly moving averages.

Solution :

We construct the following table to obtain 3-yearly moving averages for the data in problem 7 :

Year t	Production x_t	3-yearly moving total	3-yearly moving averages Trend value
1976	0	–	–
1977	4	8	2.6667
1978	4	10	3.3333
1979	2	12	4.0000
1980	6	16	5.3333
1981	8	19	6.3333
1982	5	22	7.3333
1983	9	18	6.0000
1984	4	23	7.6667
1985	10	24	8.0000
1986	10	–	–

10. The following table shows the production of gasoline in USA for the years 1962 to 1976 :

Year	Production (million barrels)	Year	Production (million barrels)
1962	0 *	1970	6
1963	0	1971	7
1964	1	1972	8
1965	1	1973	9
1966	2	1974	8
1967	3	1975	9
1968	4	1976	10
1969	5		

(i) Obtain the trend values using 5-yearly moving averages for the above data.

(ii) Plot the original time series and the trend values obtained in (i) on the same graph. (**Correction in text.*)

Solution :

(i) We construct the following table to obtain 5-yearly moving average :

Year t	Production (millions of barrels) x_t	5-yearly moving total	5-yearly moving averages Trend value
1962	0	–	–
1963	0	–	–
1964	1	4	0.8
1965	1	7	1.4
1966	2	11	2.2
1967	3	15	3.0
1968	4	20	4.0
1969	5	25	5.0
1970	6	30	6.0
1971	7	35	7.0
1972	8	38	7.6
1973	9	41	8.2
1974	8	44	8.8
1975	9	–	–
1976	10	–	–

(ii) Taking year on X-axis and production trend on Y-axis, we plot the points for production corresponding to years to get the graph of time series and plot the points for trend values corresponding to years to get the graph of trend as shown in the figure 4.4.

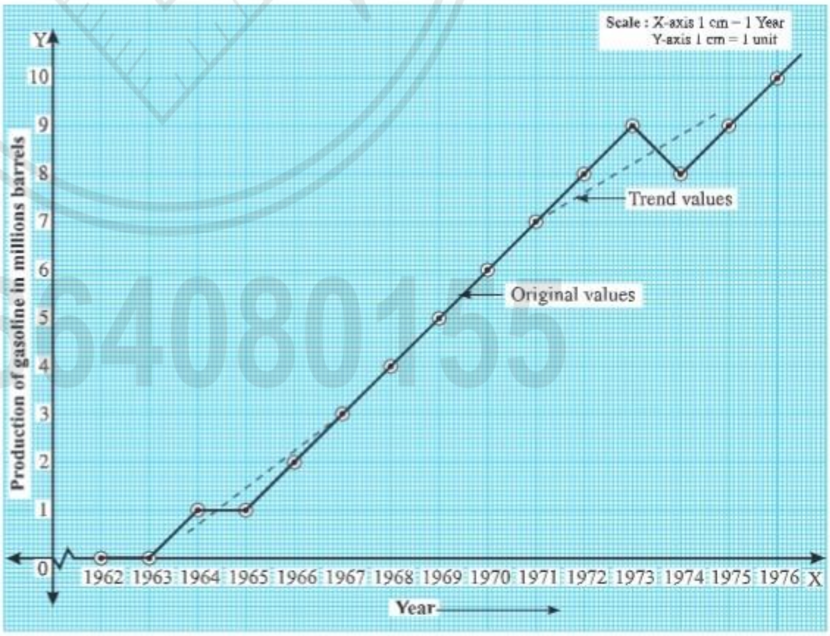


Fig. 4.4

EXAMPLES FOR PRACTICE 4.1

1. Find a trend line to the following data using the graphical method and interpret it :

Year	2008	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018
Production (in tons)	20	25	30	25	30	35	40	30	25	35	40

2. Find a trend line to the following data using the graphical method and interpret it :

Year	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014
No. of crimes (in '000)	48	45	50	48	40	45	45	40	35	30

3. Find a trend by 3 years moving averages for the following data :

Year	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015
Index number	112	104	108	121	116	111	133	125	129	139	131

4. The cost of living index numbers for different months of the year 2015–'16 are given below. Find a trend using 5 monthly moving averages :

Year/month	**Cost of living Index number**	**Year/month**	**Cost of living Index number**
2015 April	200	2015 November	218
May	194	December	196
June	181	2016 January	201
July	178	February	203
August	202	March	191
September	247	April	189
October	258	May	203

5. Find a trend value for the following time series using 4 yearly moving average :

Year	**Profit (in '000 ₹)**	**Year**	**Profit (in '000 ₹)**
2005	406	2011	1016
2006	520	2012	638
2007	936	2013	563
2008	573	2014	677
2009	488	2015	1089
2010	596	2016	718

6. Fit a straight line trend to the following data by the method of least squares. Also obtain the trend values for the given time series :

Year	2013	2014	2015	2016	2017
Production (in '000 tons)	40	50	62	58	60

7. Fit a straight line trend to the following data by method of least squares :

Year	2011	2012	2013	2014	2015	2016	2017	2018
Profit (in lakh ₹)	38	40	65	72	69	60	87	95

Also obtain the trend value for the year 2020.

Answer

1. Increasing trend **2.** Decreasing trend

3. —, 108, 111, 115, 116, 120, 123, 129, 131, 133, —

4. —, —, 191.0, 200.4, 213.2, 220.6, 224.2, 224.0, 215.2, 201.8, 196.0, 197.4, —, —.

5. —, —, 619.00, 638.75, 658.25, 676.37, 693.87, 713.37, 732.62, 751.75, —, —.

6. $x = 54 + 4.84$, $u = t - 2015$

Trend values : 44.4, 49.2, 54.0, 58.8, 63.6

7. $x = 65.75 + 3.7u$, $u = 2(t - 2014)5$

Trend for the year 2020 : 106.12.

MISCELLANEOUS EXERCISE – 4

(Textbook pages 67 to 70)

I. Choose the correct alternative :

1. Which of the following can't be a component of a time series?
   (a) Seasonality (b) Cyclical
   (c) Trend (d) Mean.

2. The first step in time series analysis is to
   (a) perform regression calculations
   (b) calculate a moving average
   (c) plot the data on a graph
   (d) identify seasonal variation.

3. Time series analysis is based on the assumption that
   (a) random error terms are normally distributed.
   (b) the variable to be forecast and other independent variables are correlated.
   (c) past patterns in the variable to be forecast will continue unchanged into the future.
   (d) the data do not exhibit a trend.

4. Moving averages are useful in identifying
   (a) seasonal component
   (b) irregular component
   (c) trend component
   (d) cyclical component.

5. We can use regression line for past data to forecast future data. We then use the line which
   (a) minimizes the sum of squared deviations of past data from the line
   (b) minimizes the sum of deviations of past data from the line
   (c) maximizes the sum of squared deviations of past data from the line
   (d) maximizes the sum of deviations of past data from the line.

6. Which of the following is a major problem for forecasting, especially when using the method of least squares?
   (a) The past cannot be known
   (b) The future is not entirely certain
   (c) The future exactly follows the patterns of the past
   (d) The future may not follow the patterns of the past.

7. An overall upward or downward pattern in an annual time series would be contained in which component of the times series
   (a) trend (b) cyclical
   (c) irregular (d) seasonal.

8. The following trend line equation was developed for annual sales from 1984 to 1990 with 1984 as base or zero year.
   $Y1 = 500 + 60X$ (in ₹ 1000). The estimated sales for 1984 (in ₹ 1000) is
   (a) ₹ 500 (b) ₹ 560
   (c) ₹ 1040 (d) ₹ 1100.

9. What is a disadvantage of the graphical method of determining a trend line?
   (a) Provides quick approximations
   (b) Is subject to human error
   (c) Provides accurate forecasts
   (d) Is too difficult to calculate.

10. Which component of time series refers to erratic time series movements that follow no recognizable or regular pattern?
    (a) Trend (b) Seasonal
    (c) Cyclical (d) Irregular.

Answers

1. (d) Mean
2. (c) plot the data on a graph
3. (c) past patterns in the variable to be forecast will continue unchanged into the future.
4. (c) trend component
5. (a) minimizes the sum of squared deviations of past data from the line
6. (d) The future may not follow the patterns of the past
7. (a) trend
8. (a) ₹ 500
9. (b) Is subject to human error
10. (a) Trend.

II. Fill in the blanks :

1. ……… component of time series is indicated by a smooth line.
2. ……… component of time series is indicated by periodic variation year after year.
3. ……… component of time series is indicated by a long wave spanning two or more years.
4. ……… component of time series is indicated by up and down movements without any pattern.
5. Additive models of time series ……… independence of its components.
6. Multiplicative models of time series ……… independence of its components.
7. The simplest method of measuring trend of time series is ………
8. The method of measuring trend of time series using only averages is ………
9. The complicated but efficient method of measuring trend of time series is ………
10. The graph of time series clearly shows ……… of it is monotone.

Answers

1. Trend 2. Seasonal 3. Cyclical 4. Irregular
5. assume 6. does not assume 7. Graphical
8. moving average 9. least square 10. trend.

III. State whether each of the following is *True* or *False* :

1. The secular trend component of time series represents irregular variations.
2. Seasonal variation can be observed over several years.
3. Cyclical variation can occur several times in a year.
4. Irregular variation is not a random component of time series.
5. Additive model of time series does not require the assumption of independence of its components.
6. Multiplicative model of time series does not require the assumption of independence of its components.
7. Graphical method of finding trend is very complicated and involves several calculations.
8. Moving average method of finding trend is very complicated and involves several calculations.
9. Least squares method of finding trend is very simple and does not involve any calculations.
10. All the three methods of measuring trend will always give the same results.

Answers

1. False 2. True 3. False 4. False 5. False
6. True 7. False 8. False 9. False 10. False.

IV. Solve the following problems :

1. The following table shows the production of pig-iron and ferro-alloys ('000 metric tonnes) :

Year	1974	1975	1976	1977	1978	1979	1980	1981	1982
Production	0	4	9	9	8	5	4	8	10

Fit a trend line to the above data by graphical method.

Solution :

Taking year on X-axis and production on Y-axis, we plot the points for production corresponding to years. Joining these points, we get the graph of time series. We fit a trend line as shown in the figure 4.5.

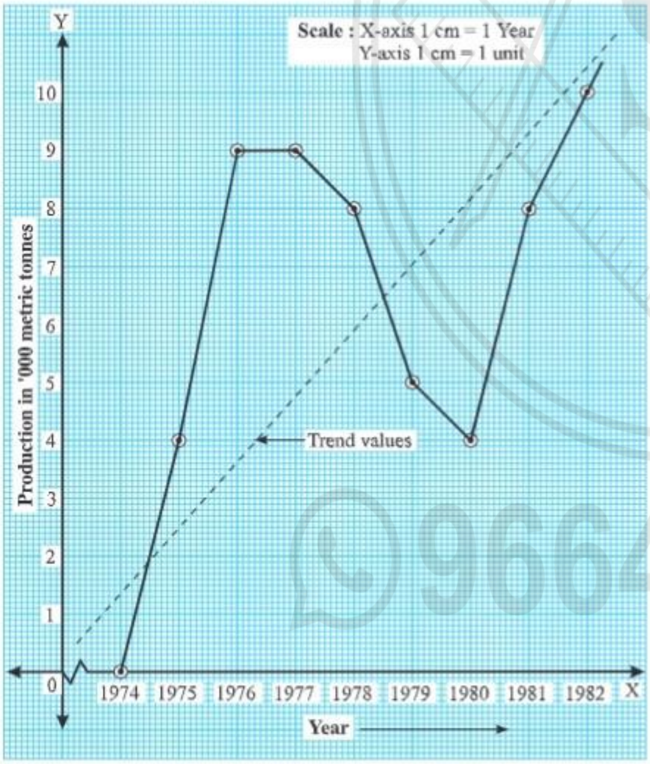


Fig. 4.5

2. Fit a trend line to the data in Problem IV (1) by the method of least squares.

Solution :

Here, $n = 9$. We transform year t to u by taking $u = t - 1978$.

We construct the following table for calculation :

Year t	Production x_t	u $=(t-1978)$	u^2	ux_t
1974	0	-4	16	0
1975	4	-3	9	-12
1976	9	-2	4	-18
1977	9	-1	1	-9
1978	8	0	0	0
1979	5	1	1	5
1980	4	2	4	8
1981	8	3	9	24
1982	10	4	16	40
Total	$\Sigma x_t = 57$	$\Sigma u = 0$	$\Sigma u^2 = 60$	77 -39 $\Sigma ux_t = 38$

The equation of trend line is $x_t = a' + b'u$.

The normal equations are

$$\Sigma x_t = na' + b' \Sigma u \quad \ldots (1)$$

$$\Sigma ux_t = a' \Sigma u + b' \Sigma u^2 \quad \ldots (2)$$

Here, $n = 9$, $\Sigma x_t = 57$, $\Sigma u = 0$, $\Sigma u^2 = 60$, $\Sigma ux_t = 38$.

Putting these values in normal equations, we get

$$57 = 9a' + b'(0) \quad \ldots (3)$$

$$38 = a'(0) + b'(60) \quad \ldots (4)$$

From equation (3), we get

$$a' = \frac{57}{9} = 6.3333$$

From equation (4), we get

$$b' = \frac{38}{60} = 0.6333$$

Put $a' = 6.3333$ and $b' = 0.6333$ in $x_t = a' + b'u$, we get the equation of trend line as

$x_t = 6.3333 + 0.6333u$, where $u = (t - 1978)$.

3. Obtain trend values for data in Problem IV (1) using 5-yearly moving averages.

Solution :

We construct the following table to obtain 5-yearly moving averages for the data in Problem 1 :

Year t	Production x_t	5-yearly moving total	5-yearly moving averages Trend value
1974	0	–	–
1975	4	–	–
1976	9	30	6.0
1977	9	35	7.0
1978	8	35	7.0
1979	5	34	6.8
1980	4	35	7.0
1981	8	–	–
1982	10	–	–

4. Following table shows the amount of sugar production (in lakh tonnes) for the years 1971 to 1982 :

Year	Production
1971	1
1972	0
1973	1
1974	2
1975	3
1976	2
1977	3
1978	6
1979	5
1980	1
1981	4
1982	10

Fit a trend line to the above data by graphical method.

Solution :

Taking year on X-axis and production on Y-axis, we plot the points for production corresponding to years. Joining these points we get the graph of time series. We fit a trend line as shown in the figure 4.6.

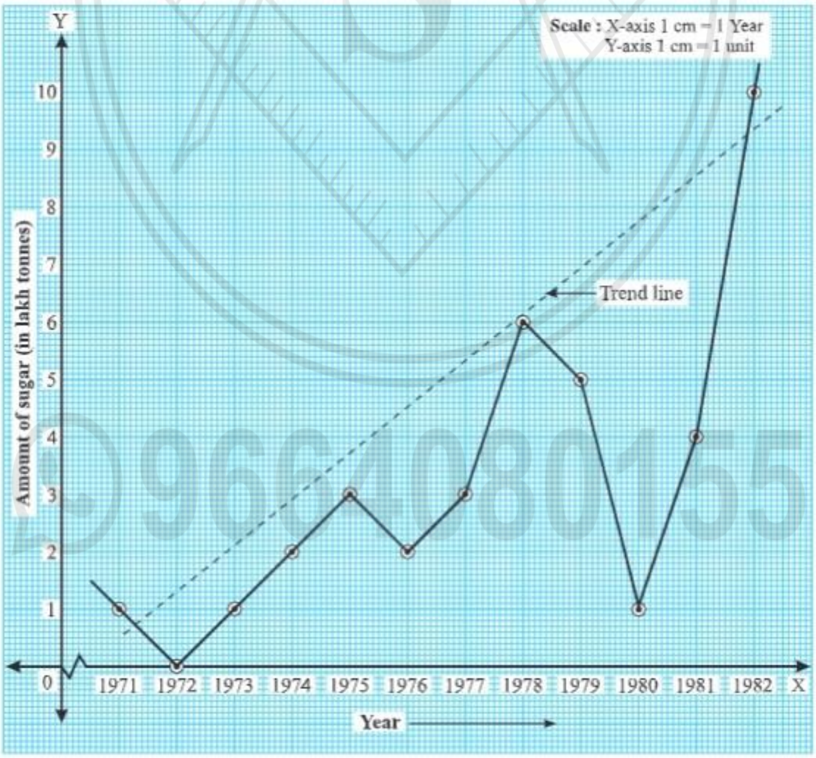


Fig. 4.6

5. Fit a trend line to data in Problem 4 by the method of least squares.

Solution :

Here, $n = 12$. We transform year t to u by taking $u = 2(t - 1976.5)$.

We construct the following table for calculation :

Year t	Production x_t	$u = 2(t - 1976.5)$	u^2	ux_t
1971	1	-11	121	-11
1972	0	-9	81	0
1973	1	-7	49	-7
1974	2	-5	25	-10
1975	3	-3	9	-9
1976	2	-1	1	-2
1977	3	1	1	3
1978	6	3	9	18
1979	5	5	25	25
1980	1	7	49	7
1981	4	9	81	36
1982	10	11	121	110
Total	$\Sigma x_t = 38$	$\Sigma u = 0$	$\Sigma u^2 = 572$	199 -39 $\Sigma ux_t = 160$

The equation of trend line is $x_t = a' + b'u$.

The normal equations are

$$\Sigma x_t = na' + b' \Sigma u \quad \ldots (1)$$

$$\Sigma ux_t = a' \Sigma u + b' \Sigma u^2 \quad \ldots (2)$$

Here, $n = 12$, $\Sigma x_t = 38$, $\Sigma u = 0$, $\Sigma u^2 = 572$.

$\Sigma ux_t = 160$

Putting these values in normal equations, we get

$$38 = 12a' + b'(0) \quad \ldots (3)$$

$$160 = a'(0) + b'(572) \quad \ldots (4)$$

From equation (3), we get

$$a' = \frac{38}{12} = 3.1667$$

From equation (4), we get

$$b' = \frac{160}{572} = 0.2797$$

Putting $a' = 3.1667$ and $b' = 0.2797$ in $y_t = a' + b'(u)$, we get the equation of trend line as $x_t = 3.1667 + 0.2797u$, where $u = 2(t - 1976.5)$

6. Obtain trend values for data in Problem 4 using 4-yearly centered moving averages.

Solution : We construct the following table to obtain 4-yearly centred moving average for the data in Problem 4 :

Yeast t	Production x_t	4-yearly moving total	4-yearly moving average	2-unit moving total	4-yearly centred moving average Trend value
1971	1	–	–	–	–
1972	0	4	1.0	–	–
1973	1	6	1.5	2.5	1.25
1974	2	8	2.0	3.5	1.75
1975	3	10	2.5	4.5	2.25
1976	2	14	3.5	6.0	3.00
1977	3	16	4.0	7.5	3.75
1978	6	15	3.75	7.75	3.75
1979	5	16	4.0	7.75	3.75
1980	1	20	5.0	9.00	4.50
1981	4	–	–	–	–
1982	10	–	–	–	–

[**Note :** *Answer given in the textbook is incorrect.*]

7. The percentage of girls' enrollment to total enrollment for years 1960–2005 is shown in the following table :

Year	1960	1965	1970	1975	1980
Percentage of girls' enrollment	0	3	3	4	4
Year	1985	1990	1995	2000	2005
Percentage of girls' enrollment	5	6	8	8	10

Fit a trend line by graphical method to the above data.

Solution :

Taking year on X-axis and Percentage of enrollment on Y-axis, we plot the points for enrollment corresponding to years. Joining these points, we get the graph of time series. We fit the trend line as shown in the figure 4.7.

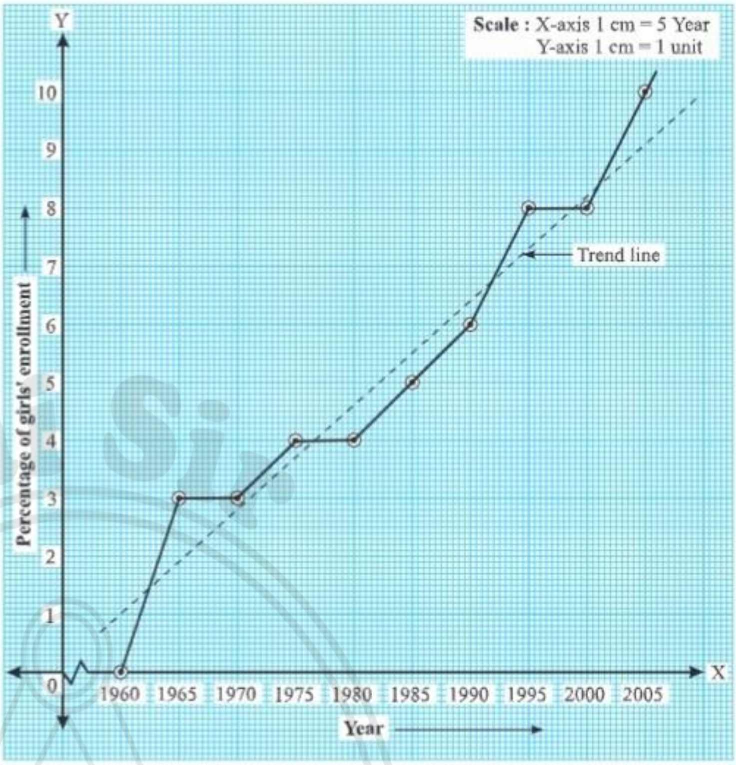


Fig. 4.7

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8. Fit a trend line to the data in Problem 7 by the method of least squares.

Solution :

Here, $n = 10$ and the interval between the year t is 5. We transform t to u by taking $u = \frac{2(t - 1982.5)}{5}$

We construct the following table for calculation :

Year t	Percentage of enrollment x_t	$u = \frac{2(t-1982.5)}{5}$	u^2	ux_t
1960	0	-9	81	0
1965	3	-7	49	-21
1970	3	-5	25	-15
1975	4	-3	9	-12
1980	4	-1	1	-4
1985	5	1	1	5
1990	6	3	9	18
1995	8	5	25	40
2000	8	7	49	56
2005	10	9	81	90
Total	$\Sigma x_t = 51$	$\Sigma u = 0$	$\Sigma u^2 = 330$	209 $-$ 52 $\Sigma ux_t = 157$

The equation of trend line is $x_t = a' + b'u$.

Two normal equations are

$\Sigma x_t = na' + b'\Sigma u$ ... (1)

$\Sigma u x_t = a'\Sigma u + b'\Sigma u^2$ ... (2)

Here, $n = 10$, $\Sigma x_t = 51$, $\Sigma u = 0$, $\Sigma u^2 = 330$, $\Sigma u x_t = 157$

Putting these values in normal equations, we get

$51 = 10a' + b'(0)$ ... (3)

$157 = a'(0) + b'(330)$ ... (4)

From equation (3), we get

$$a' = \frac{51}{10} = 5.1$$

From equation (4), we get

$$b' = \frac{157}{330} = 0.4758$$

Putting $a' = 5.1$ and $b' = 0.4758$ in $x_t = a' + b'u$, we get the equation of trend line as

$x_t = 5.1 + 0.4758\,u$,

where $u = \dfrac{2(t - 1982.5)}{5}$.

9. Obtain trend values for the data in Problem 7 using 4-yearly moving averages.

Solution :

We construct the following table to obtain 4-yearly centred moving averages for the data in Problem 7 :

Years t	Percentage of enrolment x_t	4-yearly moving total	4-yearly moving averages	2-unit moving total	4-yearly centred moving averages Trend value
1960	0	–	–	–	–
1965	3	10	2.50	–	–
1970	3	14	3.50	6.00	3.00
1975	4	16	4.00	7.50	3.75
1980	4	19	4.75	8.75	4.375
1985	5	23	5.75	10.5	5.25
1990	6	27	6.75	12.5	6.25
1995	8	32	8.00	14.75	7.375
2000	8	–	–	–	–
2005	10	–	–	–	–

[**Note :** *Answer given in the textbook is incorrect.*]

10. Following data shows the number of boxes of cereal sold in years 1977 to 1984 :

Year	1977	1978	1979	1980
Number of boxes (in ten thousands)	1	0	3	8
Year	**1981**	**1982**	**1983**	**1984**
Number of boxes (in ten thousands)	10	4	5	8

Fit a trend line to the above data by graphical method.

Solution :

Taking year on X-axis and number of boxes on Y-axis, we plot the points for number of boxes corresponding to years. Joining these points we get the graph of time series. We fit the trend line as shown in the figure 4.8.

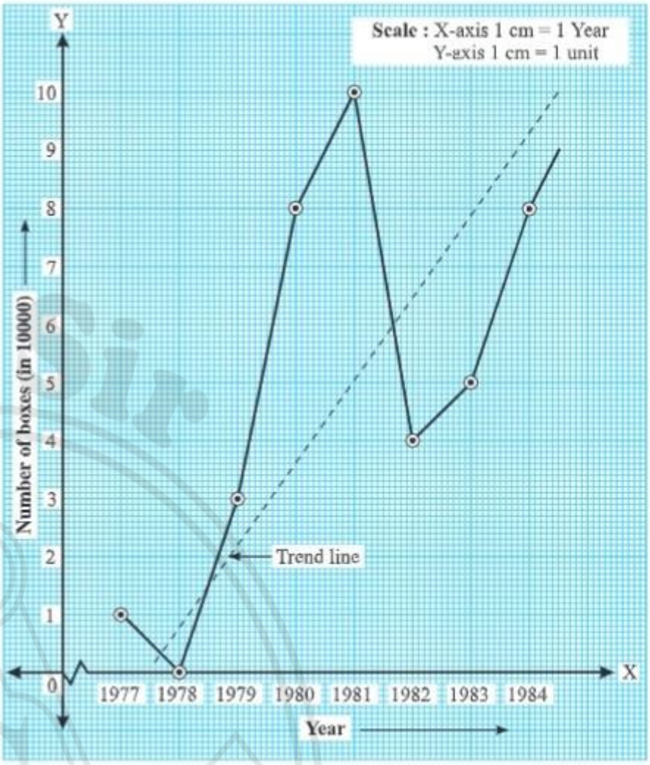

Fig. 4.8

11. Fit a trend line to data in Problem 10 by the method of least squares.

Solution :

Here, $n = 8$. We transfer year t to u, by taking $u = 2(t - 1980.5)$

We construct the following table for calculation :

Year t	Number of boxes (in '000) x_t	$u = 2(t - 1980.5)$	u^2	ux_t
1977	1	-7	49	-7
1978	0	-5	25	0
1979	3	-3	9	-9
1980	8	-1	1	-8
1981	10	1	1	10
1982	4	3	9	12
1983	5	5	25	25
1984	8	7	49	56
Total	$\Sigma x_t = 39$	$\Sigma u = 0$	$\Sigma u^2 = 168$	103 $-$ 24 $\Sigma ux_t = 79$

The equation of trend line is $x_t = a' + b'u$.

The normal equations are,

$\Sigma x_t = na' + b' \Sigma u$ … (1)

$\Sigma u x_t = a' \Sigma u + b' \Sigma u^2$ … (2)

Here, $n = 8$, $\Sigma x_t = 39$, $\Sigma u = 0$, $\Sigma u^2 = 168$, $\Sigma u x_t = 79$

Putting these values in normal equations, we get

$39 = 8a' + b'(0)$ … (3)

$79 = a'(0) + b'(168)$ … (4)

From equation (3), we get

$$a' = \frac{39}{8} = 4.8750$$

From equation (4), we get

$$b' = \frac{79}{168} = 0.4702$$

Putting $a' = 4.8750$, $b' = 0.4702$ in $x_t = a' + b'u$, we get the equation of trend line as

$x_t = 4.8750 + 0.4702\,u$, where $u = 2(t - 1980.5)$.

12. Obtain trend values for the data in Problem 10 using 3-yearly moving averages.

Solution :

We construct the following table to obtain 3-yearly moving averages for the data in Problem 10 :

Year t	Production (in '000) x_t	3-yearly moving total	3-yearly moving averages Trend value
1977	1	–	–
1978	0	4	1.3333
1979	3	11	3.6667
1980	8	21	7.0000
1981	10	22	7.3333
1982	4	19	6.3333
1983	5	17	5.6667
1984	8	–	–

13. Following table shows the number of traffic fatalities (in a state) resulting from drunken driving from years 1975 to 1983 :

Year	1975	1976	1977	1978	1979	1980	1981	1982	1983
Number of deaths	0	6	3	8	2	9	4	5	10

Fit a trend line to the above data by graphical method.

Solution :

Taking year on X-axis and number of deaths on Y-axis, we plot the points for number of deaths corresponding to years. Joining these points we get the graph of time series. We fit the trend line as shown in the figure 4.9.

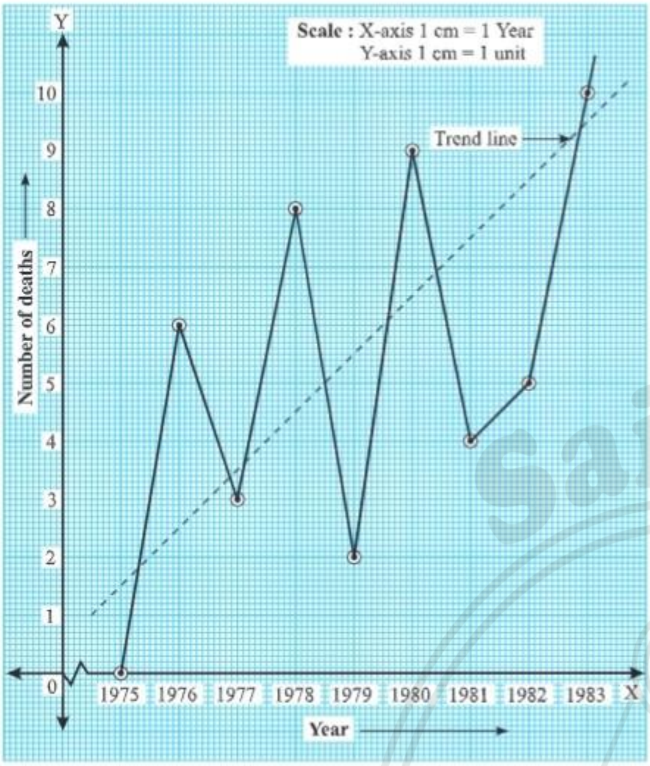

Fig. 4.9

14. Fit a trend line to data in Problem 13 by the method of least squares.

Solution : Here, $n = 9$. We transform year t to u by taking $u = t - 1979$.

We construct the following table for calculation :

Year t	Number of deaths x_t	$u = t - 1979$	u^2	ux_t
1975	0	-4	16	0
1976	6	-3	9	-18
1977	3	-2	4	-6
1978	8	-1	1	-8
1979	2	0	0	0
1980	9	1	1	9
1981	4	2	4	8
1982	5	3	9	15
1983	10	4	16	40
Total	$\Sigma x_t = 47$	$\Sigma u = 0$	$\Sigma u^2 = 60$	72 -32 $\Sigma ux_t = 40$

The equation of trend line is $x_t = a' + b'u$.

The normal equations are,

$$\Sigma x_t = na' + b' \Sigma u \quad \dots (1)$$

$$\Sigma ux_t = a' \Sigma u + b' \Sigma u^2 \quad \dots (2)$$

Here, $n = 9$, $\Sigma x_t = 47$, $\Sigma u = 0$, $\Sigma u^2 = 60$, $\Sigma ux_t = 40$.

Putting these values in normal equations, we get

$$47 = 9a' + b'(0) \quad \dots (3)$$

$$40 = a'(0) + b'(60) \quad \dots (4)$$

From equation (3), we get

$$a' = \frac{47}{9} = 5.2222$$

From equation (4), we get

$$b' = \frac{40}{60} = 0.6667$$

Putting $a' = 5.2222$ and $b' = 0.6667$ in $x_t = a' + b'u$, we get the equation of trend line as $x_t = 5.2222 + 0.6667u$,

where $u = (t - 1979)$.

15. Obtain trend values for data in Problem 13 using 4-yearly moving averages.

Solution :

We construct the following table to obtain 4-yearly moving averages for the data in Problem 13 :

Year t	Number of deaths x_t	4-yearly moving total	4-yearly moving averages
1975	0	–	–
1976	6	–	–
1977	3	17	4.25
1978	8	19	4.75
1979	2	22	5.50
1980	9	23	5.75
1981	4	20	5.00
1982	5	28	7.00
1983	10	–	–

16. Following table shows the all India infant graphical rates (per '000) for years 1980 to 2010 :

Year	1980	1985	1990	1995
IMR	10	7	5	4
Year	2000	2005	2010	
IMR	3	1	0	

Fit a trend line to the above data by graphical method.

Solution :

Taking year on X-axis and mortality rate on Y-axis, we put the points for mortality rate corresponding to years.

Joining these points we get the graph of time series. We fit the trend line as shown in the figure 4.10.

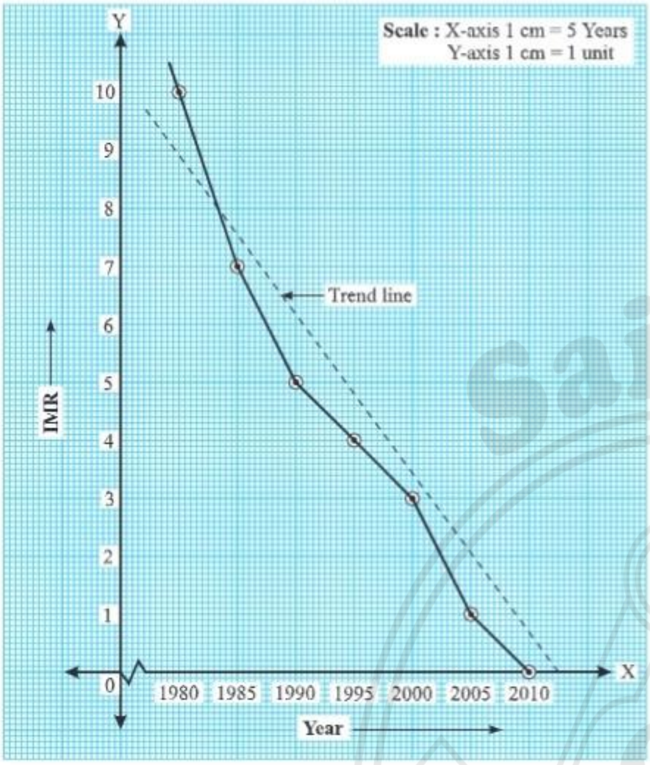

Fig. 4.10

17. Fit a trend line to data in Problem 16 by the method of least squares.

Solution :

Here, $n = 7$ and interval between the year t is 5.

We transform year t to u by taking $u = \frac{t - 1995}{5}$.

We construct the following table for calculation :

Year t	Infant Mortality Rate x_t	$u = \frac{(t-1995)}{5}$	u^2	ux_t
1980	10	-3	9	-30
1985	7	-2	4	-14
1990	5	-1	1	-5
1995	4	0	0	0
2000	3	1	1	3
2005	1	2	4	2
2010	0	3	9	0
Total	$\Sigma x_t = 30$	$\Sigma u = 0$	$\Sigma u^2 = 28$	5 -49 $\Sigma ux_t = -44$

The equation of trend line is $x_t = a' + b'u$.

The normal equations are,

$$\Sigma x_t = na' + b' \Sigma u \quad \dots (1)$$

$$\Sigma ux_t = a' \Sigma u + b' \Sigma u^2 \quad \dots (2)$$

Here, $n = 7$, $\Sigma x_t = 30$, $\Sigma u = 0$, $\Sigma u^2 = 28$, $\Sigma u x_t = -44$

Putting these values in normal equations, we get

$$30 = 7a' + b'(0) \quad \dots (3)$$

$$-44 = a'(0) + b'(28) \quad \dots (4)$$

From equation (3), we get

$$a' = \frac{30}{7} = 4.286$$

From equation (4), we get

$$b' = \frac{-44}{28} = -1.571$$

Putting $a' = 4.286$ and $b' = -1.571$ in $x_t = a' + b'u$, we get the equation of trend line as

$$x_t = 4.286 - 1.571\,u, \text{ where } u = \frac{t - 1995}{5}.$$

18. Obtain trend values for data in Problem 16 using 3-yearly moving averages.

Solution :

We construct the following table to obtain 3-yearly moving averages for the data in Problem 16 :

Year t	Infant Mortality Rate x_t	3-yearly moving total	3-yearly moving averages Trend value
1980	10	–	–
1985	7	22	7.3333
1990	5	16	5.3333
1995	4	12	4.0000
2000	3	8	2.6667
2005	1	4	1.3333
2010	0	–	–

19. Following table shows the wheat yield (in '000 tonnes) in India for years 1959 to 1968 :

Year	Yield
1959	0
1960	1
1961	2
1962	3
1963	1

Year	Yield
1964	0
1965	4
1966	1
1967	2
1968	10

Fit a trend line to the above data by the method of least squares.

Solution :

Here, $n = 10$, we transform year t to u by taking $u = 2(t - 1963.5)$. We construct the following table for calculation :

Year t	Yield (in '000 tonnes) x_t	$u = 2(t - 1963.5)$	u^2	ux_t
1959	0	−9	81	0
1960	1	−7	49	−7
1961	2	−5	25	−10
1962	3	−3	9	−9
1963	1	−1	1	−1
1964	0	1	1	0
1965	4	3	9	12
1966	1	5	25	5
1967	2	7	49	14
1968	10	9	81	90
Total	$\Sigma x_t = 24$	$\Sigma u = 0$	$\Sigma u^2 = 330$	121 − 27 $\Sigma ux_t = 94$

The equation of trend line is $x_t = a' + b'u$.

The normal equations are,

$\Sigma x_t = na' + b'\Sigma u$ ... (1)

$\Sigma ux_t = a'\Sigma u + b'\Sigma u^2$ ... (2)

Here, $n = 10$, $\Sigma x_t = 24$, $\Sigma u = 0$, $\Sigma u^2 = 330$, $\Sigma ux_t = 94$

Putting these values in the normal equation, we get

$24 = 10a' + b'(0)$ ... (3)

$94 = a'(0) + b'(330)$ ... (4)

From equation (3), we get

$$a' = \frac{24}{10} = 2.4$$

From equation (4), we get

$$b' = \frac{94}{330} = 0.2848$$

Putting $a' = 2.4$ and $b' = 0.2848$ in $x_t = a' + b'u$, we get the equation of trend line as

$x_t = 2.4 + 0.2848\,u$, where $u = 2(t - 1963.5)$

[**Note :** *Answer given in the textbook is incorrect.*]

20. Obtain trend values for data in Problem 19 using 3-yearly moving averages.

Solution :

We construct the following table to obtain 3-yearly moving average for the data in Problem 19 :

Year t	Yield (in '000 tonnes) x_t	3-yearly moving total	3-yearly moving averages Trend value
1959	0	–	–
1960	1	3	1
1961	2	6	2
1962	3	6	2
1963	1	4	1.3333
1964	0	5	1.6667
1965	4	5	1.6667
1966	1	7	2.33
1967	2	13	4.3333
1968	10	–	–

[**Note :** *Answers given in the textbook are incorrect.*]

ACTIVITIES — Textbook pages 70 and 71

[**Note :** *You may change the origin and scale in the following problems according to your convenience.*]

1. Daily SENSEX index values at opening are given for fifty days in the following table. Plot a graph from the data. Find the trend graphically, using moving averages, and by the method of least squares.

Date	Index	Date	Index
1 Jan 19	36161.8	2 Jan 19	36198.13
3 Jan 19	35934.5	4 Jan 19	35590.79
7 Jan 19	35971.18	8 Jan 19	35964.62
9 Jan 19	36181.37	10 Jan 19	36258
11 Jan 19	36191.87	14 Jan 19	36113.27
15 Jan 19	35950.08	16 Jan 19	36370.74

Date	Index	Date	Index
17 Jan 19	36413.6	18 Jan 19	36417.58
21 Jan 19	36467.12	22 Jan 19	36649.92
23 Jan 19	36494.12	24 Jan 19	36146.55
25 Jan 19	36245.77	28 Jan 19	36099.62
29 Jan 19	35716.72	30 Jan 19	35819.67
31 Jan 19	35805.51	1 Feb 19	36311.74
4 Feb 19	36456.22	5 Feb 19	36573.04
6 Feb 19	36714.54	7 Feb 19	37026.56
8 Feb 19	36873.59	11 Feb 19	36585.5
12 Feb 19	36405.72	13 Feb 19	36279.63
14 Feb 19	36065.08	15 Feb 19	35985.68
18 Feb 19	35831.18	19 Feb 19	35543.24
20 Feb 19	35564.93	21 Feb 19	35837
22 Feb 19	35906.01	25 Feb 19	35983.8
26 Feb 19	35975.75	27 Feb 19	36138.83
28 Feb 19	36025.72	1 Mar 19	36018.49
5 Mar 19	36141.07	6 Mar 19	36544.86
7 Mar 19	36744.02	8 Mar 19	36753.59
9 Mar 19	36753.59	12 Mar 19	37249.65

Solution :

We take origin at 35000, we get the approximate value of index as follows :

Date	Index	Date	Index
1 Jan 19	1	2 Jan 19	1
3 Jan 19	1	4 Jan 19	1
7 Jan 19	1	8 Jan 19	1
9 Jan 19	1	10 Jan 19	1
11 Jan 19	1	14 Jan 19	1
15 Jan 19	1	16 Jan 19	1
17 Jan 19	1	18 Jan 19	1
21 Jan 19	1	22 Jan 19	2
23 Jan 19	1	24 Jan 19	1
25 Jan 19	1	28 Jan 19	1
29 Jan 19	1	30 Jan 19	1
31 Jan 19	1	1 Feb 19	1
4 Feb 19	2	5 Feb 19	2
6 Feb 19	2	7 Feb 19	2
8 Feb 19	2	11 Feb 19	2
12 Feb 19	1	13 Feb 19	1
14 Feb 19	1	15 Feb 19	1
18 Feb 19	1	19 Feb 19	0
20 Feb 19	0	21 Feb 19	1
22 Feb 19	1	25 Feb 19	1
26 Feb 19	1	27 Feb 19	1
28 Feb 19	1	1 Mar 19	1
5 Mar 19	1	6 Mar 19	0
7 Mar 19	2	8 Mar 19	2
9 Mar 19	2	12 Mar 19	2

We take the group of 5 dates we get the data as follows :

	1	2	3	4	5
Date group	1 Jan – 7 Jan	8 Jan – 14 Jan	15 Jan – 21 Jan	22 Jan – 28 Jan	29 Jan – 4 Feb
Index	5	5	5	6	6

	6	7	8	9	10
Date group	4 Feb – 11 Feb	12 Feb – 18 Feb	19 Feb – 25 Feb	26 Feb – 5 Mar	6 Mar – 12 Mar
Index	10	5	3	5	8

Take date group on X-axis and Index on Y-axis. We get the trend line as shown in the figure 4.11.

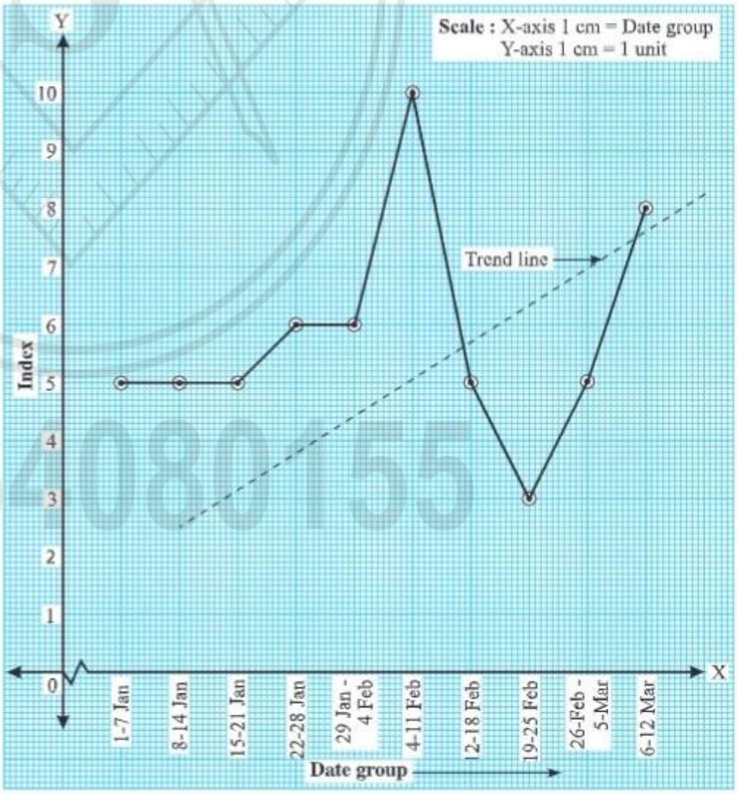

Fig. 4.11

Moving Average Method :

We find trend taking 5 dates moving average :

Date	Index	5 dates moving total	5 dates moving average trend
1 Jan 19	1	–	–
2 Jan 19	1	–	–
3 Jan 19	1	5	1
4 Jan 19	1	5	1
7 Jan 19	1	5	1
8 Jan 19	1	5	1
9 Jan 19	1	5	1
10 Jan 19	1	5	1
11 Jan 19	1	5	1
14 Jan 19	1	5	1
15 Jan 19	1	5	1
16 Jan 19	1	5	1
17 Jan 19	1	5	1
18 Jan 19	1	6	1.2
21 Jan 19	1	6	1.2
22 Jan 19	2	6	1.2
23 Jan 19	1	6	1.2
24 Jan 19	1	6	1.2
25 Jan 19	1	5	1
28 Jan 19	1	5	1
29 Jan 19	1	5	1
30 Jan 19	1	6	1.2
31 Jan 19	1	7	1.4
1 Feb 19	2	8	1.6
4 Feb 19	2	9	1.8
5 Feb 19	2	10	2
6 Feb 19	2	10	2
7 Feb 19	2	10	2
8 Feb 19	2	9	1.8
11 Feb 19	2	8	1.6
12 Feb 19	1	7	1.4
13 Feb 19	1	6	1.2
14 Feb 19	1	5	1
15 Feb 19	1	4	0.8
18 Feb 19	1	3	0.6
19 Feb 19	0	3	0.6
20 Feb 19	0	3	0.6
21 Feb 19	1	3	0.6
22 Feb 19	1	5	1
25 Feb 19	1	5	1
26 Feb 19	1	5	1
27 Feb 19	1	5	1.2
28 Feb 19	1	5	1
1 Mar 19	1	4	0.8
5 Mar 19	1	5	1
6 Mar 19	0	6	1.2
7 Mar 19	2	7	1.4
8 Mar 19	2	8	1.6
9 Mar 19	2	–	–
12 Mar 19	2	–	–

Least Square Method :

We consider the data of group of 5 dates.

Date groups	Index x_t	t	$u = t - 5.5$	u^2	$u \cdot x_t$
1 Jan– 7 Jan	5	1	-4.5	20.25	-22.5
8 Jan–14 Jan	5	2	-3.5	12.25	-17.5
15 Jan–21 Jan	5	3	-2.5	6.25	-12.5
22 Jan–28 Jan	6	4	-1.5	2.25	-9.0
29 Jan– 4 Feb	6	5	-0.5	0.25	-3.0
4 Feb–11 Feb	10	6	0.5	0.25	5.0
12 Feb–18 Feb	5	7	1.5	2.25	7.5
19 Feb–25 Feb	3	8	2.5	6.25	7.5
26 Feb– 5 Mar	5	9	3.5	12.25	17.5
6 Mar–12 Mar	8	10	4.5	20.25	36.0
Total	$\Sigma x_t = 58$	–	$\Sigma u = 0$	$\Sigma u^2 = 82.50$	73.5 -64.5 $\Sigma u x_t = 9.0$

The equation of trend line is $x_t = a' + b'u$.

The normal equations are,

$$\Sigma x_t = na' + b' \Sigma u \qquad \dots (1)$$

$$\Sigma u x_t = a' \Sigma u + b' \Sigma u^2 \qquad \dots (2)$$

Here, $n = 10$, $\Sigma x_t = 98$, $\Sigma u = 0$, $\Sigma u^2 = 82.50$, $\Sigma u x_t = 9.0$

Putting these values in normal equations, we get

$$58 = 10a' + b'(0) \qquad \therefore\ 58 = 10a' \qquad \therefore\ a' = \frac{58}{10} = 5.8$$

$$9 = a'(0) + b'(82.50) \qquad \therefore\ 9 = 82.50b' \qquad \therefore\ b' = \frac{9}{82.50} = 0.11$$

Putting $a' = 5.8$ and $b' = 0.11$ in equation $x_t = a' + b'u$, we get the equation of trend line as $x_t = 5.8 + 0.11\,(t - 5.5)$.

2. Onion prices (per quintal) in a market are given for fifteen days. Plot a graph of given data. Find the trend graphically, using moving averages, and by the method of least squares.

Date	Price
24/1/2019	400
31/1/2019	650
3/2/2019	650
7/2/2019	700
24/2/2019	550
28/2/2019	550
5/3/2019	500
7/3/2019	600

Date	Price
14/3/2019	600
28/3/2019	600
7/4/2019	800
11/4/2019	801
14/4/2019	800
25/4/2019	800
2/5/2019	800

Trend Graphically :

Take date on X-axis and Price on Y-axis. Plotting the price against date and joining the points we get the graph of the given data and we draw trend line from the graph as shown in the figure 4.12.

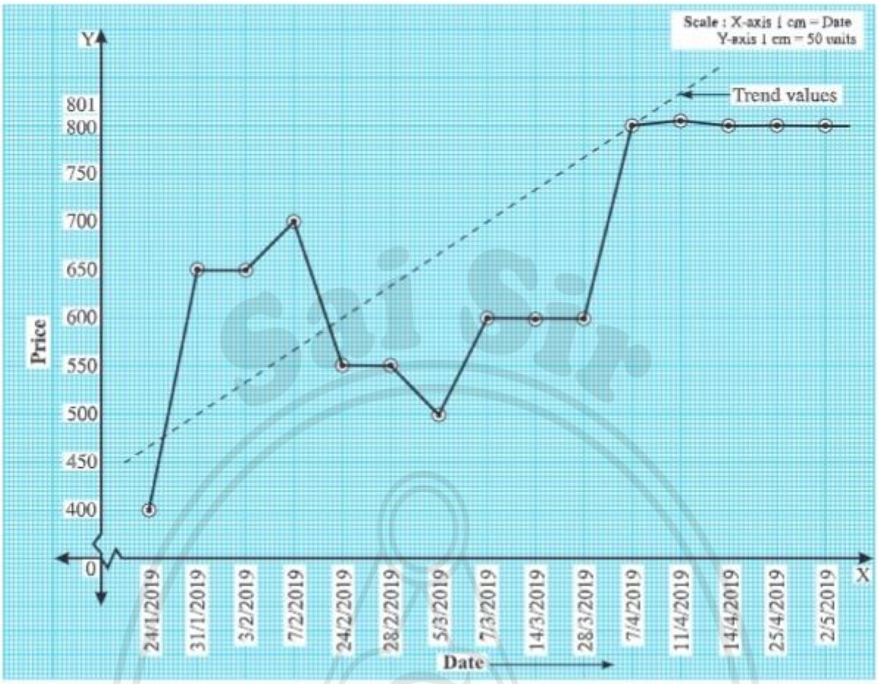

Fig. 4.12

Moving Averages Method :

We find trend by using 3-days moving averages :

Date	Price	3-days moving total	3-days moving averages Trend
24–1–2019	400	–	–
31–1–2019	650	1700	566.67
3–2–2019	650	2000	666.67
7–2–2019	700	1900	633.33
24–2–2019	550	1800	600.00
28–2–2019	550	1600	533.33
5–3–2019	500	1650	550.00
7–3–2019	600	1700	566.67
14–3–2019	600	1800	600.00
28–3–2019	600	2000	666.67
7–4–2019	800	2201	733.67
11–4–2019	801	2401	800.33
14–4–2019	800	2401	800.33
25–4–2019	800	2400	800.00
2–5–2019	800	–	–

Least Square Method :

Date	Price x_t	t	$u=t-8$	u^2	$u \cdot x_t$
24–1–2019	400	1	−7	49	−2800
31–1–2019	650	2	−6	36	−2100
3–2–2019	650	3	−5	25	−3250
7–2–2019	700	4	−4	16	−2800
24–2–2019	550	5	−3	9	−1650
28–2–2019	550	6	−2	4	−1100
5–3–2019	500	7	−1	1	−500
7–3–2019	600	8	0	0	0
14–3–2019	600	9	1	1	600
28–3–2019	600	10	2	4	1200
7–4–2019	800	11	3	9	2400
11–4–2019	801	12	4	16	3204
14–4–2019	800	13	5	25	4000
25–4–2019	800	14	6	36	4800
2–5–2019	800	15	7	49	5600
Total	$\Sigma x_t = 9801$	–	$\Sigma u = 0$	$\Sigma u^2 = 280$	21804 −14200 $\Sigma ux_t = 7604$

The equation of trend line is $x_t = a' + b'u$.

The normal equations are,

$\Sigma x_t = na' + b' \Sigma u$ … (1)

$\Sigma ux_t = a' \Sigma u + b' \Sigma u^2$ … (2)

Here, $n = 15$, $\Sigma x_t = 9801$, $\Sigma u = 0$, $\Sigma u^2 = 280$, $\Sigma u x_t = 7604$

Putting these values in normal equations, we get

$9801 = 15a' + b'(0)$ … (3)

$\therefore\ 9801 = 15a' \qquad \therefore\ a' = \frac{9801}{15} = 653.4$

$7604 = a'(0) + b'(280)$ … (4)

$\therefore\ 7604 = 280b' \qquad \therefore\ b' = \frac{7604}{280} = 27.16$

Putting $a' = 653.4$ and $b' = 27.16$ in equation $x_t = a' + b'u$, we get the equation of trend line as

$x_t = 653.4 + 27.16\,u$

Putting $u = t - 8$, we get $x_t = 653.4 + 27.16\ (t-8)$.

---

3. Following table gives the number of persons injured in road accidents for 11 years. Plot a graph from the data. Find the trend graphically, using moving averages, and by the method of least squares :

Year	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016
Number of injured	29955	27464	32224	32144	25400	28366	27185	26314	24488	23825	22072

Solution :

We take origin to 22000, so we get the number of injured approximately as follows :

Year	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016
Number of injured	8	5	10	10	3	6	5	4	2	1	0

Taking years on X-axis and number of injured on Y-axis, we get trend line as shown in the figure 4.13.

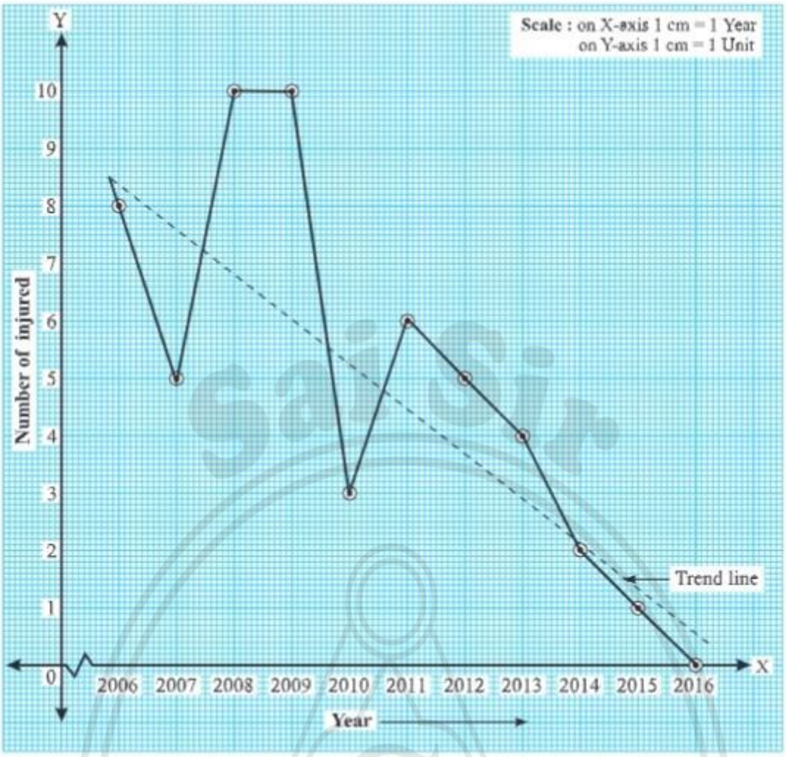

Fig. 4.13

Moving Averages Method :

We find trend by using 3 yearly moving averages :

Year	Number of injured x_t	3 yearly moving total	3 yearly moving averages trend
2006	8	–	–
2007	5	23	7.67
2008	10	25	8.33
2009	10	23	7.67
2010	3	19	6.33
2011	6	14	4.67
2012	5	15	5.00
2013	4	11	3.67
2014	2	7	2.33
2015	1	3	1.00
2016	0	–	–

Least Square Method :

Year	Number of injured x_t	t	$u = t - 6$	u^2	$u \cdot x_t$
2006	8	1	-5	25	-40
2007	5	2	-4	16	-20
2008	10	3	-3	9	-30
2009	10	4	-1	4	-20
2010	3	5	-1	1	-3
2011	6	6	0	0	0
2012	5	7	1	1	5
2013	4	8	2	4	8
2014	2	9	3	9	6
2015	1	10	4	16	4
2016	0	11	5	25	0
Total	$\Sigma x_t = 54$	–	$\Sigma u = 0$	$\Sigma u^2 =$ 110	23 -113 $\Sigma u x_t = -90$

The equation of trend is $x_t = a' + b'u$

The normal equations are,

$\Sigma x_t = na' + b' \Sigma u$ ... (1)

$\Sigma u x_t = a' \Sigma u + b' \Sigma u^2$ ... (2)

Here, $n = 11$, $\Sigma u = 0$, $\Sigma u^2 = 110$, $\Sigma x_t = 54$, $\Sigma u x_t = -90$

Putting these values in normal equations, we get

$54 = 11a' + b'(0)$ ... (3)

$\therefore\ 54 = 11a' \qquad \therefore\ a' = \dfrac{54}{11} = 4.91$

$-90 = a'(0) + b'(110)$ ... (4)

$\therefore\ -90 = 110b' \qquad \therefore\ b' = \dfrac{-90}{110} = -0.82$

Putting $a' = 4.91$ and $b' = -0.82$ in the equation $x_t = a' + b'u$, we get the equation of trend line as $x_t = 4.91 - 0.82\,u$

Putting $u = t - 6$, we get $x_t = 4.91 - 0.82\,(t - 6)$.

---

4. Following table gives the number of road accidents due to over-speeding in Maharashtra for 9 years. Plot a graph from the data. Find the trend graphically, using moving averages, and by the method of least squares :

Year	2008	2009	2010	2011	2012	2013	2014	2015	2016
Number of accidents	38680	18090	21238	28489	27054	26931	22925	24622	22071

Solution :

We take origin to 18000, we get approximate value of number of accidents as follows :

Year	2008	2009	2010	2011	2012	2013	2014	2015	2016
Number of accidents	21	0	3	10	9	9	5	7	4

Take year on X-axis and number of accidents on Y-axis, we get the trend line as shown in the figure 4.14.

Fig. 4.14

Moving Averages Method :

We find trend by using 3 years moving averages :

Year	Number of accidents x_t	3 years moving total	3 years moving averages trend
2008	21	–	–
2009	0	24	8.00
2010	3	13	4.33
2011	10	22	7.33
2012	9	28	9.33
2013	9	23	7.67
2014	5	21	7.00
2015	7	16	5.33
2016	4	–	–

Least Square Method :

Year	Number of accidents x_t	t	$u = t - 5$	u_2	$u \cdot x_t$
2008	21	1	-4	16	-84
2009	0	2	-3	9	0
2010	3	3	-2	4	-6
2011	10	4	-1	1	-10
2012	9	5	0	0	0
2013	9	6	1	1	9
2014	5	7	2	4	10
2015	7	8	3	9	21
2016	4	9	4	16	16
Total	$\Sigma x_t = 68$	–	$\Sigma u = 0$	$\Sigma u^2 =$ 60	56 -100 $\Sigma u x_t = -44$

The equation of trend is $x_t = a' + b'u$

The normal equations are,

$\Sigma x_t = na' + b' \Sigma u$ … (1)

$\Sigma u x_t = a' \Sigma u + b' \Sigma u^2$ … (2)

Here, $n = 9$, $\Sigma x_t = 68$, $\Sigma u = 0$, $\Sigma u^2 = 60$, $\Sigma u x_t = -44$

Putting these values in normal equations, we get

$68 = 9a' + b'(0)$ … (3)

$\therefore\ 68 = 9a' \qquad \therefore\ a' = \dfrac{68}{9} = 7.56$

$-44 = a'(0) + b'(60)$ … (4)

$\therefore\ -44 = 60b' \qquad \therefore\ b' = \dfrac{-44}{60} = -0.73$

Putting $a' = 7.56$ and $b' = -0.73$ in the equation $x_t = a' + b'u$, we get the equation of trend as

$x_t = 7.56 - 0.73\,u$

Putting $u = t - 5$, we get $x_t = 7.56 - 0.73\,(t - 5)$.

* * *

5 INDEX NUMBERS

CHAPTER OUTLINE

INTRODUCTION

Changes in the value of money are reflected in changes in general level of prices over a period of time. Also such changes in the value of money are found to be inversely related to changes in price level. So changes in the value of money can be understood well by observing changes in the general level of prices over a specified time period. Changes in the general level of prices are measured using a statistical tool known as Index Numbers.

Index numbers are not expressed in terms of any units of measurement as they are ratios usually expressed as percentage.

Index numbers are relative measures which indicate the changes over a specified period of time in (i) prices of commodities, (ii) industrial production, (iii) agricultural production, (iv) imports and exports, (v) prices of shares, (vi) demand and supply, etc. They are good indicators of inflationary or deflationary tendencies of the economy of a country. Thus, they are referred to as 'Economic Barometers' of the country. They are also useful for Government and Management people for taking the decisions and planning the policies to be implemented in future.

IMPORTANT FORMULAE

1. Simple Aggregate Method

(1) Price Index Number :

$$P_{01} = \frac{\Sigma p_1}{\Sigma p_0} \times 100,$$

p_1 = Price of commodity in the current year

p_0 = Price of commodity in the base year.

(2) Quantity Index Number :

$$Q_{01} = \frac{\Sigma q_1}{\Sigma q_0} \times 100,$$

where q_1 = Quantity of commodity consumed in the current year.

q_0 = Quantity of commodity consumed in the base year.

(3) Value Index Number :

$$V_{01} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_0} \times 100,$$

where p_1q_1 = Value of commodity in the current year.

p_0q_0 = Value of commodity in the base year.

2. Weighted Aggregate Method

$$P_{01} = \frac{\Sigma p_1 w}{\Sigma p_0 w} \times 100,$$

where w = weight assigned to commodity.

(1) Laspeyre's Price Index Number :

$$P_{01}(L)=\frac{\Sigma p_1q_0}{\Sigma p_0q_0}\times 100,$$

where base year's quantities (q_0) are weights.

(2) Paasche's Price Index Number :

$$P_{01}(P)=\frac{\Sigma p_1q_1}{\Sigma p_0q_1}\times 100,$$

where current year's quantities (q_1) are weights.

(3) Dorbish-Bowley's Price Index Number :

$$P_{01}(D\text{-}B)=\frac{\dfrac{\Sigma p_1q_0}{\Sigma p_0q_0}+\dfrac{\Sigma p_1q_1}{\Sigma p_0q_1}}{2}\times 100=\frac{P_{01}(L)+P_{01}(P)}{2}$$

Note : P_{01}(D-B) is the mean of P_{01}(L) and P_{01}(P).

(4) Fisher's Price Index Number :

$$P_{01}(F)=\sqrt{\frac{\Sigma p_1q_0}{\Sigma p_0q_0}\times\frac{\Sigma p_1q_1}{\Sigma p_0q_1}}\times 100=\sqrt{P_{01}(L)\times P_{01}(P)}$$

Note : P_{01}(F) is the geometric mean of P_{01}(L) and P_{01}(P).

(5) Marshall-Edgeworth's Price Index Number :

$$P_{01}(M\text{-}E)=\frac{\Sigma p_1(q_0+q_1)}{\Sigma p_0(q_0+q_1)}\times 100$$

$$=\frac{\Sigma p_1q_0+\Sigma p_1q_1}{\Sigma p_0q_0+\Sigma p_0q_1}\times 100$$

Note : In P_{01}(M-E), the mean of q_0 and q_1 is taken as weight.

(6) Walsch's Price Index Number :

$$P_{01}(W)=\frac{\Sigma p_1\sqrt{q_0q_1}}{\Sigma p_0\sqrt{q_0q_1}}\times 100$$

Note : In P_{01}(W), the geometric mean of q_0 and q_1 is taken as weight.

3. Cost of Living Index Number

(1) Aggregate Expenditure Method :

$$CLI=\frac{\Sigma p_1q_0}{\Sigma p_0q_0}\times 100$$

Note : This formula is equivalent to formula of Laspeyre's formula.

(2) Family Budget Method :

$$CLI=\frac{\Sigma IW}{\Sigma W},\ \text{where } I=\frac{p_1}{p_0}\times 100,\ W=p_0q_0$$

Note : Both the formulae are equivalent.

5.1 : DEFINITION OF INDEX NUMBERS

Index number can be defined as follows :

- "An index number is a statistical measure designed to show changes in a variable or a group of related variables with respect to time, geographic location or other characteristics such as production, income, etc."
- "An index number is a measure which measures the changes in some quantity which we cannot observe directly."
- Index number is a percentage ratio which measures the average change in several variables between two different times, places or situations.

[Notes :

- Index Number is an 'economic indicator of business activities.'
- "An Index Number is a numerical value characterizing the change in a complex economic phenomenon over a period of time." – **Maslow.**
- "An Index Number is a statistical measure designed to show changes in a variable or a group of related variables with respect to time, geographical location or some other characteristics." – **Spiegel.**
- "An Index Number is a measure designed to show an average change over time in the price, quantity or value of a group of items." – **Gregory and Ward.**
- "An Index Number is a device that measures differences in the magnitude of a group of related variables." – **Croxton and Cowden.**
- "An Index Number is a series that reflects in its trend and fluctuations the movements of some quantity to which it is related." – **B.L. Bowley.**
- "An Index Number is a special kind of average." – **Blair.**]

Examples of Index Numbers :

- **NIFTY 50** is the benchmark index of the National Stock Exchange (NSE) in India that represents the weighted average of top 50 Indian companies across 13 different sectors.
- **S & P BSE Sensex** is the benchmark index of the Bombay Stock Exchange (BSE) in India. It is the weighted average of 30 largest and most actively traded stocks of Indian companies on the BSE.

5.2 : TYPES OF INDEX NUMBERS

Index numbers are classified into the following three categories :

(1) **Price Index Number :** It is the most common index number which measures the general changes in the

prices of the goods. It is very good measure of inflation in the economy. It compares the prices of current year with the base year which represents relative variation.

(2) **Quantity Index Number :** It measures the changes in the quantity of goods produced, consumed, sold or purchased, etc. over the specified years. It is a good indication of the output of an economy.

(3) **Value Index Number :** It measures the combined effect of changes in prices as well as quantities. It takes product of price and quantity and measures the percentage change in the value of a commodity or a group of commodities.

5.3 : TERMINOLOGY AND NOTATIONS

1. Terminology :

(1) **Base Period :** The period with respect to which comparisons are made is called Base Period. It is denoted by suffix zero (0).

(2) **Current Period :** The period for which comparisons are required to be made is called 'Current Period'. It is denoted by suffix one (1).

Note : The period may be 'a day', 'a week', 'a month' or 'a year'.

2. Notations :

p_0 : The price of a commodity in the base year.

q_0 : Quantity of a commodity consumed during the base year.

p_1 : The price of a commodity in the current year.

q_1 : Quantity of a commodity consumed during the current year.

w : Weight assigned to a commodity according to its relative importance in the group of commodities.

I : Price relative. It is given by $I = \frac{p_1}{p_0} \times 100$. Simple Index Number.

P_{01} : Price Index Number for the current year with respect to base year.

Q_{01} : Quantity Index Number for the current year with respect to base year.

V_{01} : Value Index Number for the current year with respect to base year.

5.4 : CONSTRUCTION OF INDEX NUMBERS

There are two methods of constructing Index Numbers :

5.4.1 : Simple Aggregate Method

In this method every commodity is given equal importance.

(1) **Price Index Number :**

$$P_{01} = \frac{\Sigma p_1}{\Sigma p_0} \times 100$$

(2) **Quantity Index Number :**

$$Q_{01} = \frac{\Sigma q_1}{\Sigma q_0} \times 100$$

(3) **Value Index Number :**

Value = Price × Quantity

Base year's value $= p_0 q_0$

Current year's value $= p_1 q_1$

$$V_{01} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_0} \times 100$$

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Find the Price Index Number using Simple Aggregate Method in each of the following examples :

1. Use 1995 as base year in the following problem :

Commodity	P	Q	R	S	T
Price (in ₹) in 1995	15	20	24	22	28
Price (in ₹) in 2000	27	38	32	40	45

Solution :

Here, Base year = 1995

∴ p_0 = Price in the year 1995 and

p_1 = Price in the year 2000.

Commodity	Price (in ₹)	
	p_0	p_1
P	15	27
Q	20	38
R	24	32
S	22	40
T	28	45
Total	$\Sigma p_0 = 109$	$\Sigma p_1 = 182$

Price Index Number by Simple Aggregate Method :

$$P_{01} = \frac{\Sigma p_1}{\Sigma p_0} \times 100$$

$$= \frac{182}{109} \times 100$$

$$= 1.6697 \times 100$$

$= 166.97$

Hence, price index number is 166.97.

---

2. Use 1995 as base year in the following problem :

Commodity	A	B	C	D	E
Price (in ₹) in 1995	42	30	54	70	120
Price (in ₹) in 2005	60	55	74	110	140

Solution :

Here, Base year = 1995

$\therefore\ p_0$ = Price in the year 1995 and

p_1 = Price in the year 2005.

Commodity	Price (in ₹)	
	p_0	p_1
A	42	60
B	30	55
C	54	74
D	70	110
E	120	140
Total	$\Sigma p_0 = 316$	$\Sigma p_1 = 439$

Price Index Number by Simple Aggregate Method :

$$P_{01} = \frac{\Sigma p_1}{\Sigma p_0} \times 100$$
$$= \frac{439}{316} \times 100$$
$$= 1.3892 \times 100$$
$$= 138.92$$

Hence, price index number is 138.92.

3.

Commodity	Units	Base Year Price (in ₹)	Current Year Price (in ₹)
Wheat	kg	28	36
Rice	kg	40	56
Milk	litre	35	45
Clothing	metre	82	104
Fuel	litre	58	72

Solution :

Here, p_0 = Price in Base year

p_1 = Price in current year.

Commodity	Unit	Price (in ₹)	
		p_0	p_1
Wheat	kg	28	36
Rice	kg	40	56
Milk	litre	35	45
Clothing	metre	82	104
Fuel	litre	58	72
Total		$\Sigma p_0 = 243$	$\Sigma p_1 = 313$

Price Index Number by Simple Aggregate Method :

$$P_{01} = \frac{\Sigma p_1}{\Sigma p_0} \times 100$$
$$= \frac{313}{243} \times 100$$
$$= 1.2881 \times 100$$
$$= 128.81$$

Hence, price index number is 128.81.

---

4. Use 2000 as base year in the following problem :

Commodity	Price (in ₹) for year 2000	Price (in ₹) for year 2006
Watch	900	1475
Shoes	1760	2300
Sunglasses	600	1040
Mobiles	4500	8500

Solution :

Here, Base year = 2000

$\therefore\ p_0$ = Price in the year 2000 and

p_1 = Price in the year 2006.

Commodity	Price (in ₹)	
	p_0	p_1
Watch	900	1475
Shoes	1760	2300
Sunglasses	600	1040
Mobiles	4500	8500
Total	$\Sigma p_0 = 7760$	$\Sigma p_1 = 13315$

Price Index Number by Simple Aggregate Method :

$$P_{01} = \frac{\Sigma p_1}{\Sigma p_0} \times 100$$
$$= \frac{13315}{7760} \times 100$$
$$= 1.7159 \times 100 = 171.59$$

Hence, price index number is 171.59.

5. Use 1990 as base year in the following problem :

Commodity	Unit	Price in 1990 (in ₹)	Price in 1997 (in ₹)
Butter	kg	27	33
Cheese	kg	30	36
Milk	litre	25	29
Bread	loaf	10	14
Eggs	doz	24	36
Ghee	tin	250	320

Solution :

Here, Base year = 1990

$\therefore$ p_0 = Price in the year 1990 and

p_1 = Price in the year 1997.

Commodity	Unit	Price (in ₹)	
		p_0	p_1
Butter	kg	27	33
Cheese	kg	30	36
Milk	litre	25	29
Bread	loaf	10	14
Eggs	doz	24	36
Ghee	tin	250	320
Total		$\Sigma p_0 = 366$	$\Sigma p_1 = 468$

Price Index Number :

$$P_{01} = \frac{\Sigma p_1}{\Sigma p_0} \times 100$$

$$= \frac{468}{366} \times 100$$

$$= 1.2787 \times 100$$

$$= 127.87$$

Hence, price index number is 127.87.

6. Assume 2000 to be base year in the following problem :

Fruits	Unit	Price in 2000 (in ₹)	Price in 2007 (in ₹)
Mango	doz	250	300
Banana	doz	12	24
Apple	kg	80	110
Peach	kg	75	90
Orange	doz	36	65
Sweet Lime	doz	30	45

Solution :

Here, base year = 2000

$\therefore$ p_0 = Price in the year 2000 and

p_1 = Price in the year 2007.

Fruits	Unit	Price (in ₹)	
		p_0	p_1
Mango	doz	250	300
Banana	doz	12	24
Apple	kg	80	110
Peach	kg	75	90
Orange	doz	36	65
Sweet Lime	doz	30	45
Total		$\Sigma p_0 = 483$	$\Sigma p_1 = 634$

Price Index Number :

$$P_{01} = \frac{\Sigma p_1}{\Sigma p_0} \times 100$$

$$= \frac{634}{483} \times 100$$

$$= 1.3126 \times 100 = 131.26$$

Hence, price index number is 131.26.

7. Use 2005 as base year in the following problem :

Vegetables	Unit	Price in 2005 (in ₹)	Price in 2012 (in ₹)
Ladyfinger	kg	32	38
Capsicum	kg	30	36
Brinjal	kg	40	60
Tomato	kg	40	62
Potato	kg	16	28

Solution :

Here, Base year = 2005

$\therefore$ p_0 = Price in the year 2005 and p_1 = Price in the year 2012.

Vegetables	Unit	Price (in ₹)	
		p_0	p_1
Ladyfinger	kg	32	38
Capsicum	kg	30	36
Brinjal	kg	40	60
Tomato	kg	40	62
Potato	kg	16	28
Total		$\Sigma p_0 = 158$	$\Sigma p_1 = 224$

Price Index Number :

$$P_{01} = \frac{\Sigma p_1}{\Sigma p_0} \times 100$$

$$= \frac{224}{158} \times 100$$

$$= 1.4177 \times 100$$

$$= 141.77$$

Hence, price index number is 141.77.

Find the Quantity Index Number using Simple Aggregate Method in each of the following examples :

8.

Commodity	I	II	III	IV	V
Base Year quantities	140	120	100	200	225
Current Year quantities	100	80	70	150	185

Solution :

Let, q_0 = Quantity of base year and

q_1 = Quantity of current year.

Commodity	Quantity	
	q_0	q_1
I	140	100
II	120	80
III	100	70
IV	200	150
V	225	185
Total	$\Sigma q_0 = 785$	$\Sigma q_1 = 585$

Quantity Index number by Simple Aggregate Method :

$$Q_{01} = \frac{\Sigma q_1}{\Sigma q_0} \times 100$$

$$= \frac{585}{785} \times 100$$

$$= 0.7452 \times 100$$

$$= 74.52$$

Hence, quantity index number is 74.52.

9.

Commodity	A	B	C	D	E
Base Year quantities	360	280	340	160	260
Current Year quantities	440	320	470	210	300

Solution :

Let, q_0 = Quantity of base year and

q_1 = Quantity of current year.

Commodity	Quantity	
	q_0	q_1
A	360	440
B	280	320
C	340	470
D	160	210
E	260	300
Total	$\Sigma q_0 = 1400$	$\Sigma q_1 = 1740$

Quantity Index number by Simple Aggregate method :

$$Q_{01} = \frac{\Sigma q_1}{\Sigma q_0} \times 100 = \frac{1740}{1400} \times 100$$

$$= 1.2429 \times 100 = 124.29$$

Hence, quantity index number is 124.29.

Find the Value Index Number using Simple Aggregate Method in each of the following examples :

10.

Commodity	Base Year		Current Year	
	Price (in ₹)	Quantity	Price (in ₹)	Quantity
A	30	22	40	18
B	40	16	60	12
C	10	38	15	24
D	50	12	60	16
E	20	28	25	36

Solution :

Here, p_0 = Price in base year, p_1 = Price in current year,

q_0 = Quantity of base year and

q_1 = Quantity of current year.

Commodity	Base year		Current year		p_0q_0	p_1q_1
	p_0	q_0	p_1	q_1		
A	30	22	40	18	660	720
B	40	16	60	12	640	720
C	10	38	15	24	380	360
D	50	12	60	16	600	960
E	20	28	25	36	560	900
Total					$\Sigma p_0q_0 = 2840$	$\Sigma p_1q_1 = 3660$

Value Index Number by Simple Aggregate Method :

$$V_{01} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_0} \times 100$$

$$= \frac{3660}{2840} \times 100$$

$$= 1.2887 \times 100$$

$$= 128.87$$

Hence, value index number is 128.87.

11.

Commodity	Base Year		Current Year	
	Price (in ₹)	Quantity	Price (in ₹)	Quantity
A	50	22	70	14
B	70	16	90	22
C	60	18	105	14
D	120	12	140	15
E	100	22	155	28

Solution :

Let p_0 = Price in base year,

p_1 = Price in current year,

q_0 = Quantity of base year,

q_1 = Quantity of current year.

Commodity	Base year		Current year		p_0q_0	p_1q_1
	p_0	q_0	p_1	q_1		
A	50	22	70	14	1100	980
B	70	16	90	22	1120	1980
C	60	18	105	14	1080	1470
D	120	12	140	15	1440	2100
E	100	22	155	28	2200	4340
Total					Σp_0q_0 $= 6940$	Σp_1q_1 $= 10870$

Value Index Number by Simple Aggregate Method :

$$V_{01} = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_0} \times 100$$

$$= \frac{10870}{6940} \times 100$$

$$= 1.5663 \times 100$$

$$= 156.63$$

Hence, value index number is 156.63.

12. Find x, if Price Index Number by Simple Aggregate Method is 125 :

Commodity	P	Q	R	S	T
Base Year Price (in ₹)	8	12	16	22	18
Current Year Price (in ₹)	12	18	x	28	22

Solution :

Given : $P_{01} = 125$, $x = ?$

Commodity	Price (in ₹)	
	Base year p_0	Current year p_1
P	8	12
Q	12	18
R	16	x
S	22	28
T	18	22
Total	$\Sigma p_0 = 76$	$\Sigma p_1 = 80 + x$

Now, Price Index Number

$$P_{01} = \frac{\Sigma p_1}{\Sigma p_0} \times 100$$

$$\therefore\ 125 = \frac{80 + x}{76} \times 100$$

$$\therefore\ \frac{125 \times 76}{100} = 80 + x$$

$$\therefore\ 95 = 80 + x$$

$$\therefore\ 95 - 80 = x$$

$$\therefore\ x = 15$$

Hence, the value of x is ₹ 15.

13. Find y, if the Price Index Number by Simple Aggregate Method is 120, takin 1995 as base year :

Commodity	A	B	C	D
Price (in ₹) in 1995	95	y	80	35
Price (in ₹) in 2003	116	74	92	42

Solution :

Here, Base year = 1995

$\therefore$ p_0 = Price in 1995 and

p_1 = Price in 2003.

Given : $P_{01} = 120$, $y = ?$

Commodity	Price (in ₹)	
	p_0	p_1
A	95	116
B	y	74
C	80	92
D	35	42
Total	$\Sigma p_0 = 210 + y$	$\Sigma p_1 = 324$

Now, Price Index Number

$$P_{01} = \frac{\Sigma p_1}{\Sigma p_0} \times 100$$

$$\therefore \ 120 = \frac{324}{210 + y} \times 100$$

$$\therefore \ 120(210 + y) = 32400$$

$$\therefore \ 210 + y = \frac{32400}{120}$$

$$\therefore \ 210 + y = 270$$

$$\therefore \ y = 270 - 210$$

$$\therefore \ y = 60$$

Hence, the value of y is ₹ 60.

EXAMPLES FOR PRACTICE 5.1

1. Calculate the Price Index Number using Simple Aggregate Method with respect to the base year 2010 :

Commodity	Price in 2010	Price in 2020
A	12	40
B	30	40
C	10	25
D	25	30

2. Calculate Price Index Number using Simple Aggregate Method taking 2012 as the base year :

Commodity	Price in 2012	Price in 2015
A	12	38
B	28	42
C	10	24
D	26	30
E	24	46

3. Calculate the Quantity Index Number for the year 2020 using Simple Aggregate Method from the following data :

Commodity	Quantity consumed in the year	
	2019	2020
Wheat	50	60
Rice	40	90
Potato	70	70
Sugar	40	50

4. Calculate Value Index Number for the year 2020 using Simple Aggregate Method :

Commodity	2016		2020	
	Price (in ₹)	Quantity (in kg)	Price (in ₹)	Quantity (in kg)
A	5	60	10	50
B	6	40	18	60
C	4	15	6	20
D	10	25	30	20

5. Find x from the following data if Price Index Number is 132 :

Commodity		A	B	C	D	E
Price (in ₹)	Base year	100	20	x	50	80
	Current year	130	30	30	80	93

6. Find y from the following data if the Quantity Index Number is 120 :

Commodity		A	B	C	D
Quantity (in kg)	Base year	50	40	20	5
	Current year	60	40	15	y

7. Find x from the following data if the Value Index Number is 200 :

Commodity	Base Year		Current Year	
	Price (in ₹)	Quantity (in kg)	Price (in ₹)	Quantity (in kg)
A	10	10	20	10
B	8	20	22	15
C	2	x	8	10
D	9	10	16	10
E	5	6	3	10

Answers

1. 175.32. **2.** 180. **3.** 135. **4.** 270.59. **5.** $x = 25$.
6. $y = 23$. **7.** $x = 10$.

5.4.2 : Weighted Aggregate Method

In this method, suitable weights are assigned to various commodities according to their relative importance in the group.

If w is the weight assigned to a commodity,

Price Index Number $P_{01} = \dfrac{\Sigma p_1 w}{\Sigma p_0 w} \times 100$

In most of the cases w = Quantity consumed.

Methods of constructing Weighted Index Number :

(1) Laspeyre's Price Index Number :

Here, w = base year's quantity

$$P_{01}(L) = \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100$$

(2) Paasche's Price Index Number :

Here, w = current year's quantity

$$P_{01}(P) = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times 100$$

(3) Dorbish-Bowley's Price Index Number :

$$P_{01}(D\text{-}B) = \frac{\dfrac{\Sigma p_1 q_0}{\Sigma p_0 q_0} + \dfrac{\Sigma p_1 q_1}{\Sigma p_0 q_1}}{2} \times 100$$

Relation among Laspeyre's, Paasche's and Dorbish-Bowley's Index Numbers :

Dorbish-Bowley's Index Number is the arithmetic mean of Laspeyre's and Paasche's Index Numbers, i.e.

$$P_{01}\ (D\text{-}B) = \frac{P_{01}(L) + P_{01}(P)}{2}$$

$$= \frac{\dfrac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100 + \dfrac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times 100}{2}$$

$$P_{01}(D\text{-}B) = \frac{\dfrac{\Sigma p_1 q_0}{\Sigma p_0 q_0} + \dfrac{\Sigma p_1 q_1}{\Sigma p_0 q_1}}{2} \times 100$$

(4) Fisher's Price Index Number :

$$P_{01}(F) = \sqrt{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}} \times 100$$

Relation among Laspeyre's, Paasche's and Fisher's Index Numbers :

Fisher's Index Number is the geometric mean of Laspeyre's and Paasche's Index Numbers, i.e.

$$P_{01}(F) = \sqrt{P_{01}(L) \cdot P_{01}(P)}$$

$$= \sqrt{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100 \times \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times 100}$$

$$P_{01}(F) = \sqrt{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}} \times 100$$

(5) Marshall-Edgeworth's Price Index Number :

$$P_{01}(M\text{-}E) = \frac{\Sigma p_1(q_0 + q_1)}{\Sigma p_0(q_0 + q_1)} \times 100$$

$$= \frac{\Sigma p_1 q_0 + \Sigma p_1 q_1}{\Sigma p_0 q_0 + \Sigma p_0 q_1} \times 100$$

(6) Walsch's Price Index Number :

$$P_{01}(W) = \frac{\Sigma p_1 \sqrt{q_0 q_1}}{\Sigma p_0 \sqrt{q_0 q_1}} \times 100$$

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Calculate Laspeyre's Paasche's, Dorbish-Bowley's and Marshall-Edgeworth's Price Index Numbers in Problems 1 and 2 :

1.

Commodity	Base Year		Current Year	
	Price	Quantity	Price	Quantity
A	8	20	11	15
B	7	10	12	10
C	3	30	5	25
D	2	50	4	35

Solution :

Commodity	Base Year		Current Year		p_1q_0	p_0q_0	p_1q_1	p_0q_1
	p_0	q_0	p_1	q_1				
A	8	20	11	15	220	160	165	120
B	7	10	12	10	120	70	120	70
C	3	30	5	25	150	90	125	75
D	2	50	4	35	200	100	140	70
Total					Σp_1q_0 $= 690$	Σp_0q_0 $= 420$	Σp_1q_1 $= 550$	Σp_0q_1 $= 335$

Laspeyre's Price Index Number :

$$P_{01}(L) = \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100$$

$$= \frac{690}{420} \times 100$$

$$= 1.6429 \times 100$$

$$= 164.29.$$

Paasche's Price Index Number :

$$P_{01}(P) = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times 100$$

$$= \frac{550}{335} \times 100$$

$$= 1.6418 \times 100 = 164.18$$

Dorbish-Bowley's Price Index Number :

$$P_{01}(\text{D-B}) = \frac{\dfrac{\Sigma p_1q_0}{\Sigma p_0q_0} + \dfrac{\Sigma p_1q_1}{\Sigma p_0q_1}}{2} \times 100$$

$$= \frac{1.6429 + 1.6418}{2} \times 100$$

$$= \frac{3.2847}{2} \times 100$$

$$= 1.6424 \times 100$$

$$= 164.24$$

[*Alternative Method :*

$$P_{01}\ (\text{D-B}) = \frac{P_{01}(\text{L}) + P_{01}(\text{P})}{2} = \frac{164.29 + 164.18}{2}$$

$$= \frac{328.47}{2} = 164.24]$$

Marshall-Edgeworth's Price Index Number :

$$P_{01}(\text{M-E}) = \frac{\Sigma p_1q_0 + \Sigma p_1q_1}{\Sigma p_0q_0 + \Sigma p_0q_1} \times 100$$

$$= \frac{690 + 550}{420 + 335} \times 100$$

$$= \frac{1240}{755} \times 100$$

$$= 1.6424 \times 100$$

$$= 164.24$$

Hence, $P_{01}(\text{L}) = 164.29$, $P_{01}(\text{P}) = 164.18$, $P_{01}(\text{D-B}) = 164.24$ and $P_{01}(\text{M-E}) = 164.24$.

[**Note :** *Answers given in the textbook are incorrect.*]

2.

Commodity	Base Year		Current Year	
	Price	Quantity	Price	Quantity
I	10	9	20	8
II	20	5	30	4
III	30	7	50	5
IV	40	8	60	6

Solution :

Commodity	Base Year		Current Year		p_1q_0	p_0q_0	p_1q_1	p_0q_1
	p_0	q_0	p_1	q_1				
I	10	9	20	8	180	90	160	80
II	20	5	30	4	150	100	120	80
III	30	7	50	5	350	210	250	150
IV	40	8	60	6	480	320	360	240
Total					Σp_1q_0 $= 1160$	Σp_0q_0 $= 720$	Σp_1q_1 $= 890$	Σp_0q_1 $= 550$

Laspeyre's Price Index Number :

$$P_{01}(\text{L}) = \frac{\Sigma p_1q_0}{\Sigma p_0q_0} \times 100$$

$$= \frac{1160}{720} \times 100$$

$$= 1.6111 \times 100$$

$$= 161.11$$

Paasche's Price Index Number :

$$P_{01}(\text{P}) = \frac{\Sigma p_1q_1}{\Sigma p_0q_1} \times 100$$

$$= \frac{890}{550} \times 100$$

$$= 1.6152 \times 100$$

$$= 161.82$$

Dorbish-Bowley's Price Index Number :

$$P_{01}\ (\text{D-B}) = \frac{P_{01}(\text{L}) + P_{01}(\text{P})}{2}$$

$$= \frac{161.11 + 161.82}{2}$$

$$= \frac{322.93}{2}$$

$$= 161.46$$

Marshall-Edgeworth's Price Index Number :

$$P_{01}(\text{M-E}) = \frac{\Sigma p_1q_0 + \Sigma p_1q_1}{\Sigma p_0q_0 + \Sigma p_0q_1} \times 100$$

$$= \frac{1160 + 890}{720 + 550} \times 100$$

$$= \frac{2050}{1270} \times 100$$

$$= 1.6142 \times 100$$

$$= 161.42$$

Hence, $P_{01}(\text{L}) = 161.11$, $P_{01}(\text{P}) = 161.82$, $P_{01}(\text{D-B}) = 161.46$ and $P_{01}(\text{M-E}) = 161.42$.

Calculate Walsch's Price Index Number in problem 3 and 4 :

3.

Commodity	Base Year		Current Year	
	Price	Quantity	Price	Quantity
L	4	16	3	19
M	6	16	8	14
N	8	28	7	32

Solution :

Commodity	Base Year		Current Year		q_0q_1	$\sqrt{q_0q_1}$	$p_1\sqrt{q_0q_1}$	$p_0\sqrt{q_0q_1}$
	p_0	q_0	p_1	q_1				
L	4	16	3	19	304	17.44	52.32	69.76
M	6	16	8	14	224	14.97	119.76	89.82
N	8	28	7	32	896	29.93	209.51	239.44
Total							$\Sigma p_1\sqrt{q_0q_1}$ $=381.59$	$\Sigma p_0\sqrt{q_0q_1}$ $=399.02$

Walsch's Price Index Number :

$$P_{01}(W)=\frac{\Sigma p_1\sqrt{q_0q_1}}{\Sigma p_0\sqrt{q_0q_1}}\times 100$$

$$=\frac{381.59}{399.02}\times 100$$

$$=0.9563\times 100=95.63$$

Hence, Walsch's index number is 95.63.

[**Note :** *Answer given in the textbook is incorrect.*]

4.

Commodity	Base Year		Current Year	
	Price	Quantity	Price	Quantity
I	10	12	20	9
II	20	4	25	8
III	30	13	40	27
IV	60	29	75	36

Solution :

Commodity	Base Year		Current Year		q_0q_1	$\sqrt{q_0q_1}$	$p_1\sqrt{q_0q_1}$	$p_0\sqrt{q_0q_1}$
	p_0	q_0	p_1	q_1				
I	10	12	20	9	108	10.39	207.8	103.9
II	20	4	25	8	32	5.66	141.5	113.2
III	30	13	40	27	351	18.73	749.2	561.9
IV	60	29	75	36	1044	32.31	2423.2	1938.6
Total							$\Sigma p_1\sqrt{q_0q_1}$ $=3521.75$	$\Sigma p_0\sqrt{q_0q_1}$ $=2717.6$

Walsch's Price Index Number :

$$P_{01}(W)=\frac{\Sigma p_1\sqrt{q_0q_1}}{\Sigma p_0\sqrt{q_0q_1}}\times 100$$

$$=\frac{3521.75}{2717.6}\times 100$$

$$=1.2959\times 100=129.59$$

Hence, Walsch's index number is 129.59.

[**Note :** *Answer given in the textbook is incorrect.*]

5. If $P_{01}(L)=90$ and $P_{01}(P)=40$, find $P_{01}(D\text{-}B)$ and $P_{01}(F)$.

Solution :

Given : $P_{01}(L)=90$, $P_{01}(P)=40$, $P_{01}(D\text{-}B)=?$, $P_{01}(F)$

We have, $P_{01}(D\text{-}B)=\dfrac{P_{01}(L)+P_{01}(P)}{2}$

$$=\frac{90+40}{2}=\frac{130}{2}$$

$$=65$$

We know that, $P_{01}(F)$ is geometric mean of $P_{01}(L)$ and $P_{01}(P)$.

$\therefore\ P_{01}(F)=\sqrt{P_{01}(L)\times P_{01}(P)}$

$$=\sqrt{90\times 40}$$

$$=\sqrt{3600}$$

$$=60$$

Hence, $P_{01}(D\text{-}B)=65$ and $P_{01}(F)=60$.

6. If $\Sigma p_0q_0=140$, $\Sigma p_0q_1=200$, $\Sigma p_1q_0=350$ and $\Sigma p_1q_1=460$.

Find Laspeyre's, Paasche's, Dorbish-Bowley's and Marshall-Edgeworth's Price Index Numbers.

Solution :

Given : $\Sigma p_0q_0=140$, $\Sigma p_0q_1=200$, $\Sigma p_1q_0=350$, $\Sigma p_1q_1=460$.

Laspeyre's Price Index Number :

$$P_{01}(L)=\frac{\Sigma p_1q_0}{\Sigma p_0q_0}\times 100$$

$$=\frac{350}{140}\times 100$$

$$=2.5\times 100$$

$$=250$$

Paasche's Price Index Number :

$$P_{01}(P)=\frac{\Sigma p_1q_1}{\Sigma p_0q_1}\times 100$$

$$=\frac{460}{200}\times 100$$

$$=2.3\times 100$$

$$=230$$

Dorbish-Bowley's Price Index Number :

$$P_{01}(D\text{-}B)=\frac{\dfrac{\Sigma p_1q_0}{\Sigma p_0q_0}+\dfrac{\Sigma p_1q_1}{\Sigma p_0q_1}}{2}\times 100$$

$$=\frac{2.5+2.3}{2}\times 100$$

$$=\frac{4.8}{2}\times 100$$

$$=2.4\times 100=240$$

Marshall-Edgeworth's Price Index Number :

$$P_{01}(\text{M-E}) = \frac{\Sigma p_1q_0 + \Sigma p_1q_1}{\Sigma P_0q_0 + \Sigma p_0q_1} \times 100$$

$$= \frac{350 + 460}{140 + 200} \times 100$$

$$= \frac{810}{340} \times 100$$

$$= 2.3824 \times 100$$

$$= 238.24.$$

7. Given that, the Laspeyre's and Dorbish-Bowley's Price Index Numbers are 160.32 and 164.18 respectively. Find the Paasche's Price Index Number.

Solution :

Given : $P_{01}(L) = 160.32$, $P_{01}(\text{D-B}) = 164.18$, $P_{01}(P) = ?$

We know that,

$$P_{01}(\text{D-B}) = \frac{P_{01}(L) + P_{01}(P)}{2}$$

$$\therefore\ 164.18 = \frac{160.32 + P_{01}(P)}{2}$$

$$\therefore\ 2(164.18) = 160.32 + P_{01}(P)$$

$$\therefore\ 328.36 - 160.32 = P_{01}(P)$$

$$\therefore\ 168.04 = P_{01}\ (P)$$

$$\therefore\ P_{01}(P) = 168.04$$

Hence, $P_{01}(P)$ is 168.04.

8. Given that, $\Sigma p_0q_0 = 220$, $\Sigma p_0q_1 = 380$, $\Sigma p_1q_1 = 350$ and Marshall-Edgeworth's Price Index Number is 150. Find Laspeyre's Price Index Number.

Solution :

Given : $\Sigma p_0q_0 = 220$, $\Sigma p_0q_1 = 380$, $\Sigma p_1q_1 = 350$.

$P_{01}(\text{M-E}) = 150$, $P_{01}(L) = ?$

We have,

$$P_{01}(\text{M-E}) = \frac{\Sigma p_1q_0 + \Sigma p_1q_1}{\Sigma P_0q_0 + \Sigma p_0q_1} \times 100$$

$$\therefore\ 150 = \frac{\Sigma p_1q_0 + 350}{220 + 380} \times 100$$

$$\therefore\ 150 = \frac{\Sigma p_1q_0 + 350}{600} \times 100$$

$$\therefore\ 150 \times 6 = \Sigma p_1q_0 + 350$$

$$\therefore\ 900 - 350 = \Sigma p_1q_0$$

$$\therefore\ \Sigma p_1q_0 = 550$$

Now,

$$P_{01}(L) = \frac{\Sigma p_1q_0}{\Sigma p_0q_0} \times 100$$

$$= \frac{550}{220} \times 100$$

$$= 2.5 \times 100$$

$$= 250$$

Hence, Laspeyre's Price Index Number is 250.

9. Find x in the following table if Laspeyre's and Paasche's Price Index Numbers are equal :

Commodity	Base Year		Current Year	
	Price	Quantity	Price	Quantity
A	2	10	2	5
B	2	5	x	2

Solution :

Given : $P_{01}(L) = P_{01}(P)$

Commodity	Base Year		Current Year		p_1q_0	p_0q_0	p_1q_1	p_0q_1
	p_0	q_0	p_1	q_1				
A	2	10	2	5	20	20	10	10
B	2	5	x	2	$5x$	10	$2x$	4
Total					Σp_1q_0 $= 20 + 5x$	Σp_0q_0 $= 30$	Σp_1q_1 $= 10 + 2x$	Σp_0q_1 $= 14$

We have,

$$P_{01}(L) = \frac{\Sigma p_1q_0}{\Sigma p_0q_0} \times 100$$

$$= \frac{20 + 5x}{30} \times 100$$

$$P_{01}(P) = \frac{\Sigma p_1q_1}{\Sigma p_0q_1} \times 100$$

$$= \frac{10 + 2x}{14} \times 100$$

Given : $P_{01}(L) = P_{01}(P)$

$$\therefore\ \frac{20 + 5x}{30} \times 100 = \frac{10 + 2x}{14} \times 100$$

$$\therefore\ \frac{20 + 5x}{30} = \frac{10 + 2x}{14}$$

$$\therefore\ 14(20 + 5x) = 30(10 + 2x)$$

$$\therefore\ 280 + 70x = 300 + 60x$$

$$\therefore\ 70x - 60x = 300 - 280$$

$$\therefore\ 10x = 20$$

$$\therefore\ x = 2$$

Hence, $x = 2$.

10. If Laspeyre's Price Index Number is four times Paasche's Price Index Number, then find the relation between Dorbish-Bowley's and Fisher's Price Index Numbers.

Solution :

Given : $P_{01}(L) = 4P_{01}(P)$

We have, $P_{01}(\text{D-B}) = \dfrac{P_{01}(L) + P_{01}(P)}{2}$

$$= \frac{4P_{01}(P) + P_{01}(P)}{2}$$

$$= \frac{5P_{01}(P)}{2}$$

$$= 2.5P_{01}(P) \qquad \dots (1)$$

Also, we have,

$$P_{01}(F) = \sqrt{P_{01}(L) \times P_{01}(P)}$$

$$= \sqrt{4P_{01}(P) \times P_{01}(P)}$$

$$= 2P_{01}(P) \qquad \dots (2)$$

From (1), $P_{01}(P) = \dfrac{P_{01}(\text{D-B})}{2.5}$ ... (3)

From (2), $P_{01}(P) = \dfrac{P_{01}(F)}{2}$ ... (4)

From (3) and (4),

$$\frac{P_{01}(\text{D-B})}{2.5} = \frac{P_{01}(F)}{2}$$

$\therefore$ $P_{01}(\text{D-B}) = \dfrac{2.5P_{01}(F)}{2} = \dfrac{5}{4}P_{01}(F)$

Hence, $P_{01}(\text{D-B}) = \dfrac{5}{4}P_{01}(F)$.

[**Note :** *Answer given in the textbook is incorrect.*]

11. If Dorbish-Bowley's and Fisher's Price Index Numbers are 5 and 4 respectively, then find Laspeyre's and Paasche's Price Index Numbers.

Solution :

Given : $P_{01}(\text{D-B}) = 5$ and $P_{01}(F) = 4$, $P_{01}(L) = ?$, $P_{01}(P) = ?$

We have, $P_{01}(\text{D-B}) = \dfrac{P_{01}(L) + P_{01}(P)}{2}$

$\therefore$ $5 = \dfrac{P_{01}(L) + P_{01}(P)}{2}$

$\therefore$ $P_{01}(L) + P_{01}(P) = 10$ ... (1)

Also, $P_{01}(F) = \sqrt{P_{01}(L) \times P_{01}(P)}$

$\therefore$ $4 = \sqrt{P_{01}(L) \times P_{01}(P)}$

$\therefore$ $P_{01}(L) \times P_{01}(P) = 16$ ... (2)

Now, we have,

$$[P_{01}(L) - P_{01}(P)]^2$$

$$= [P_{01}(L) + P_{01}(P)]^2 - 4P_{01}(L) \cdot P_{01}(P)$$

$$[\because\ (\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta]$$

$$= (10)^2 - 4(16)$$

$$= 100 - 64$$

$$= 36$$

$\therefore$ $P_{01}(L) - P_{01}(P) = \pm 6$ ... (3)

Adding equations (1) and (3), we get

$2P_{01}(L) = 16$

$\therefore$ $P_{01}(L) = 8$

Using this result in (1), we get

$P_{01}(P) = 2$

If we take $P_{01}(L) - P_{01}(P) = -6$ then

$P_{01}(L) = 2$ and $P_{01}(P) = 8$

Hence, $P_{01}(L) = 8$ and $P_{01}(P) = 2$ ... ($\because$ AM > GM)

[**Note :** *Answer given in the textbook is incorrect.*]

EXAMPLES FOR PRACTICE 5.2

1. If $\Sigma p_0q_0 = 700$, $\Sigma p_0q_1 = 900$, $\Sigma p_1q_0 = 1070$ and Marshall-Edgeworth's Price Index Number is 140, find Paasche's Price Index Number.

2. Calculate Walsch's Price Index Number from the following data :

Commodity	Base Year		Current Year	
	Price	Quantity	Price	Quantity
A	5	40	8	10
B	3	50	7	32

3. Calculate Laspeyre's Price Index Number for the year 2002 from the following data :

Commodity	Year 2001		Year 2002
	Price (p_0)	Quantity (q_0)	Price (p_1)
A	20	8	40
B	50	10	60
C	40	15	50
D	20	20	20

4. If for a data, Laspeyre's Index Number is 86.02 and Paasche's Index Number is 81.25, then calculate Dorbish-Bowley's Index Number.

5. Compute Walsch's Price Index Number for the following data :

Commodity	Base Year		Current Year	
	Price	Quantity	Price	Quantity
A	4	9	5	4
B	5	8	6	2
C	6	2	8	2

6. Compute Marshall-Edgeworth's Price Index Number from the following :

Commodity	Base Year		Current Year	
	Price	Quantity	Price	Quantity
A	5	10	6	5
B	4	15	5	10
C	2	20	2	15

7. If Laspeyre's and Paasche's Price Index Numbers are 50 and 72 respectively, find Dorbish-Bowley's and Fisher' Index Numbers.

8. Find the missing price, if Laspeyre's Price Index Number is equal to Paasche's Price Index Number for the following data :

Commodity	Base Year		Current Year	
	Price	Quantity	Price	Quantity
A	1	10	2	5
B	1	5	–	2

9. If $\Sigma p_0q_0 = 150$, $\Sigma p_0q_1 = 250$, $\Sigma p_1q_1 = 375$ and Laspeyre's Price Index Number is 140, find Marshall-Edgeworth's Index Number.

10. If $\Sigma p_1q_0 : \Sigma p_0q_0 = 5 : 4$ and $\Sigma p_1q_1 : \Sigma p_0q_1 = 8 : 5$, find Laspeyre's, Paasche's, Fisher's and Dorbish-Bowley's Index Numbers.

11. The ratio of Laspeyre's Index Number to Paasche's Index Number is $\frac{4}{5}$. If Fisher's Index Number is 120, find Dorbish-Bowley's Index Number.

Answers

1. 130. **2.** 200. **3.** 124.7. **4.** 83.635. **5.** 125.
6. 116.33. **7.** P_{01}(D-B) = 61, P_{01}(F) = 60.
8. Missing price = ₹ 2. **9.** 146.25.
10. P_{01}(L) = 125, P_{01}(P) = 160, P_{01}(F) = 141.42, P_{01}(D-B) = 142.5.
11. P_{01}(L) = 107.33, P_{01}(P) = 134.16, P_{01}(D-B) = 120.745.

5.5 : COST OF LIVING INDEX NUMBER

The percentage change occurring in the cost of living of a particular section of people at a given place during a certain period in relation to base period is called Cost of Living Index Number of that particular section of people.

- It measures the effect of change in prices on cost of living of people.
- The standard of living depends on the type and quantity of commodities used by different sections of people. Hence, the Cost of Living Index Number is different for different class of people.
- Cost of Living Index Number is also known as Consumer Price Index Number.

5.5.1 : Steps in Construction of Cost of Living Index Number

1. **Choice of Base year :** The base year should be a normal year free from abnormal condition like wars, femines, floods, political instability, etc.

   Base year can be chosen either using (i) Fixed base method or (ii) Chain base method.

2. **Choice of Commodities :** Since, all commodities cannot be included, only representatives should be chosen according to the purpose of the index number.

   The commodities chosen must (i) represent the tastes, habits and customs of the people (ii) be recognizable (iii) have the same quality over different periods and places (iv) be sufficiently large in numbers (v) have in common use and stable in nature (vi) have economic and social importance.

3. **Collection of Prices :** (i) Prices must be collected from places where a particular commodity is traded in large quantities (ii) If published data on prices is available, it must be used (iii) Prices collected from individuals or institutions must correct (iv) Retail prices of commodities are collected (v) If prices are collected from several sources must be averaged.

4. **Choice of Averages :** Geometric mean is theoritically the best but arithmatic mean is used in practice.

5. **Choice of Weights :** Proper weights must be assigned to the commodities according to their relative importance. Weights should be chosen rationally and not arbitrarily.

6. Purpose of Index Number : All steps are to be viewed in light of the purpose for which index number is being prepared. It is important to have a clear idea about the purpose of the index number before it is constructed.

5.5.2 : Methods to Construct Cost of Living Index Number

1. Aggregate Expenditure Method (Weighted Aggregate Method) : In this method, quantities consumed in the base year are used as weights.

Hence, Cost of Living Index Number

$$\text{CLI} = \frac{\text{Total expenditure in current year}}{\text{Total expenditure in base year}} \times 100$$

$$= \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100$$

> **Note :** The above formula is equivalent to formula of **Laspeyre's Index Number.**

2. Family Budget Method (Weighted Relative Method) : In this method, Price relatives (I) for all items are obtained, i.e.

$$\text{I} = \frac{p_1}{p_0} \times 100$$

The base year's expenditure of items ($p_0 q_0$) are taken as weights of Price relatives I.

Hence, Cost of Living Index Number

$$\text{CLI} = \frac{\Sigma \text{IW}}{\Sigma \text{W}}, \quad \text{where } \text{I} = \frac{p_1}{p_0} \times 100 \text{ and } \text{W} = p_0 q_0.$$

> **Note :** This formula is equivalent to formula of Aggregate expenditure method.

$$\text{CLI} = \frac{\Sigma \text{IW}}{\Sigma \text{W}}$$

Put $\text{I} = \frac{p_1}{p_0} \times 100$ and $\text{W} = p_0 q_0$

$$\therefore \ \text{CLI} = \frac{\Sigma \frac{p_1}{p_0} \times 100 (p_0 q_0)}{\Sigma p_0 q_0}$$

$$\therefore \ \text{CLI} = \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100$$

5.6 : USES OF COST OF LIVING INDEX NUMBER

(1) Cost of living index number provides a realistic picture of the economic position of different classes of people. Hence, it is used to regulate the dearness allowance or the grant of bonus to the employees so as to enable them to meet the increased cost of living.

(2) It is used to settle the disputes related to salary and wages.

(3) It is used to calculate purchasing power of money.

Purchasing power of money

$$= \frac{1}{\text{Cost of Living Index Number}}$$

(4) It is used to determine the real wages

$$\text{Real wages} = \frac{\text{Actual wage}}{\text{Cost of Living Index Number}} \times 100$$

(5) It helps the government in divising the taxation policy.

EXERCISE 5.3 Textbook page 87

Calculate the cost of living index in problems 1 to 3 :

1.

Group	Base Year		Current Year
	Price	Quantity	Price
Food	120	15	170
Clothing	150	20	190
Fuel and Lighting	130	30	220
House Rent	160	10	180
Miscellaneous	200	12	200

Solution :

Group	Base Year		Current Year	$p_1 q_0$	$p_0 q_0$
	p_0	q_0	p_1		
Food	120	15	170	2550	1800
Clothing	150	20	190	3800	3000
Fuel and Lighting	130	30	220	6600	3900
House rent	160	10	180	1800	1600
Miscellaneous	200	12	200	2400	2400
Total				$\Sigma p_1 q_0$ = 17150	$\Sigma p_0 q_0$ = 12700

By aggregate expenditure method,

$$\text{CLI} = \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100 = \frac{17150}{12700} \times 100$$

$$= 1.3504 \times 100$$

$$= 135.04$$

Hence, Cost of Living Index Number is 135.04.

2.

Group	Base Year		Current Year
	Price	Quantity	Price
Food	40	15	45
Clothing	30	10	35
Fuel and Lighting	20	17	25
House Rent	60	22	70
Miscellaneous	70	25	80

Solution :

Group	Base Year		Current Year	p_1q_0	p_0q_0
	p_0	q_0	p_1		
Food	40	15	45	675	600
Clothing	30	10	35	350	300
Fuel and Lighting	20	17	25	425	340
House rent	60	22	70	1540	1320
Miscellaneous	70	25	80	2000	1750
Total				Σp_1q_0 $= 4990$	Σp_0q_0 $= 4310$

By aggregate expenditure method,

$$\text{CLI} = \frac{\Sigma p_1q_0}{\Sigma p_0q_0} \times 100 = \frac{4990}{4310} \times 100$$

$$= 1.1578 \times 100 = 115.78$$

Hence, Cost of Living Index Number is 115.78.

3.

Group	Base Year		Current Year
	Price	Quantity	Price
Food	132	10	170
Clothing	154	12	160
Fuel and Lighting	164	20	180
House Rent	175	18	195
Miscellaneous	128	5	120

Solution :

Group	Base year		Current year	p_1q_0	p_0q_0
	p_0	q_0	p_1		
Food	132	10	170	1700	1320
Clothing	154	12	160	1920	1848
Fuel and Lighting	164	20	180	3600	3280
House rent	175	18	195	3510	3150
Miscellaneous	128	5	120	600	640
Total				Σp_1q_0 $= 11330$	Σp_0q_0 $= 10238$

By aggregate expenditure method,

$$\text{CLI} = \frac{\Sigma p_1q_0}{\Sigma p_0q_0} \times 100 = \frac{11330}{10238} \times 100$$

$$= 1.1067 \times 100 = 110.67$$

Hence, Cost of Living Index Number is 110.67.

Base year weights (W) and current year price relatives (I) are given in Problems 4 to 8. Calculate the cost of living index in each case :

4.

Group	Food	Clothing	Fuel and Lighting	House Rent	Miscellaneous
I	70	90	100	60	80
W	5	3	2	4	6

Solution :

Group	I	W	IW
Food	70	5	350
Clothing	90	3	270
Fuel and Lighting	100	2	200
House Rent	60	4	240
Miscellaneous	80	6	480
Total		$\Sigma W = 20$	$\Sigma IW = 1540$

By Family budget method,

$$\text{CLI} = \frac{\Sigma IW}{\Sigma W} = \frac{1540}{20} = 77$$

Hence, Cost of Living Index Number is 77.

5.

Group	Food	Clothing	Fuel and Lighting	House Rent	Miscellaneous
I	400	300	150	120	100
W	3	3	4	5	2

Solution :

Group	I	W	IW
Food	400	3	1200
Clothing	300	3	900
Fuel and Lighting	150	4	600
House Rent	120	5	600
Miscellaneous	100	2	200
Total		$\Sigma W = 17$	$\Sigma IW = 3500$

By Family budget method,

$$CLI = \frac{\Sigma IW}{\Sigma W} = \frac{3500}{17} = 205.88$$

Hence, Cost of Living Index Number is 205.88.

6.

Group	Food	Clothing	Fuel and Lighting	House Rent	Miscellaneous
I	200	150	120	180	160
W	30	20	10	40	50

Solution :

Group	I	W	IW
Food	200	30	6000
Clothing	150	20	3000
Fuel and Lighting	120	10	1200
House Rent	180	40	7200
Miscellaneous	160	50	8000
Total		$\Sigma W = 150$	$\Sigma IW = 25400$

By Family budget method,

$$CLI = \frac{\Sigma IW}{\Sigma W} = \frac{25400}{150} = 169.33$$

Hence, Cost of Living Index Number is 169.33.

7. Find *x*, if cost of living index is 150 :

Group	Food	Clothing	Fuel and Lighting	House Rent	Miscellaneous
I	180	120	300	100	160
W	4	5	6	x	3

Solution :

Group	I	W	IW
Food	180	4	720
Clothing	120	5	600
Fuel and Lighting	300	6	1800
House Rent	100	x	$100x$
Miscellaneous	160	3	480
Total		$\Sigma W = 18 + x$	$\Sigma IW = 3600 + 100x$

By Family budget method,

$$CLI = \frac{\Sigma IW}{\Sigma W}$$

Given : $CLI = 150$

$\therefore\ 150 = \dfrac{3600 + 100x}{18 + x}$

$\therefore\ 150(18 + x) = 3600 + 100x$

$\therefore\ 2700 + 150x = 3600 + 100x$

$\therefore\ 150x - 100x = 3600 - 2700$

$\therefore\ 50x = 900$

$\therefore\ x = 18$

Hence, x is 18.

8. Find *y*, if the cost of living index is 200 :

Group	Food	Clothing	Fuel and Lighting	House Rent	Miscellaneous
I	180	120	160	300	200
W	4	5	3	y	2

Solution :

Group	I	W	IW
Food	180	4	720
Clothing	120	5	600
Fuel and Lighting	160	3	480
House Rent	300	y	$300y$
Miscellaneous	200	2	400
Total		$\Sigma W = 14 + y$	$\Sigma IW = 2200 + 300y$

By Family budget method,

$$\text{CLI} = \frac{\Sigma IW}{\Sigma W}$$

Given : CLI = 200

$$\therefore\ 200 = \frac{2200 + 300y}{14 + y}$$

$$\therefore\ 200(14 + y) = 2200 + 300y$$

$$\therefore\ 2800 + 200y = 2200 + 300y$$

$$\therefore\ 2800 - 2200 = 300y - 200y$$

$$\therefore\ 600 = 100y$$

$$\therefore\ y = 6$$

Hence, y is 6.

9. The Cost of Living Index Number for the years 1995 and 1999 are 140 and 200 respectively. A person earns ₹ 11,200 per month in the year 1995. What should be his monthly earnings in the year 1999 in order to maintain his standard of living as in the year 1995 ?

Solution :

Here, given that,

CLI for 1995 = 140

CLI for 1999 = 200

Earning of a person in the year 1995 = ₹ 11200 p.m.

Earning for the year 1999 = ?

Earning of a person in the year 1999

$$= \frac{\text{His earning in 1995} \times \text{CLI for 1999}}{\text{CLI for 1995}}$$

$$= \frac{11200 \times 200}{140}$$

= ₹ 16000

Hence, the earning p.m. of a person in the year 1999 should be ₹ 16,000 to maintain his former standard of living.

[Alternative Method :

Real income of a person in the year 1995

$$= \frac{\text{Income}}{\text{CLI}} \times 100$$

$$= \frac{11200}{140} \times 100$$

= ₹ 8000

Now, in the year 1999, CLI = 200.

Real income = ₹ 8000.

$$\text{Real income} = \frac{\text{Income}}{\text{CLI}} \times 100$$

$$\therefore\ 8000 = \frac{\text{Income}}{200} \times 100$$

$$\therefore\ \text{Income} = \frac{8000 \times 200}{100}$$

= ₹ 16,000]

EXAMPLES FOR PRACTICE 5.3

1. Compute the Cost of Living Index Number using the following data :

Group	Base Year		Current Year
	Price	Quantity	Price
A	5	50	7
B	12	100	15
C	8	30	10
D	4	40	8
E	10	10	13

2. Find the value of k, if the Cost of Living Index Number for the following data is 246 :

Group	Food	Clothing	Fuel and Lighting	House Rent	Miscellaneous
I	300	240	220	k	250
W	6	3	5	4	2

3. Find the Cost of Living Number from the following data :

Group	Index Number	Weight
A	120	52
B	150	14
C	170	6
D	110	12
E	130	16

II. Find x, if the Cost of Living Index from the following data is 150 :

Group	Food	Clothing	Fuel and Lighting	House Rent	Miscellaneous
I	200	150	140	100	120
W	6	4	x	3	4

5. Compute the Cost of Living Index Number from the following data :

Group	Index Number	Weight
Food	125	52
Clothing	200	36
House Rent	110	12

6. If the Cost of Living Index Number for the following data is 150, find x :

Group	Group Index (I)	Weight (W)
Food	150	4
Fuel	140	3
Clothing	100	3
House Rent	120	4
Miscellaneous	200	x

7. Construct Cost of Living Index Number for the following data :

Group	Base Year		Current Year
	Price (in ₹)	Quantity	Price (in ₹)
Food and clothing	40	3	70
Fuel and Lighting	30	5	60
House Rent	50	2	50
Miscellaneous	60	3	90

8. The Cost of Living Index Numbers for the year 2001 and 2007 are 150 and 200 respectively. A person earned ₹ 18,000 per month in the year 2001, what should be his earning per month in the year 2007, so as to maintain his former (i.e. of the year 2001) standard of living?

9. For the following data compute the Cost of Living Index Number. If the monthly income of a person was ₹ 12500 in the year 2005, find his expected monthly income in the year 2008 :

Group		Food	Clothing	Fuel	Rent	Misc.
Weights		45	20	15	10	10
Price	2005	75	20	15	50	30
(in ₹)	2008	90	30	18	80	36

10. The expenditure for the five groups have been increased by 50%, 90%, 110%, 160% and 200% respectively in the year 2008 as compared to the year 2004. If their relative importance in the group is in the ratio of 10 : 6 : 4 : 3 : 2, compute the Cost of Living Index Number for the year 2008.

Answers

1. 133.33. **2.** $k = 200$. **3.** 127.60. **4.** $x = 3$. **5.** 150.20. **6.** $x = 6$. **7.** 160. **8.** ₹ 24000. **9.** CLI = 130, ₹ 16250. **10.** [**Hint :** I = 100 + % increase] CLI = 194.4.

MISCELLANEOUS EXERCISE – 5

(Textbook pages 89 to 94)

I. Choose the correct alternative :

1. Price Index Number by Simple Aggregate Method is given by

(a) $\sum\frac{p_1}{p_0}\times 100$ (b) $\sum\frac{p_0}{p_1}\times 100$

(c) $\frac{\Sigma p_1}{\Sigma p_0}\times 100$ (d) $\frac{\Sigma p_0}{\Sigma p_1}\times 100$.

2. Quantity Index Number by Simple Aggregate Method is given by

(a) $\sum\frac{q_1}{q_0}\times 100$ (b) $\sum\frac{q_0}{q_1}\times 100$

(c) $\frac{\Sigma q_1}{\Sigma q_0}\times 100$ (d) $\frac{\Sigma q_0}{\Sigma q_1}\times 100$.

3. Value Index Number by Simple Aggregate Method is given by

(a) $\sum\frac{p_1q_0}{p_0q_1}\times 100$ (b) $\sum\frac{p_0q_1}{p_1q_0}\times 100$

(c) $\frac{\Sigma p_1q_1}{\Sigma p_1q_0}\times 100$ (d) $\frac{\Sigma p_1q_1}{\Sigma p_0q_0}\times 100$.

4. Price Index Number by Weighted Aggregate Method is given by

(a) $\sum\frac{p_1w}{p_0w}\times 100$ (b) $\sum\frac{p_0w}{p_1w}\times 100$

(c) $\frac{\Sigma p_1w}{\Sigma p_0w}\times 100$ (d) $\frac{\Sigma p_0w}{\Sigma p_1w}\times 100$.

5. Quantity Index Number by Weighted Aggregate Method is given by

(a) $\sum\frac{q_1w}{q_0w}\times 100$ (b) $\sum\frac{q_0w}{q_1w}\times 100$

(c) $\frac{\Sigma q_1w}{\Sigma q_0w}\times 100$ (d) $\frac{\Sigma q_0}{\Sigma q_1}\times 100$.

6. Value Index Number by Weighted Aggregae Method is given by

(a) $\sum \frac{p_1 q_0 w}{p_0 q_0 w} \times 100$ (b) $\sum \frac{p_0 q_1 w}{p_0 q_0 w} \times 100$

(c) $\frac{\Sigma p_1 q_1 w}{\Sigma p_0 q_1 w} \times 100$ (d) $\frac{\Sigma p_1 q_1 w}{\Sigma p_0 q_0 w} \times 100.$

7. Laspeyre's Price Index Number is given by

(a) $\frac{\Sigma p_0 q_0}{\Sigma p_1 q_0} \times 100$ (b) $\frac{\Sigma p_0 q_1}{\Sigma p_1 q_1} \times 100$

(c) $\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100$ (d) $\frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times 100.$

8. Paasche's Price Index Number is given by

(a) $\frac{\Sigma p_0 q_0}{\Sigma p_1 q_0} \times 100$ (b) $\frac{\Sigma p_0 q_1}{\Sigma p_1 q_1} \times 100$

(c) $\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100$ (d) $\frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times 100.$

9. Dorbish-Bowley's Price Index Number is given by

(a) $\frac{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_1} + \frac{\Sigma p_0 q_1}{\Sigma p_1 q_0}}{2} \times 100$ (b) $\frac{\frac{\Sigma p_1 q_1}{\Sigma p_0 q_0} + \frac{\Sigma p_0 q_0}{\Sigma p_1 q_1}}{2} \times 100$

(c) $\frac{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} + \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}}{2} \times 100$ (d) $\frac{\frac{\Sigma p_0 q_0}{\Sigma p_1 q_0} + \frac{\Sigma p_0 q_1}{\Sigma p_1 q_1}}{2} \times 100.$

10. Fisher's Price Number is given by

(a) $\sqrt{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}} \times 100$

(b) $\sqrt{\frac{\Sigma p_0 q_0}{\Sigma p_1 q_0} \times \frac{\Sigma p_0 q_1}{\Sigma p_1 q_1}} \times 100$

(c) $\sqrt{\frac{\Sigma p_0 q_1}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\Sigma p_1 q_0}} \times 100$

(d) $\sqrt{\frac{\Sigma p_1 q_0}{\Sigma p_1 q_1} \times \frac{\Sigma p_0 q_0}{\Sigma p_0 q_1}} \times 100.$

11. Marshall-Edgeworth's Price Index Number is given by

(a) $\frac{\Sigma p_1(q_0+q_1)}{\Sigma p_0(q_0+q_1)} \times 100$ (b) $\frac{\Sigma p_0(q_0+q_1)}{\Sigma p_1(q_0+q_1)} \times 100$

(c) $\frac{\Sigma q_1(p_0+p_1)}{\Sigma q_0(p_0+p_1)} \times 100$ (d) $\frac{\Sigma q_0(p_0+p_1)}{\Sigma q_1(p_0+p_1)} \times 100.$

[**Note :** *Option (d) is modified.*]

12. Walsh's Price Index Number is given by

(a) $\frac{\Sigma p_1 \sqrt{q_0 q_1}}{\Sigma p_0 \sqrt{q_0 q_1}} \times 100$ (b) $\frac{\Sigma p_0 \sqrt{q_0 q_1}}{\Sigma p_1 \sqrt{q_0 q_1}} \times 100$

(c) $\frac{\Sigma q_1 \sqrt{p_0 p_1}}{\Sigma q_0 \sqrt{p_0 p_1}} \times 100$ (d) $\frac{\Sigma q_0 \sqrt{p_0 p_1}}{\Sigma q_1 \sqrt{p_0 p_1}} \times 100.$

13. The Cost of Living Index Number using Aggregate Expenditure Method is given by

(a) $\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100$ (b) $\sum \frac{p_1 q_1}{p_0 q_1} \times 100$

(c) $\frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times 100$ (d) $\sum \frac{p_1 q_0}{p_0 q_0} \times 100.$

14. The Cost of Living Index Number using Weighted Relative Method is given by

(a) $\frac{\Sigma IW}{\Sigma W}$ (b) $\sum \frac{W}{IW}$

(c) $\frac{\Sigma W}{\Sigma IW}$ (d) $\sum \frac{IW}{W}.$

Answers

1. (c) $\frac{\Sigma p_1}{\Sigma p_0} \times 100$ **2.** (c) $\frac{\Sigma q_1}{\Sigma q_0} \times 100$

3. (d) $\frac{\Sigma p_1 q_1}{\Sigma p_0 q_0} \times 100$ **4.** (c) $\frac{\Sigma p_1 w}{\Sigma p_0 w} \times 100$

5. (c) $\frac{\Sigma q_1 w}{\Sigma q_0 w} \times 100$ **6.** (d) $\frac{\Sigma p_1 q_1 w}{\Sigma p_0 q_0 w} \times 100$

7. (c) $\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100$ **8.** (d) $\frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times 100$

9. (c) $\frac{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} + \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}}{2} \times 100$

10. (a) $\sqrt{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}} \times 100$

11. (a) $\frac{\Sigma p_1(q_0+q_1)}{\Sigma p_0(q_0+q_1)} \times 100$ **12.** (a) $\frac{\Sigma p_1 \sqrt{q_0 q_1}}{\Sigma p_0 \sqrt{q_0 q_1}} \times 100$

13. (a) $\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100$ **14.** (a) $\frac{\Sigma IW}{\Sigma W}.$

II. Fill in the blanks :

1. Price Index Number by Simple Aggregate Method is given by ....
2. Quantity Index Number by Simple Aggregate Method is given by ....
3. Value Index Number by Simple Aggregate Method is given by ....

4. Price Index Number by Weighted Aggregate Method is given by ....

5. Quantity Index Number by Weighted Aggregate Method is given by ....

6. Value Index Number by Weighted Aggregate Method is given by ....

7. Laspeyre's Price Index Number is given by ....

8. Paache's Price Index Number is given by ....

9. Dorbish-Bowley's Price Index Number is given by ....

10. Fisher's Price Index Number is given by ....

11. Marshall-Edgeworth's Price Index Number is given by ....

12. Walsh's Price Index Number is given by ....

Answers

1. $\frac{\Sigma p_1}{\Sigma p_0} \times 100$
2. $\frac{\Sigma q_1}{\Sigma q_0} \times 100$
3. $\frac{\Sigma p_1 q_1}{\Sigma p_0 q_0} \times 100^*$
4. $\frac{\Sigma p_1 w}{\Sigma p_0 w} \times 100$
5. $\frac{\Sigma q_1 w}{\Sigma q_0 w} \times 100$
6. $\frac{\Sigma p_1 q_1 w}{\Sigma p_0 q_0 w} \times 100^*$
7. $\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100$
8. $\frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times 100$
9. $\frac{1}{2}\left[\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} + \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}\right] \times 100$
10. $\sqrt{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}} \times 100^*$
11. $\frac{\Sigma p_1 (q_0 + q_1)}{\Sigma p_0 (q_0 + q_1)} \times 100^*$
12. $\frac{\Sigma p_1 \sqrt{q_0 q_1}}{\Sigma p_0 \sqrt{q_0 q_1}} \times 100.$

[*Note : *Answers given in the textbook are incorrect.*]

III. State whether each of the following is *True* or *False* :

1. $\frac{\Sigma p_1}{\Sigma p_0} \times 100$ is the Price Index Number by Simple Aggregate Method.

2. $\frac{\Sigma q_0}{\Sigma q_1} \times 100$ is the Quantity Index Number by Simple Aggregate Method.

3. $\sum \frac{p_0 q_0}{p_1 q_1} \times 100$ is Value Index Number by Simple Aggregate Method.

4. $\sum \frac{p_1 q_0}{p_0 q_0} \times 100$ is Paasche's Price Index Number.

5. $\frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times 100$ is Laspeyre's Price Index Number.

6. $\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100$ is Dorbish-Bowley's Price Index Number.

7. $\frac{1}{2}\left[\sqrt{\frac{\Sigma p_1 q_0}{\Sigma p_0 q_0}} + \frac{\sqrt{p_1 q_1}}{\sqrt{p_0 q_1}}\right] \times 100$ is Fisher's Price Index Number.

8. $\frac{\Sigma p_0 (q_0 + q_1)}{\Sigma p_1 (q_0 + q_1)} \times 100$ is Marshall-Edgeworth's Price Index Number.

9. $\frac{\Sigma p_0 \sqrt{q_0 q_1}}{\Sigma p_1 \sqrt{q_0 q_1}} \times 100$ is Walsh's Price Index Number.

10. $\sqrt{\frac{p_1 q_0}{\Sigma p_0 q_0}} \times \sqrt{\frac{\Sigma p_1 q_1}{\Sigma p_0 q_1}} \times 100$ is Fisher's Price Index Number.

Answers

1. True 2. False 3. False 4. False 5. False
6. False 7. False 8. False 9. False 10. False.

IV. Solve the following problems :

1. Find Price Index Number using Simple Aggregate Method. Consider 1980 as base year :

Commodity	Price in 1980 (in ₹)	Price in 1985 (in ₹)
I	22	46
II	38	36
III	20	28
IV	18	44
V	12	16

Solution :

Here, Base year = 1980.

Commodity	Base Year 1980	Current Year 1985
	p_0 (in ₹)	p_1 (in ₹)
I	22	46
II	38	36
III	20	28
IV	18	44
V	12	16
Total	$\Sigma p_0 = 110$	$\Sigma p_1 = 170$

Price Index Number by Simple Aggregate Method :

$$P_{01} = \frac{\Sigma p_1}{\Sigma p_0} \times 100$$

$$= \frac{170}{110} \times 100$$

$$= 1.5455 \times 100$$

$$= 154.55.$$

---

2. Find the Quantity Index Number using Simple Aggregate Method :

Commodity	Base Year Quantity	Current Year Quantity
A	100	130
B	170	200
C	210	250
D	90	110
E	50	150

Solution :

Commodity	Quantity	
	Base Year	Current Year
	q_0	q_1
A	100	130
B	170	200
C	210	250
D	90	110
E	50	150
Total	$\Sigma q_0 = 620$	$\Sigma q_1 = 840$

Quantity Index Number by Simple Aggregate Method :

$$Q_{01} = \frac{\Sigma q_1}{\Sigma q_0} \times 100$$

$$= \frac{840}{620} \times 100$$

$$= 1.3548 \times 100$$

$$= 135.48.$$

---

3. Find the Value Index Number using Simple Aggregate Method :

Commodity	Base Year		Current Year	
	Price	Quantity	Price	Quantity
I	20	42	22	45
II	35	60	40	58
III	50	22	55	24
IV	60	56	70	62
V	25	40	30	41

Solution :

Commodity	Base Year		Current Year		p_1q_1	p_0q_0
	p_0	q_0	p_1	q_1		
I	20	42	22	45	990	840
II	35	60	40	58	2320	2100
III	50	22	55	24	1320	1100
IV	60	56	70	62	4340	3360
V	25	40	30	41	1230	1000
Total					$\Sigma p_1q_1 = 10200$	$\Sigma p_0q_0 = 8400$

Value Index Number by Simple Aggregate Method :

$$V_{01} = \frac{\Sigma p_1q_1}{\Sigma p_0q_0} \times 100$$

$$= \frac{10200}{8400} \times 100 = 1.2143 \times 100 = 121.43.$$

---

4. Find x, if Price Index Number using Simple Aggregate Method is 200 :

Commodity	P	Q	R	S	T
Base Year Price	20	12	22	23	13
Current Year Price	30	x	38	51	19

Solution :

Commodity	Base Year price	Current Year price
	p_0	p_1
P	20	30
Q	12	x
R	22	38
S	23	51
T	13	19
Total	$\Sigma p_0 = 90$	$\Sigma p_1 = 138 + x$

Price Index Number by Simple Aggregate Method :

$$P_{01} = \frac{\Sigma p_1}{\Sigma p_0} \times 100$$

Given, $P_{01} = 200$

$$\therefore\ 200 = \frac{138 + x}{90} \times 100$$

$$\therefore\ \frac{200 \times 90}{100} = 138 + x$$

$$\therefore\ 180 = 138 + x$$

$$\therefore\ 180 - 138 = x$$

$$\therefore\ x = 42$$

Hence, x is 42.

5. Calculate Laspeyre's and Paasche's Price Index Number for the following data :

Commodity	Base Year		Current Year	
	Price p_0	Quantity q_0	Price p_1	Quantity q_1
A	20	18	30	15
B	25	8	28	5
C	32	5	40	7
D	12	10	18	10

Solution :

Commodity	Base Year		Current Year		p_1q_0	p_0q_0	p_1q_1	p_0q_1
	p_0	q_0	p_1	q_1				
A	20	18	30	15	540	360	450	300
B	25	8	28	5	224	200	140	125
C	32	5	40	7	200	160	280	224
D	12	10	18	10	180	120	180	120
Total					Σp_1q_0 $= 1144$	Σp_0q_0 $= 840$	Σp_1q_1 $= 1050$	Σp_0q_1 $= 769$

Laspeyre's Price Index Number :

$$P_{01}(L) = \frac{\Sigma p_1q_0}{\Sigma p_0q_0} \times 100$$

$$= \frac{1144}{840} \times 100$$

$$= 1.3619 \times 100$$

$$= 136.19$$

Paasche's Price Index Number :

$$P_{01}(P) = \frac{\Sigma p_1q_1}{\Sigma p_0q_1} \times 100$$

$$= \frac{1050}{769} \times 100$$

$$= 1.3654 \times 100$$

$$= 136.54$$

Hence, $P_{01}(L)$ is 136.19 and $P_{01}(P)$ is 136.54.

[**Note :** *Answers given in the textbook are incorrect.*]

6. Calculate Dorbish-Bowley's Price Index Number for the following data :

Commodity	Base Year		Current Year	
	Price p_0	Quantity q_0	Price p_1	Quantity q_1
I	8	30	11	28
II	9	25	12	22
III	10	15	13	11

Solution :

Commodity	Base Year		Current Year		p_1q_0	p_0q_0	p_1q_1	p_0q_1
	p_0	q_0	p_1	q_1				
I	8	30	11	28	330	240	308	224
II	9	25	12	22	300	225	264	198
III	10	15	13	11	195	150	143	110
Total					Σp_1q_0 $= 825$	Σp_0q_0 $= 615$	Σp_1q_1 $= 715$	Σp_0q_1 $= 532$

Dorbish-Bowley's Price Index Number :

$$P_{01}(D\text{-}B) = \frac{\frac{\Sigma p_1q_0}{\Sigma p_0q_0} + \frac{\Sigma p_1q_1}{\Sigma p_0q_1}}{2} \times 100$$

$$= \frac{\frac{825}{615} + \frac{715}{532}}{2} \times 100$$

$$= \frac{1.3415 + 1.3440}{2} \times 100$$

$$= \frac{2.6855}{2} \times 100$$

$$= 1.3427 \times 100 = 134.27.$$

7. Calculate Marshall-Edgeworth's Price Index Number for the following data :

Commodity	Base Year		Current Year	
	Price p_0	Quantity q_0	Price p_1	Quantity q_1
X	12	35	15	25
Y	29	50	30	70

Solution :

Commodity	Base Year		Current Year		p_1q_0	p_0q_0	p_1q_1	p_0q_1
	p_0	q_0	p_1	q_1				
X	12	35	15	25	525	420	375	300
Y	29	50	30	70	1500	1450	2100	2030
Total					Σp_1q_0 $=2025$	Σp_0q_0 $=1870$	Σp_1q_1 $=2475$	Σp_0q_1 $=2330$

Marshall-Edgeworth's Price Index Number :

$$P_{01}(\text{M-E}) = \frac{\Sigma p_1q_0 + \Sigma p_1q_1}{\Sigma p_0q_0 + \Sigma p_0q_1} \times 100$$

$$= \frac{2025 + 2475}{1870 + 2330} \times 100$$

$$= \frac{4500}{4200} \times 100$$

$$= 1.0714 \times 100 = 107.14.$$

8. Calculate Walsch's Price Index Number for the following data :

Commodity	Base Year		Current Year	
	Price p_0	Quantity q_0	Price p_1	Quantity q_1
I	8	30	12	25
II	10	42	20	16

Solution :

Commodity	Base year		Current year		q_0q_1	$\sqrt{q_0q_1}$	$p_1\sqrt{q_0q_1}$	$p_0\sqrt{q_0q_1}$
	p_0	q_0	p_1	q_1				
I	8	30	12	25	750	27.39	328.68	219.12
II	10	42	20	16	672	25.92	518.40	259.20
Total							$\Sigma p_1\sqrt{q_0q_1}$ $=847.08$	$\Sigma p_0\sqrt{q_0q_1}$ $=478.32$

Walsch's Price Index Number :

$$P_{01}(W) = \frac{\Sigma p_1\sqrt{q_0q_1}}{\Sigma p_0\sqrt{q_0q_1}} \times 100$$

$$= \frac{847.08}{478.32} \times 100$$

$$= 1.7709 \times 100$$

$$= 177.09 \approx 177.10$$

Hence, Walsch's Index Number is 177.09 ≈ 177.10.

9. Calculate Laspeyre's and Paasche's Price Index Number for the following data :

Commodity	Base Year		Current Year	
	Price p_0	Quantity q_0	Price p_1	Quantity q_1
I	8	30	12	25
II	10	42	20	16

Solution :

Commodity	Base Year		Current Year		p_1q_0	p_0q_0	p_1q_1	p_0q_1
	p_0	q_0	p_1	q_1				
I	8	30	12	25	360	240	300	200
II	10	42	20	16	840	420	320	160
Total					Σp_1q_0 $=1200$	Σp_0q_0 $=660$	Σp_1q_1 $=620$	Σp_0q_1 $=360$

Laspeyre's Price Index Number :

$$P_{01}(L) = \frac{\Sigma p_1q_0}{\Sigma p_0q_0} \times 100$$

$$= \frac{1200}{660} \times 100$$

$$= 1.8181 \times 100$$

$$= 181.82.$$

Paasche's Price Index Number :

$$P_{01}(P) = \frac{\Sigma p_1q_1}{\Sigma p_0q_1} \times 100$$

$$= \frac{620}{360} \times 100$$

$$= 1.7222 \times 100$$

$$= 172.22$$

Hence, Laspeyre's and Paasche's Price Index Numbers are 181.82 and 172.22 respectively.

10. Find x, if Laspeyre's Price Index Number is same as Paasche's Price Index Number for the following data :

Commodity	Base Year		Current Year	
	Price p_0	Quantity q_0	Price p_1	Quantity q_1
A	3	x	2	5
B	4	6	3	5

Solution :

Commodity	Base Year		Current Year		p_1q_0	p_0q_0	p_1q_1	p_0q_1
	p_0	q_0	p_1	q_1				
A	3	x	2	5	$2x$	$3x$	10	15
B	4	6	3	5	18	24	15	20
Total					Σp_1q_0 $= 18+2x$	Σp_0q_0 $= 24+3x$	Σp_1q_1 $= 25$	Σp_0q_1 $= 35$

Laspeyre's Price Index Number :

$$P_{01}(L) = \frac{\Sigma p_1q_0}{\Sigma p_0q_0} \times 100$$

$$= \frac{18+2x}{24+3x} \times 100$$

Paasche's Price Index Number :

$$P_{01}(P) = \frac{\Sigma p_1q_1}{\Sigma p_0q_1} \times 100$$

$$= \frac{25}{35} \times 100$$

Given : $P_{01}(L) = P_{01}(P)$

$\therefore \frac{18+2x}{24+3x} \times 100 = \frac{25}{35} \times 100$

$\therefore \frac{18+2x}{24+3x} = \frac{25}{35}$

$\therefore 35(18+2x) = 25(24+3x)$

$\therefore 630 + 70x = 600 + 75x$

$\therefore 630 - 600 = 75x - 70x$

$\therefore 30 = 5x$

$\therefore x = \frac{30}{5} \quad \therefore x = 6$

Hence, x is 6.

11. Find x, if Walsch's Price Index Number is 150 for the following data :

Commodity	Base Year		Current Year	
	Price p_0	Quantity q_0	Price p_1	Quantity q_1
A	5	3	10	3
B	x	4	16	9
C	15	5	23	5
D	10	2	26	8

Solution :

Commodity	Base year		Current year		q_0q_1	$\sqrt{q_0q_1}$	$p_1\sqrt{q_0q_1}$	$p_0\sqrt{q_0q_1}$
	p_0	q_0	p_1	q_1				
A	5	3	10	3	9	3	30	15
B	x	4	16	9	36	6	96	$6x$
C	15	5	23	5	25	5	115	75
D	10	2	26	8	16	4	104	40
Total							$\Sigma p_1\sqrt{q_0q_1}$ $= 345$	$\Sigma p_0\sqrt{q_0q_1}$ $= 130+6x$

Walsch's Price Index Number :

$$P_{01}(W) = \frac{\Sigma p_1\sqrt{q_0q_1}}{\Sigma p_0\sqrt{q_0q_1}} \times 100$$

$$= \frac{345}{130+6x} \times 100$$

But, given, $P_{01}(W) = 150$

$\therefore 150 = \frac{345}{130+6x} \times 100$

$\therefore 150(130+6x) = 345 \times 100$

$\therefore 19500 + 900x = 34500$

$\therefore 900x = 34500 - 19500$

$\therefore 900x = 15000 \quad \therefore x = \frac{15000}{900} \quad \therefore x = 16.66$

Hence, x is 16.66.

12. Find x, if the Paasche's Price Index Number is 140 for the following data :

Commodity	Base Year		Current Year	
	Price p_0	Quantity q_0	Price p_1	Quantity q_1
A	20	8	40	7
B	50	10	60	10
C	40	15	60	x
D	12	15	15	15

Solution :

Commodity	Base Year		Current Year		p_1q_1	p_0q_1
	p_0	q_0	p_1	q_1		
A	20	8	40	7	280	140
B	50	10	60	10	600	500
C	40	15	60	x	$60x$	$40x$
D	12	15	15	15	225	180
Total					Σp_1q_1 $= 1105 + 60x$	Σp_0q_1 $= 820 + 40x$

Paasche's Price Index Number :

$$P_{01}(P) = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times 100$$

$$= \frac{1105 + 60x}{820 + 40x} \times 100$$

But, it is given, $P_{01}(P) = 140$

$$\therefore \ 140 = \frac{1105 + 60x}{820 + 40x} \times 100$$

$$\therefore \ \frac{140}{100} = \frac{1105 + 60x}{820 + 40x}$$

$$\therefore \ 1.40(820 + 40x) = 1105 + 60x$$

$$\therefore \ 1148 + 56x = 1105 + 60x$$

$$\therefore \ 1148 - 1105 = 60x - 56x$$

$$\therefore \ 43 = 4x$$

$$\therefore \ x = \frac{43}{4}$$

$$\therefore \ x = 10.75$$

Hence, x is 10.75.

13. Given that, Laspeyre's and Paasche's Price Index Numbers are 25 and 16 respectively. Find Dorbish-Bowley's and Fisher's Price Index Numbers.

Solution :

Given : $P_{01}(L) = 25$, $P_{01}(P) = 16$, $P_{01}(D\text{-}B) = ?$ and $P_{01}(F) = ?$

We have, $P_{01}(D\text{-}B) = \dfrac{P_{01}(L) + P_{01}(P)}{2}$

$$= \frac{25 + 16}{2}$$

$$= \frac{41}{2}$$

$$= 20.5$$

Also, we have,

$$P_{01}(F) = \sqrt{P_{01}(L) \times P_{01}(P)}$$

$$= \sqrt{25 \times 16}$$

$$= \sqrt{400}$$

$$= 20$$

Hence, Dorbish-Bowley's and Fisher's Index Numbers are 20.5 and 20 respectively.

14. If Laspeyre's and Dorbish's Price Index Numbers are 150.2 and 152.8 respectively. Find Paasche's Price Index Number.

Solution :

Given, $P_{01}(L) = 150.2$, $P_{01}(D\text{-}B) = 152.8$, $P_{01}(P) = ?$

We have,

$$P_{01}(D\text{-}B) = \frac{P_{01}(L) + P_{01}(P)}{2}$$

$$\therefore \ 152.8 = \frac{150.2 + P_{01}(P)}{2}$$

$$\therefore \ 2 \times 152.8 = 150.2 + P_{01}(P)$$

$$\therefore \ 305.6 - 150.2 = P_{01}(P)$$

$$\therefore \ 155.4 = P_{01}(P)$$

$$\therefore \ P_{01}(P) = 155.4$$

Hence, Paasche's Index Number is 155.4.

15. If $\Sigma p_0 q_0 = 120$, $\Sigma p_0 q_1 = 160$, $\Sigma p_1 q_1 = 140$ and $\Sigma p_1 q_0 = 200$, find Laspeyre's, Paasche's, Dorbish-Bowley's and Marshall-Edgeworth's Price Index Numbers. [Note : *Question is modified.*]

Solution :

Given : $\Sigma p_0 q_0 = 120$, $\Sigma p_1 q_1 = 140$, $\Sigma p_0 q_1 = 160$, $\Sigma p_1 q_0 = 200$

Laspeyre's Price Index Number :

$$P_{01}(L) = \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100$$

$$= \frac{200}{120} \times 100$$

$$= 1.6667 \times 100 = 166.67$$

Paasche's Price Index Number :

$$P_{01}(P) = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times 100$$

$$= \frac{140}{160} \times 100 = 0.875 \times 100$$

$$= 87.5$$

Dorbish-Bowley's Price Index Number :

$$P_{01}(D\text{-}B) = \frac{\dfrac{\Sigma p_1 q_0}{\Sigma p_0 q_0} + \dfrac{\Sigma p_1 q_1}{\Sigma p_0 q_1}}{2} \times 100$$

$$= \frac{\dfrac{200}{120} + \dfrac{140}{160}}{2} \times 100$$

$$= \frac{1.6667 + 0.875}{2} \times 100$$

$$= \frac{2.5417}{2} \times 100$$

$$= 1.27085 \times 100 = 127.085$$

Marshall-Edgeworth's Price Index Number :

$$P_{01}(M\text{-}E) = \frac{\Sigma p_1 q_0 + \Sigma p_1 q_1}{\Sigma p_0 q_0 + \Sigma p_0 q_1} \times 100$$

$$= \frac{200 + 140}{120 + 160} \times 100$$

$$= \frac{340}{280} \times 100$$

$$= 1.2142 \times 100 = 121.42$$

Hence, Laspeyre's Price Index Numbers $= 166.67$

Paasche's Price Index Number $= 87.5$

Dorbish-Bowley's Price Index Number $= 127.085$

Marshall-Edgeworth's Price Index Number $= 121.42$.

16. Given : $\Sigma p_0 q_0 = 130$, $\Sigma p_1 q_1 = 140$, $\Sigma p_0 q_1 = 160$ and $\Sigma p_1 q_0 = 200$. Find Laspeyre's, Paasche's, Dorbish-Bowley's and Marshall-Edgeworth's Price Index Numbers. [Note : *Question is modified.*]

Solution :

Given : $\Sigma p_0 q_0 = 130$, $\Sigma p_1 q_1 = 140$, $\Sigma p_0 q_1 = 160$, $\Sigma p_1 q_0 = 200$,

Laspeyre's Price Index Number :

$$P_{01}(L) = \frac{\Sigma p_1 q_0}{\Sigma p_0 q_0} \times 100$$

$$= \frac{200}{130} \times 100$$

$$= 1.5385 \times 100$$

$$= 153.85$$

Paasche's Price Index Number :

$$P_{01}(P) = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times 100$$

$$= \frac{140}{160} \times 100 = 0.875 \times 100$$

$$= 87.5$$

Dorbish-Bowley's Price Index Number :

$$P_{01}(D\text{-}B) = \frac{(L) + P_{01}(P)}{2}$$

$$= \frac{153.85 + 87.5}{2}$$

$$= \frac{241.35}{2}$$

$$= 120.68$$

Marshall-Edgeworth's Price Index Number :

$$P_{01}(M\text{-}E) = \frac{\Sigma p_1 q_0 + \Sigma p_1 q_1}{\Sigma p_0 q_0 + \Sigma p_0 q_1} \times 100$$

$$= \frac{200 + 140}{130 + 160} \times 100$$

$$= \frac{340}{290} \times 100$$

$$= 1.1724 \times 100$$

$$= 117.24.$$

17. Given that $\Sigma p_1 q_1 = 300$, $\Sigma p_0 q_1 = 320$, $\Sigma p_0 q_0 = 120$ and Marshall-Edgeworth's Price Index Number is 120. Find Paasche's Price Index Number.

Solution :

Given, $\Sigma p_1 q_1 = 300$, $\Sigma p_0 q_1 = 320$, $\Sigma p_0 q_0 = 120$,

$P_{01}(M\text{-}E) = 120$, $P_{01}(P) = ?$

we have, $P_{01}(M\text{-}E) = \dfrac{\Sigma p_1 q_0 + \Sigma p_1 q_1}{\Sigma p_0 q_0 + \Sigma p_0 q_1} \times 100$

$$\therefore\ 120 = \frac{\Sigma p_1 q_0 + 300}{120 + 320} \times 100$$

$$\therefore\ 120 = \frac{\Sigma p_1 q_0 + 300}{440} \times 100$$

$$\therefore\ \frac{120 \times 440}{100} = \Sigma p_1 q_0 + 300$$

$$\therefore\ \frac{52800}{100} - 300 = \Sigma p_1 q_0$$

$$\therefore\ \Sigma p_1 q_0 = 528 - 300 = 228$$

Paasche's Price Index Number :

$$P_{01}(P) = \frac{\Sigma p_1 q_1}{\Sigma p_0 q_1} \times 100$$

$$= \frac{300}{228} \times 100$$

$$= 1.3158 \times 100$$

$$= 131.58$$

Hence, $\Sigma p_1 q_0 = 228$ and Paasche's Price Index Number is 131.58.

18. Calculate the Cost of Living Index Number for the following data :

Group	Base Year		Current Year
	Price p_0	Quantity q_0	Price p_1
Food	150	13	160
Clothing	170	18	150
Fuel and Lighting	175	10	190
House Rent	200	12	210
Miscellaneous	210	15	260

Solution :

Group	Base Year		Current Year	p_1q_0	p_0q_0
	p_0	q_0	p_1		
Food	150	13	160	2080	1950
Clothing	170	18	150	2700	3060
Fuel and Lighting	175	10	190	1900	1750
House rent	200	12	210	2520	2400
Miscellaneous	210	15	260	3900	3150
Total				$\Sigma p_1q_0 = 13100$	$\Sigma p_0q_0 = 12310$

Cost of Living Index Number :

$$\text{CLI} = \frac{\Sigma p_1q_0}{\Sigma p_0q_0} \times 100$$

$$= \frac{13100}{12310} \times 100$$

$$= 1.0642 \times 100$$

$$= 106.42.$$

19. Find the Cost of Living Index number by the weighted aggregate method :

Group	Food	Clothing	Fuel and Lighting	House Rent	Miscellaneous
I	78	80	110	60	90
W	5	3	4	2	6

Solution :

Group	I	W	IW
Food	78	5	390
Clothing	80	3	240
Fuel and Lighting	110	4	440
House Rent	60	2	120
Miscellaneous	90	6	540
Total		$\Sigma W = 20$	$\Sigma IW = 1730$

Cost of Living Index Number :

$$\text{CLI} = \frac{\Sigma IW}{\Sigma W}$$

$$= \frac{1730}{20}$$

$$= 86.5.$$

20. Find Cost of Living Index Number by Family Budget Method for the following data. Also, find the expenditure of a person in the year 2008 if his expenditure in 2005 was ₹ 10,000 :

Group	2005 Base Year Price	2008 Current Year Price	Weight
Food	12	60	25
Clothing	10	45	20
Fuel and Lighting	20	35	15
House Rent	25	20	30
Miscellaneous	16	48	10

Solution :

Group	2005 Base year	2008 Current year	$I = \frac{p_1}{p_0} \times 100$	W	IW
	p_0	p_1			
Food	12	60	$\frac{60}{12} \times 100 = 500$	25	12500
Clothing	10	45	$\frac{45}{10} \times 100 = 450$	20	9000
Fuel and Lighting	20	35	$\frac{35}{20} \times 100 = 175$	15	2625
House rent	25	20	$\frac{20}{25} \times 100 = 80$	30	2400
Miscellaneous	16	48	$\frac{48}{16} \times 100 = 300$	10	3000
Total				$\Sigma W = 100$	$\Sigma IW = 29525$

Cost of Living Index Number :

$$\text{CLI} = \frac{\Sigma IW}{\Sigma W}$$

$$= \frac{29525}{100}$$

$$= 295.25$$

Given : Expenditure in 2005 = ₹ 10,000

Now, expenditure in the year 2008

$$= \frac{\text{CLI for 2008} \times \text{Expenditure in 2005}}{100}$$

$$= \frac{295.25 \times 10000}{100} = ₹\ 29525$$

Hence, the expenditure of a person in the year 2008 is ₹ 29525.

21. Find x, if the Cost of Living Index number is 193 for the following data :

Group	Food	Clothing	Fuel and Lighting	House Rent	Miscellaneous
I	221	198	171	183	161
W	35	14	x	8	20

Solution :

Group	I	W	IW
Food	221	35	7735
Clothing	198	14	2772
Fuel and Lighting	171	x	$171x$
House Rent	183	8	1464
Miscellaneous	161	20	3220
Total		$\Sigma W = 77 + x$	$\Sigma IW = 15191 + 171x$

Cost of Living Index Number :

$$\text{CLI} = \frac{\Sigma IW}{\Sigma W} = \frac{15191 + 171x}{77 + x}$$

But it is given, CLI = 193

$$\therefore \ 193 = \frac{15191 + 171x}{77 + x}$$

$$\therefore \ 193(77 + x) = 15191 + 171x$$

$$\therefore \ 14861 + 193x = 15191 + 171x$$

$$\therefore \ 193x - 171x = 15191 - 14861$$

$$\therefore \ 22x = 330$$

$$\therefore \ x = \frac{330}{22}$$

$$\therefore \ x = 15$$

Hence, x is 15.

22. The Cost of Living Index Number for the years 2000 and 2003 are 150 and 210 respectively. A person earns ₹ 13,500 per month in the year 2000. What should be his monthly earning in the year 2003 in order to maintain the same standard of living?

Solution :

Given : CLI for 2000 = 150

CLI for 2003 = 210

The earning p.m. of a person in the year 2000 = ₹ 13500.

Now, the earning of a person in the year 2003

$$= \frac{\text{Earning in the year 2000} \times \text{CLI for 2003}}{\text{CLI for 2000}}$$

$$= \frac{13500 \times 210}{150}$$

= ₹ 18900

Hence, the earning of a person should be ₹ 18,900 p.m. so as to maintain his former standard of living.

ACTIVITIES | **Textbook page 94**

Try each of the following activities for better understanding of index numbers.

1. Find weekly prices of any five vegetables for at least six months. Taking the first week of observation as the base period, find Price Index Numbers for the remaining five months for every vegetable :

Vegetables	Weekly price per kg (in ₹)					
	Jan. 20	Feb. 20	Mar. 20	April 20	May 20	June 20
Ladyfinger	30	36	36	39	42	45
Capsicum	25	30	35	40	45	50
Brinjal	40	42	42	45	40	48
Tomato	25	30	32	35	35	40
Potato	20	22	22	25	30	35

Base period : Jan. 20 $\qquad \therefore \ p_0$ = Price of Jan. 20

$\therefore$ Price Index is 100 for Jan. 20 $\qquad p_1$ = Prices of the rest months.

Vegetables	Price Index $= \frac{p_1}{p_0} \times 100$					
	Jan. 20	Feb. 20	Mar. 20	April 20	May 20	June 20
Ladyfinger	100	$\frac{36}{30} \times 100 = 120$	$\frac{36}{30} \times 100 = 120$	$\frac{39}{30} \times 100 = 130$	$\frac{42}{30} \times 100 = 140$	$\frac{45}{30} \times 100 = 150$
Capsicum	100	$\frac{30}{25} \times 100 = 120$	$\frac{35}{25} \times 100 = 140$	$\frac{40}{25} \times 100 = 160$	$\frac{45}{25} \times 100 = 180$	$\frac{50}{25} \times 100 = 200$
Brinjal	100	$\frac{42}{40} \times 100 = 105$	$\frac{42}{40} \times 100 = 105$	$\frac{45}{40} \times 100 = 112.5$	$\frac{40}{40} \times 100 = 100$	$\frac{48}{40} \times 100 = 120$
Tomato	100	$\frac{30}{25} \times 100 = 120$	$\frac{32}{25} \times 100 = 128$	$\frac{35}{25} \times 100 = 140$	$\frac{35}{25} \times 100 = 140$	$\frac{40}{25} \times 100 = 160$
Potato	100	$\frac{22}{20} \times 100 = 110$	$\frac{22}{20} \times 100 = 110$	$\frac{25}{20} \times 100 = 125$	$\frac{30}{20} \times 100 = 150$	$\frac{35}{20} \times 100 = 175$

2. Note the SENSEX for six months. Talking the first month as the base period. Find price index numbers for the remaining five months :

Months	SENSEX	Index numbers Base : 32500
Oct. 19	32500	$= 100$
Nov. 19	33150	$\frac{33150}{32500} \times 100 = 102$
Dec. 19	33300	$\frac{33300}{32500} \times 100 = 102.46$
Jan. 20	32800	$\frac{32800}{32500} \times 100 = 100.92$
Feb. 20	34000	$\frac{34000}{32500} \times 100 = 104.62$
Mar. 20	33900	$\frac{33900}{32500} \times 100 = 104.31$

3. Note inflation rate for six months. Taking the first month as the base period. Find price index numbers for the remaining five months :

Months	Inflation rate %	Index numbers Base : 1st month
1st	8.4	$= 100$
2nd	8.5	$\frac{8.5}{8.4} \times 100 = 101.19$
3rd	9.0	$\frac{9.0}{8.4} \times 100 = 107.14$
4th	8.8	$\frac{8.8}{8.4} \times 100 = 104.76$
5th	9.0	$\frac{9.0}{8.4} \times 100 = 107.14$
6th	9.3	$\frac{9.3}{8.4} \times 100 = 110.71$

4. Note petrol prices for six months. Taking the first month as the base period, find price index numbers for the remaining five months :

Months	Petrol prices per litre (in ₹)	Index numbers Base : Jan. 20
Jan. 20	72.00	$= 100$
Feb. 20	72.60	$\frac{72.60}{72} \times 100 = 100.83$
Mar. 20	72.62	$\frac{72.62}{72} \times 100 = 100.86$
April 20	72.70	$\frac{72.70}{72} \times 100 = 100.97$
May 20	72.75	$\frac{72.75}{72} \times 100 = 101.04$
June 20	72.80	$\frac{72.80}{72} \times 100 = 101.11$

5. Note gold prices for six months. Taking the first month as the base period, find price index numbers for the remaining five months :

Months	Gold prices per gram (in ₹)	Index numbers Base : June '19
June '19	3300	$= 100$
July '19	3350	$\frac{3350}{3300} \times 100 = 101.52$
Aug. '19	3400	$\frac{3400}{3300} \times 100 = 103.03$
Sept. '19	3280	$\frac{3280}{3300} \times 100 = 99.39$
Oct. '19	3370	$\frac{3370}{3300} \times 100 = 102.12$
Nov. '19	3520	$\frac{3520}{3300} \times 100 = 107.67$

* * *

6 LINEAR PROGRAMMING

CHAPTER OUTLINE

INTRODUCTION

Linear programming was first introduced for American Air Force in 1939 by George B. Dantzig for planning war strategies hence to optimise the utilization of limited resources to get maximum returns. After the Second World War its scope widened. Many industries took its advantage as an optimization technique for different fields such as management, planning, production, transportation, decision-making, etc. Thus, today linear programming is one of the best and widely used developed optimization techniques. The different optimization techniques are known as Operation Research.

Linear programming problems are related to efficient use of limited resources such as man power, raw material, cost of material, availability of machine hours, warehouse space and so on. Linear programming is a mathematical technique designed to help the managers in their planning and decision-making. It is usually used to optimize a certain function called objective function subject to given conditions or restrictions known as constraints.

Applications of Linear Programming : Linear programming under the caption Operation Research, is the technique widely used in many fields.

(1) In planning the best strategy in a war and to plan expenditures and returns in order to reduce cost to the army and increase losses to the enemy.

(2) Linear Programming helps managers of organizations in their planning and decision-making.

(3) It is used in micro-economics and company management such as planning, production, transportation, technology and other issues for optimizing the object like to maximize profit or minimize cost with limited resources.

(4) In developing production schedule to minimize total production and inventory costs.

(5) In establishing an investment portfolio to maximize return on investment.

(6) In determining a distribution system of transportation that will minimize the total shipping cost from several warehouses to various market locations. Such problem is referred to as Transportation problem.

(7) In allocating different jobs to different persons that will minimize the time and cost. Such problem is referred to as Assignment problem.

(8) In allocation of a limited advertising budget in order to maximize advertising effectiveness.

(9) In diet problem where the object is to minimize the cost of diet with a certain minimum amount of each nutrient required.

In the earliear classes, we have studied graphical solution of linear inequations in two variables. Now, we shall study the graphical solutions to find the maximum / minimum value of a linear expression.

IMPORTANT POINTS TO REMEMBER

1. **Linear Inequations in two variables :**

$ax+by+c \geqslant 0$, $ax+by+c \leqslant 0$, where, a b, c are real numbers and $a, b \neq 0$.

Solution : The set of all ordered pairs (x, y) which satisfy the given inequations is called the solution set of the inequations.

2. **Graphical representation :** Graphical representation (graphs) is a region on either side of straight line $ax+by=c$ in the Cartesian Coordinate System.

3. **Common region and Feasible region :** The set of ordered pairs (x, y) that satisfy all the inequations in the given system of linear inequations is the common solution and the region represented by the common solution is called common region. If it lies in the first quadrant, it is called feasible region.

4. **Linear Programming Problem (LPP) :** It is defined as the problem of maximising or minimizing a linear function (objective function) subject to linear constraints. The constraints may be equalities or inequalities.

(1) The mathematical structure of the general LPP with two variables :

Optimize $z=c_1x_1+c_2x_2$

Subject to linear constraints

$a_{11}x_1+a_{12}x_2\ (\leqslant = \geqslant)\ b_1$

$a_{21}x_1+a_{22}x_2\ (\leqslant = \geqslant)\ b_2$

....

$a_{m1}x_1+a_{m2}x_2\ (\leqslant = \geqslant) b_m$

where, x_1, x_2 are two decision variables

c_1, c_2 are profit or cost parameters of the problem,

b_1, b_2, ... b_m are requirement parameters

$z=$ objective function.

The problem is to determine the values of x_1 and x_2 which satisfy all the constraints and non-negativity restriction of the LPP so that z is either maximized or minimized.

(2) Solution of LPP :

(a) Methods : LPP can be solved by two methods: (i) Graphical method (ii) Simplex method.

[Note : *We will discuss about graphical method only in this chapter.*]

(b) Solution : A set of values the variables which satisfies all the constraints of LPP is called solution of the LPP.

(c) Feasible Solution : A solution of the LPP which also satisfies non-negativity restriction of the LPP, if present, is called feasible solution.

(d) Optimum Feasible Solution : A feasible solution which optimises, i.e. either maximizes or minimizes the objective function of the LPP is called optimum feasible solution.

(e) Convex Set : The set of all feasible solutions of LPP is a convex set.

(f) Optimum Value : The objective function of LPP attains its optimum value (i.e., either maximum or minimum) at, at least one of the vertices of the convex polygon.

6.1. MEANING OF LINEAR PROGRAMMING PROBLEM (LPP)

A linear programming problem may be defined as the problem of maximizing or minimizing (i.e., optimizing) a linear function subject to linear constraints. The constraints may be equations or inequations.

Here, the word 'linear' is used to indicate that all the mathematical functions are linear and the word 'programming' (planning) refers to the process of determining a particular programme of action out of several alternatives available.

[Note : We shall study LPP with at most two variables.]

Some terms used in LPP :

(1) **Decision variables :** The variables involved in LPP are called decision variables. If all the expressions of LPP are linear functions of two variables x and y, then these variables are decision variables.

(2) **Objective function :** A linear function of variables which is to be optimized, i.e. either maximized or minimized is called objective function of LPP. Usually it is denoted by z.

e.g. Find the values of x and y such that $z=2x+5y$ is optimized subject to given constraints.

(3) **Constraints :** Conditions under which the objective function is to be optimized are called constraints. These constraints are in the form of linear equations or inequations.

e.g. $x \geqslant 0$, $y \geqslant 0$, $x+y \leqslant 6$, $2x+3y \geqslant 15$.

(4) **Non-negativity constraints :** The values of variables under consideration may be positive or zero but not negative, such constraints are referred to as non-negativity restrictions. e.g. $x \geqslant 0$, $y \geqslant 0$.

6.2 : MATHEMATICAL FORMULATION OF LPP

The mathematical structure of the general LPP with two variables is as follows :

Optimize $z = c_1x_1 + c_2x_2$

Subject to constraints

$$a_{11}x_1 + a_{12}x_2 \;(\leqslant = \geqslant)\; b_1$$
$$a_{21}x_1 + a_{22}x_2 \;(\leqslant = \geqslant)\; b_2$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 \;(\leqslant = \geqslant)\; b_m$$

where, x_1 and x_2 are decision variables

c_1 and c_2 are profit or cost parameters

b_i are known values

a_{ij} are activity parameters

z is objective function

The problem is to determine the values of x_1 and x_2 which satisfy all the constraints, non-negativity restrictions of the LPP so that the value of the objective function z is either maximized or minimized.

EXERCISE 6.1 | Textbook pages 97 to 99

1. A manufacturing firm produces two types of gadgets A and B, which are first processed in the foundry and then sent to machine shop for finishing. The number of manhours of labour required in each shop for production of A and B and the number of manhours available for the firm are as follows :

Gadgets	Foundry	Machine shop
A	10	5
B	6	4
Time available (in hour)	60	35

Profit on the sale of A is ₹ 30 and B is ₹ 20 per unit. Formulate the LPP to have maximum profit.

Solution : Let the number of gadgets A produced by the firm be x and the number of gadgets B produced by the firm be y.

The profit on the sale of A is ₹ 30 per unit and on the sale of B is ₹ 20 per unit.

$\therefore$ the total profit is $z = 30x + 20y$.

This is a linear function which is to be maximized. Hence, it is the objective function.

The constraints are as per the following table :

	Gadget A (x)	Gadget B (y)	Total available Time (in hour)
Foundry	10	6	60
Machine shop	5	4	35

From the above table total manhours of labour required for x units of gadget A and y units of gadget B in foundry is $(10x + 6y)$ hours and total manhours of labour required in machine shop is $(5x + 4y)$ hours.

Since, maximum time avilable in foundry and machine shops are 60 hours and 35 hours respectively,

therefore, the constraints are $10x + 6y \leqslant 60$, $5x + 4y \leqslant 35$.

Since, x and y cannot be negative, we have $x \geqslant 0$, $y \geqslant 0$.

Hence, the given LPP can be formulated as :

Maximize $z = 30x + 20y$, subject to $10x + 6y \leqslant 60$, $5x + 4y \leqslant 35$, $x \geqslant 0$, $y \geqslant 0$.

2. In a cattle breeding firm, it is prescribed that the food ration for one animal must contain 14, 22 and 1 unit of nutrients A, B and C respectively. Two different kinds of fodder are available. Each unit weight of these two contains the following amounts of these three nutrients :

Fodder → Nutrients ↓	Fodder 1	Fodder 2
Nutrients A	2	1
Nutrients B	2	3
Nutrients C	1	1

The cost of fodder 1 is ₹ 3 per unit and that of fodder 2 is ₹ 2 per unit. Formulate the LPP to minimize the cost.

Solution : Let x units of fodder 1 and y units of fodder 2 be prescribed.

The cost of fodder 1 is ₹ 3 per unit and cost of fodder 2 is ₹ 2 per unit.

$\therefore$ the total cost is $z = 3x + 2y$

This is the linear function which is to be minimized. Hence, it is the objective function. The constraints are as per the following table :

Fodder → Nutrient ↓	Fodder 1	Fodder 2	Minimum requirements
Nutrients A	2	1	14
Nutrients B	2	3	22
Nutrients C	1	1	1

From the above table fodder contains $(2x+y)$ units of nutrients A, $(2x+3y)$ units of nutrients B and $(x+y)$ units of nutrients C. The minimum requirements of these nutrients are 14 units, 22 units and 1 unit respectively.

Therefore, the constraints are

$2x+y \geqslant 14,\ \ 2x+3y \geqslant 22,\ \ x+y \geqslant 1$

Since, number of units (i.e. x and y) cannot be negative, we have, $x \geqslant 0,\ y \geqslant 0$.

Hence, the given LPP can be formulated as

Minimize $z = 3x+2y$, subject to $2x+y \geqslant 14$, $2x+3y \geqslant 22$, $x+y \geqslant 1$, $x \geqslant 0$, $y \geqslant 0$.

3. A company manufactures two types of chemicals A and B. Each chemical requires two types of raw material P and Q. The table below shows number of units of P and Q required to manufacture one unit of A and one unit of B.

Chemical → Raw Material ↓	A	B	Availability
P	3	2	120
Q	2	5	160

The company gets profits of ₹ 350 and ₹ 400 by selling one unit of A and one unit of B respectively. Formulate the problem as LPP to maximize profit.

Solution : Let the company manufactures x units of chemical A and y units of chemical B. Then the total profit to the company is $p =$ ₹ $(350x+400y)$.

This is a linear function which is to be maximized. Hence, it is the objective function.

The constraints are as per the following table :

Chemical → Raw Material ↓	A (x)	B (y)	Availability
P	3	2	120
Q	2	5	160

The raw material P required for x units of chemical A and y units of chemical B is $3x+2y$. Since, the maximum availability of P is 120, we have the first constraint as $3x+2y \leqslant 120$.

Similarly, considering the raw material Q, we have $2x+5y \leqslant 160$.

Since, x and y cannot be negative, we have, $x \geqslant 0, y \geqslant 0$.

Hence, the given LPP can be formulated as :

Maximize $p = 350x+400y$, subject to $3x+2y \leqslant 120$, $2x+5y \leqslant 160$, $x \geqslant 0$, $y \geqslant 0$.

4. A printing company prints two types of magazines A and B. The company earns ₹ 10 and ₹ 15 on magazines A and B per copy. These are processed on three Machines I, II, III. Magazine A requires 2 hours on Machine I, 5 hours on Machine II, and 2 hours on machine III. Magazine B requires 3 hours on machine I, 2 hours on machine II, and 6 hours on Machine III. Machines I, II, III are available for 36, 50 and 60 hours per week respectively. Formulate the Linear programming problem to maximize the profit.

Solution : Let the company prints x magazine of type A and y magazine of type B.

Profit on sale of magazine A is ₹ 10 per copy and magazine B is ₹ 15 per copy.

Therefore, the total earning z of the company is $z =$ ₹ $(10x+15y)$.

This is a linear function which is to be maximized. Hence, it is the objective function.

The constraints are as per the following table :

Magazine type → Machine type ↓	Time required per unit		Available time per week (in hours)
	Magazine A (x)	Magazine B (y)	
Machine I	2	3	36
Machine II	5	2	50
Machine III	2	6	60

From the above table, the total time required for Machine I is $(2x+3y)$ hours, for Machine II is $(5x+2y)$ hours and for Machine III is $(2x+6y)$ hours. The machines I, II, III are available for 36, 50 and 60 hours per week. Therefore, the constraints are $2x+3y \leqslant 36$, $5x+2y \leqslant 50$, $2x+6y \leqslant 60$. Since, x and y cannot be negative, we have,

$x \geqslant 0$, $y \geqslant 0$. Hence, the given LPP can be formulated as :

Maximize $z = 10x + 15y$, subject to $2x + 3y \leqslant 36$, $5x + 2y \leqslant 50$, $2x + 6y \leqslant 60$, $x \geqslant 0$, $y \geqslant 0$.

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5. A manufacturer produces bulbs and tubes. Each of these must be processed through two machines M_1 and M_2. A package of bulbs require 1 hour of work on Machine M_1 and 3 hours of work on Machine M_2. A package of tubes require 2 hours on Machine M_1 and 4 hours on Machine M_2. He earns a profit of ₹ 13.5 per package of bulbs and ₹ 55 per package of tubes. Formulate the LLP to maximize the profit.

Solution : Let the number of packages of bulbs produced by manufacturer be x and packages of tubes be y. The manufacturer earns a profit of ₹ 13.5 per package of bulbs and ₹ 55 per package of tubes.

Therefore, his total profit is $p = ₹\ (13.5x + 55y)$

This is a linear function which is to be maximized. Hence, it is the objective function.

The constraints are as per the following table :

	Bulbs (x)	Tubes (y)	Available Time
Machine M_1	1	2	10
Machine M_2	3	4	12

From the above table, the total time required for Machine M_1 is $(x + 2y)$ hours and for Machine M_2 is $(3x + 4y)$ hours. Given Machine M_1 and M_2 are available for at most 10 hours and 12 hours a day respectively.

Therefore, the constraints are $x + 2y \leqslant 10$, $3x + 4y \leqslant 12$.

Since, x and y cannot be negative, we have, $x \geqslant 0$, $y \geqslant 0$.

Hence, the given LPP can be formulated as :

Maximize $p = 13.5x + 55y$, subject to $x + 2y \leqslant 10$, $3x + 4y \leqslant 12$, $x \geqslant 0$, $y \geqslant 0$.

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6. A company manufactures two types of fertilizers F_1 and F_2. Each type of fertilizer requires two raw materials A and B. The number of units of A and B required to manufacture one unit of fertilizer F_1 and F_2 and availability of the raw materials A and B per day are given in the following table :

Fertilizers → Raw Materials ↓	F_1	F_2	Availability
A	2	3	40
B	1	4	70

By selling one unit of F_1 and one unit of F_2, company gets a profit of ₹ 500 and ₹ 750 respectively. Formulate the problem as LPP to maximize the profit.

Solution : *Refer to the solution of Q. 3.*

Ans. Maximize $z = 500x + 750y$, subject to $2x + 3y \leqslant 40$, $x + 4y \leqslant 70$, $x \geqslant 0$, $y \geqslant 0$.

---

7. A doctor has prescribed two different kinds of feeds A and B to form a weekly diet for a sick person. The minimum requirement of fats, carbohydrates and proteins are 18, 28, 14 units respectively. One unit of food A has 4 units of fat, 14 units of carbohydrates and 8 units of protein. One unit of food B has 6 units of fat, 12 units of carbohydrates and 8 units of protein. The price of food A is ₹ 4.5 per unit and that of food B is ₹ 3.5 per unit. Form the LPP, so that the sick person's diet meets the requirements at a minimum cost.

Solution : Let the diet of sick person include x units of food A and y units of food B.

Then $x \geqslant 0$, $y \geqslant 0$.

The prices of food A and B is ₹ 4.5 and ₹ 3.5 per unit respectively.

Therefore, the total cost is $z = ₹\ (4.5x + 3.5y)$.

This is the linear function which is to be minimized. Hence, it is objective function.

The constraints are as per the following table :

Food Type→ Ingredients↓	Food A (x)	Food B (y)	Minimum requirements
Fats	4	6	18
Carbohydrates	14	12	28
Proteins	8	8	14

From the above table, the sick person's diet will include $(4x + 6y)$ units of fats, $(14x + 12y)$ units of carbohydrates and $(8x + 8y)$ units of proteins. The minimum requirements of these ingredients are 18 units, 28 units and 14 units respectively.

Therefore, the constraints are

$4x+6y \geqslant 18,\ \ 14x+12y \geqslant 28,\ \ 8x+8y \geqslant 14.$

Hence, the given LPP can be formulated as

Minimize $z = 4.5x + 3.5y$, subject to $4x+6y \geqslant 18$, $14x+12y \geqslant 28$, $8x+8y \geqslant 14$, $x \geqslant 0$, $y \geqslant 0$.

[**Note :** *The inequality $7x+8y \geqslant 14$ in the textbook is incorrect.*]

8. If John drives a car at a speed of 60 km/hour, he has to spend ₹ 5 per km on petrol. If he drives at a faster speed of 90 km/hour, the cost of petrol increases to ₹ 8 per km. He has ₹ 600 to spend on petrol and wishes to travel the maximum distance within an hour. Formulate the above problem as LPP.

Solution : Let John travel x_1 km at a speed of 60 km/hour and x_2 km at a speed of 90 km/hour.

Therefore, time required to travel a distance of x_1 km is $\frac{x_1}{60}$ hours and the time required to travel a distance of x_2 km is $\frac{x_2}{90}$ hours.

Then total time required to travel is $\left(\frac{x_1}{60}+\frac{x_2}{90}\right)$ hours.

Since, he wishes to travel the maximum distance within an hour,

$$\frac{x_1}{60}+\frac{x_2}{90} \leqslant 1$$

He has to spend ₹ 5 per km on petrol at a speed of 60 km/hour and ₹ 8 per km at a speed of 90 km/hour.

$\therefore$ the total cost of travelling is ₹ $(5x_1+8x_2)$

Since, he has ₹ 600 to spend on petrol,

$$5x_1+8x_2 \leqslant 600$$

Since, distance is never negative, $x_1 \geqslant 0$, $x_2 \geqslant 0$.

Total distance travelled by John is $z = (x_1+x_2)$ km.

This is the linear function which is to be maximized.

Hence, it is objective function.

Hence, the given LPP can be formulated as :

Maximize $z = x_1 + x_2$, subject to $\frac{x_1}{60}+\frac{x_2}{90} \leqslant 1$, $5x_1+8x_2 \leqslant 600$, $x_1 \geqslant 0$, $x_2 \geqslant 0$.

9. The company makes concrete bricks made up of cement and sand. The weight of a concrete brick has to be at least 5 kg. Cement costs ₹ 20 per kg and sand costs ₹ 6 per kg. Strength consideration dictate that a concrete brick should contain minimum 4 kg of cement and not more than 2 kg of sand. Formulate the LPP for the cost to be minimum.

Solution : Let the company use x_1 kg of cement and x_2 kg of sand to make concrete bricks.

Cement costs ₹ 20 per kg and sand costs ₹ 6 per kg.

$\therefore$ the total cost $c =$ ₹ $(20x_1+6x_2)$

This is a linear function which is to be minimized.

Hence, it is the objective function.

Total weight of brick $= (x_1+x_2)$ kg

Since, the weight of concrete brick has to be at least 5 kg,

$\therefore\ x_1+x_2 \geqslant 5.$

Since, concrete brick should contain minimum 4 kg of cement and not more than 2 kg of sand,

$x_1 \geqslant 4$ and $0 \leqslant x_2 \leqslant 2$

Hence, the given LPP can be formulated as :

Minimize $c = 20x_1+6x_2$, subject to $x_1+x_2 \geqslant 5$, $x_1 \geqslant 4$, $0 \leqslant x_2 \leqslant 2$, $x_1 \geqslant 0$, $x_2 \geqslant 0$.

EXAMPLES FOR PRACTICE 6.1

1. A company manufactures two models of cars : model A and model B. To stay in business it must produce at least 50 cars of model A per month. It has facilities to produce at most 200 cars of model A and 150 cars of model B per month. The total demand for both models together is at most 300 cars per month.

   The profit per car is ₹ 4000 for model A and ₹ 3000 for model B. It is required to determine the number of cars of each type to be manufactured so as to get maximum profit. Formulate this problem as a LPP.

2. The manager of an oil refinery must decide on the optimal mix of two possible blending processes of which the inputs and outputs per production run are as follows :

Process	Input		Output	
	Crude A	Crude B	Gasoline X	Gasoline Y
1	5	3	5	8
2	4	5	4	4

   The maximum amounts available of crudes A and B are 200 units and 150 units respectively. Market requirements show that at least 100 units of gasoline X and 80 units of gasoline Y must be produced. The profit per production run from process 1 and process 2 are 3 and 4 respectively. Formulate this problem as a LPP so that profit is maximum.

3. Give the mathematical form of the following LPP. A manufacturer produces two types of products A and B. There are two sections in the factory. Each product of type A is to be processed for 2 hours in section I and 1 hour in section II. Each product of type B is to be processed for 1 hour in section I and 2 hours in section II. The section I and section II can be used for maximum 104 hours and 76 hours per month respectively. Each product of A gives a profit of ₹ 6 and that of product B gives a profit of ₹ 11. The manufacturer wants to maximize the profit.

4. A person likes to decide on the constituents of a diet which will satisfy his daily needs of proteins, fat and carbohydrates at the minimum cost. Choices from three different types of foods can be made. The yields per unit of this food are given in the following table :

Food type	**Yields per unit**			**Cost per unit (in ₹)**
	Proteins	**Fat**	**Carbohydrates**	
A	4	1	3	2
B	2	4	1	7
C	4	2	1	3
Daily requirement	6	2	3	

Formulate the LPP.

5. A dealer wishes to purchase a number of fans and sewing machines. He has only ₹ 5760 to invest and has space for atmost 20 items. A fan costs him ₹ 360 and a sewing machine ₹ 240. His expectation is to sell at least 5 fans at a profit of ₹ 22 and sewing machine at a profit of ₹ 18. Assuming that he can sell all the items that he can buy, how should he invest his money in order to maximize his profit? Formulate this problem as LPP.

6. A company manufactures two types of toys A and B. Each toy of type A requires 2 minutes for cutting and 1 minute for assembling. Each toy of type B requires 3 minutes for cutting and 4 minutes for assembling. There are 3 hours available for cutting and 2 hours and 40 minutes available for assembling. On selling a toy of type A, he gets a profit of ₹ 10 and that on selling a toy of type B, he gets a profit of ₹ 20. Formulate this problem to maximize the profit.

7. A machine is used for producing two products I and II. Product I is produced by using 3 units of chemical salt and 2 units of chemical mixture. Product II is produced by using 2 units of chemical salt and 4 units of chemical mixture. Only 1000 units of chemical salt and 1500 units of mixture are available. The profit on product I is ₹ 25 and on II, it is ₹ 20. Give the mathematical formulation for this LPP to maximize the profit.

8. A firm produces two products A and B. The profit on each unit of product A is ₹ 40 and that on each unit of product B is ₹ 50. Both the products are processed on three machines M_1, M_2 and M_3. The time required in hours by each product and the total time available in hours per week on each machine is as given below in the table. Formulate this problem as LPP to maximize the profit.

Product	**Machines**		
	M_1	**M_2**	**M_3**
A	3	4	2
B	4	5	8
Time available in hours	36	48	70

Answers

1. $x =$ Cars of model A, $y =$ Cars of model B.
Maximize $z = 4000x + 3000y$
subject to constraints, $50 \leqslant x \leqslant 200$. $y \leqslant 150$, $x + y \leqslant 300$, $x \geqslant 0$, $y \geqslant 0$.

2. $x_1 =$ Production run in process–1,
$x_2 =$ Production run in process–2.
Maximize $z = 3x_1 + 4x_2$
subject to constraints, $5x_1 + 4x_2 \leqslant 200$, $3x_1 + 5x_2 \leqslant 150$, $5x_1 + 4x_2 \geqslant 100$, $8x_1 + 4x_2 \geqslant 80$, $x_1, x_2 \geqslant 0$.

3. $x_1 =$ Number of product A, $x_2 =$ Number of product B
Maximize $z = 6x_1 + 11x_2$

subject to constraints, $2x_1 + x_2 \leqslant 104$, $x_1 + 2x_2 \leqslant 76$, $x_1, x_2 \geqslant 0$.

4. $x_1 =$ Units of food A, $x_2 =$ Units of food B, $x_3 =$ Units of food C
Minimize $z = 2x_1 + 7x_2 + 3x_3$
subject to constraints,
$4x_1 + 2x_2 + 4x_3 \geqslant 6$, $x_1 + 4x_2 + 2x_3 \geqslant 2$,
$3x_1 + x_2 + x_3 \geqslant 3$, $x_1, x_2, x_3 \geqslant 0$.

5. $x_1 =$ Number of fans, $x_2 =$ Number of sewing machines.
Maximize $z = 22x_1 + 18x_2$
subject to constraints, $360x_1 + 240x_2 \leqslant 5760$,
$x_1 + x_2 \leqslant 20$. $x_1 \geqslant 5$. $x_1, x_2 \geqslant 0$.

6. Maximize $z = 10x + 20y$, subject to $2x + 3y \leqslant 180$, $x + 4y \leqslant 160$, $x \geqslant 0$, $y \geqslant 0$.

7. Maximize $z = 25x + 20y$, subject to $3x + 2y \leqslant 1000$, $2x + 4y \leqslant 1500$, $x \geqslant 0$, $y \geqslant 0$.

8. Maximize $z = 40x + 50y$, subject to $3x + 4y \leqslant 36$, $4x + 5y \leqslant 48$, $2x + 8y \leqslant 70$, $x \geqslant 0$, $y \geqslant 0$.

6.3 : SOLUTION OF LPP BY GRAPHICAL METHOD

1. Convex set : A set of points in a plane is said to be a convex set if the line joining any two points of the set entirely lies within the set, i.e. convex set is a geometrical figure having no holes in it.

The following sets are convex sets :

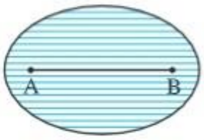


Fig. 6.1

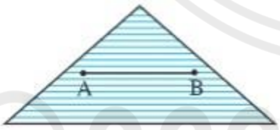


Fig. 6.2

The following sets are not convex sets :

Fig. 6.3

Fig. 6.4

[Notes :

(1) The convex sets may be bounded. Following are bounded convex sets.

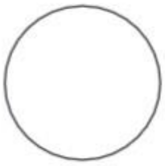


Fig. 6.5

Fig. 6.6

(2) The convex sets may be unbounded. Following are unbounded convex sets.

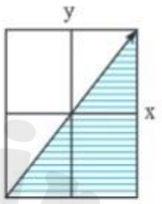


Fig. 6.7

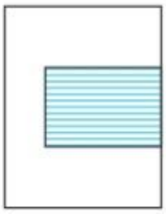


Fig. 6.8]

2. Feasible Region : The common region determined by all the constraints and non-negativity restrictions of the linear programming problem is called the feasible region.

[Note : The boundaries of the region may or may not be included in the feasible region.]

Theorem 1 : The set of all feasible solutions of LPP is a convex set.

Theorem 2 : The objective function of LPP attains its optimum value (either maximum or minimum) at least one of the vertices of the convex polygon. This is known as convex polygon theorem.

3. Solution of LPP : Two different methods of obtaining the solution of LPP are (i) Graphical method and (ii) Simplex method. We restrict to grpahical method only.

Graphical method : LPP with two decision variables can easily be solved graphically.

Some definitions :

★ **Solution :** A set of values of variables which satisfies all the constraints of the LPP is called solution of the LPP.

★ **Feasible Solution :** A solution of the LPP which also satisfies non-negativity restrictions of the LPP, if present, is called feasible solution.

★ **Optimum Feasible Solution :** A feasible solution which optimizes, i.e. either maximizes or minimizes the objective function of the LPP is called optimum feasible solution.

[**Notes :**

(1) If a LPP has optimum solutions at more than one point then the entire line joining those points will give optimum solutions. Hence the problem will have infinite solution.

(2) Graphical method of solving a LPP is also known as Corner Point Method (Convex Polygon Theorem).]

Algorithm of Graphical Method :

(1) Convert all inequations of constraints into equations.

(2) Draw the lines of the equations in XY plane, find at least two points $(x, 0)$ and $(0, y)$.

(3) Locate common region indicated by the constraints, which is feasible region.

(4) Find the vertices of the feasible region.

(5) Find the value of the objective function z at all vertices of the feasible region.

(6) The coordinates of the vertex for which z is maximum (minimum) gives (give) the optimal solution / solutions.

EXERCISE 6.2 | Textbook page 101

Solve the following LPP by graphical method :

1. Maximize $z = 11x + 8y$,

subject to $x \leqslant 4$, $y \leqslant 6$, $x + y \leqslant 6$, $x \geqslant 0$, $y \geqslant 0$.

Solution : First we draw the lines AB, CD and ED whose equations are $x = 4$, $y = 6$ and $x + y = 6$ respectively.

Line	Equation	Points on the X-axis	Points on the Y-axis	Sign	Region
AB	$x = 4$	A (4, 0)	—	$\leqslant$	origin side of the line AB
CD	$y = 6$	—	D (0, 6)	$\leqslant$	origin side of the line CD
ED	$x + y = 6$	E (6, 0)	D (0, 6)	$\leqslant$	origin side of the line ED

The feasible region is shaded portion OAPDO in the figure.

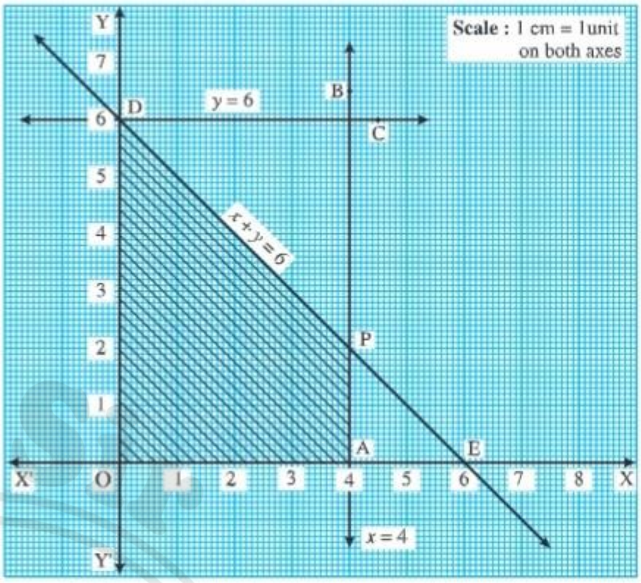


Fig. 6.9

The vertices of the feasible region are O (0, 0), A (4, 0), P and D (0, 6).

P is the point of intersection of the lines $x + y = 6$ and $x = 4$. Substituting $x = 4$ in $x + y = 6$, we get

$4 + y = 6 \quad \therefore\ y = 2 \quad \therefore$ P is (4, 2).

$\therefore$ the corner points of feasible region are O (0, 0), A (4, 0), P (4, 2) and D (0, 6).

The values of the objective function $z = 11x + 8y$ at these vertices are

$z\,(\mathrm{O}) = 11(0) + 8(0) = 0 + 0 = 0$

$z(\mathrm{A}) = 11(4) + 8(0) = 44 + 0 = 44$

$z\,(\mathrm{P}) = 11(4) + 8(2) = 44 + 16 = 60$

$z\,(\mathrm{D}) = 11(0) + 8(2) = 0 + 16 = 16$

$\therefore$ z has maximum value **60**, when $x = \mathbf{4}$ and $y = \mathbf{2}$.

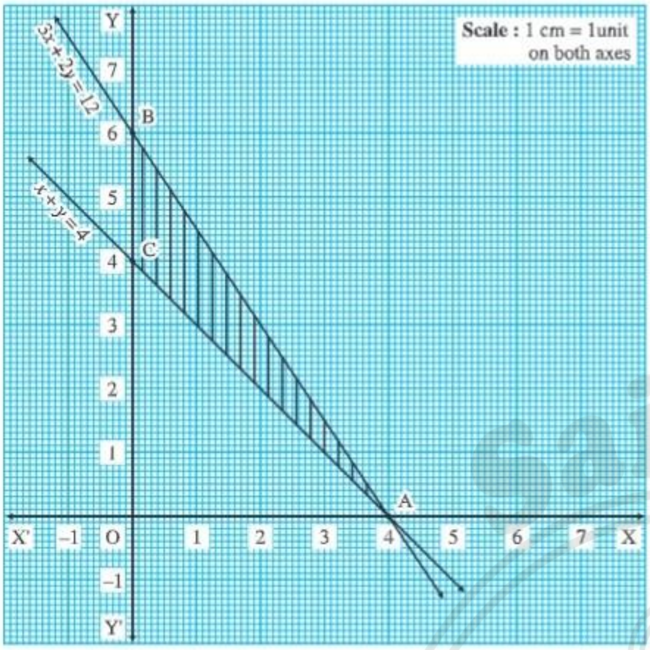
---

2. Maximize $z = 4x + 6y$,

subject to $3x + 2y \leqslant 12$, $x + y \geqslant 4$, $x, y \geqslant 0$.

Solution : First we draw the lines AB and AC whose equations are $3x + 2y = 12$ and $x + y = 4$ respectively.

Line	Equation	Points on the X-axis	Points on the Y-axis	Sign	Region
AB	$3x + 2y = 12$	A (4, 0)	B (0, 6)	$\leqslant$	origin side of the line AB
AC	$x + y = 4$	A (4, 0)	C (0, 4)	$\geqslant$	non-origin side of the line AC



Scale : 1 cm = 1unit on both axes

Fig. 6.10

The feasible region is the △ABC which is shaded in the figure.

The vertices of the feasible region (i.e. corner points) are A (4, 0), B (0, 6) and C (0, 4).

The values of the objective function $z = 4x + 6y$ at these vertices are

$z(A) = 4(4) + 6(0) = 16 + 0 = 16$

$z(B) = 4(0) + 6(6) = 0 + 36 = 36$

$z(C) = 4(0) + 6(4) = 0 + 24 = 24$

$\therefore$ z has maximum value **36**, when $x = 0$, $y = 6$.

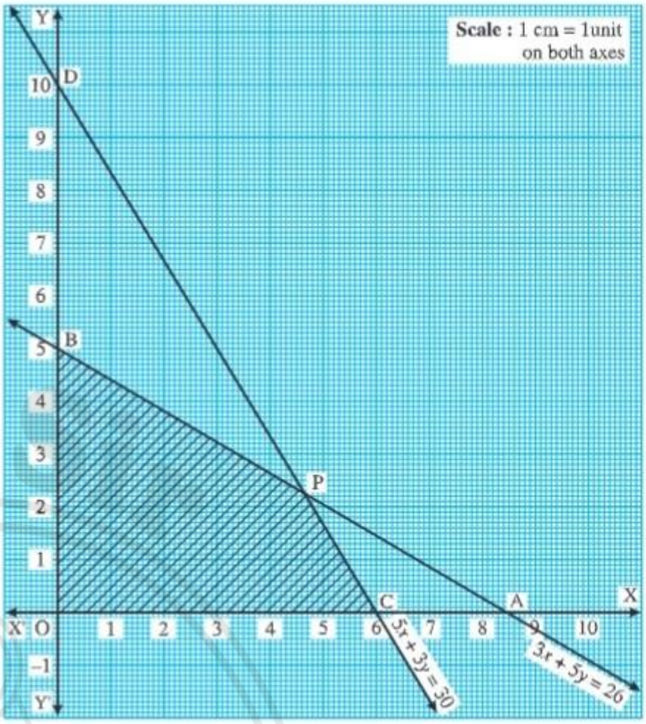
---

3. Maximize $z = 7x + 11y$,

subject to $3x + 5y \leqslant 26$, $5x + 3y \leqslant 30$, $x \geqslant 0$, $y \geqslant 0$.

Solution : First we draw the lines AB and CD whose equations are $3x + 5y = 26$ and $5x + 3y = 30$ respectively.

Line	Equation	Points on the X-axis	Points on the Y-axis	Sign	Region
AB	$3x + 5y = 26$	$A\left(\frac{26}{3}, 0\right)$	$B\left(0, \frac{26}{5}\right)$	$\leqslant$	origin side of the line AB
CD	$5x + 3y = 30$	C (6, 0)	D (0, 10)	$\leqslant$	origin side of the line CD



Scale : 1 cm = 1unit on both axes

Fig. 6.11

The feasible region is OCPBO which is shaded in the figure.

The vertices of the feasible region are O (0, 0), C (6, 0), P and $B\left(0, \frac{26}{5}\right)$.

The vertex P is the point of intersection of the lines

$3x + 5y = 26$ ... (1)

and $5x + 3y = 30$ ... (2)

Multiplying equation (1) by 3 and equation (2) by 5, we get

$9x + 15y = 78$

and $25x + 15y = 150$

On subtracting, we get

$16x = 72 \quad \therefore x = \frac{72}{16} = \frac{9}{2} = 4.5$

Substituting $x = 4.5$ in equation (2), we get

$5(4.5) + 3y = 30$

$22.5 + 3y = 30$

$\therefore 3y = 7.5 \quad \therefore y = 2.5$

$\therefore$ P is (4.5, 2.5)

The values of the objective function $z = 7x + 11y$ at these corner points are

$z(O) = 7(0) + 11(0) = 0 + 0 = 0$

$z(C) = 7(6) + 11(0) = 42 + 0 = 42$

$z(P) = 7(4.5) + 11(2.5) = 31.5 + 27.5 = 59.0 = 59$

$z(B) = 7(0) + 11\left(\frac{26}{5}\right) = \frac{286}{5} = 57.2$

$\therefore$ z has maximum value **59**, when $x = 4.5$ and $y = 2.5$.

4. Maximize $z = 10x + 25y$,

subject to $0 \leqslant x \leqslant 3$, $0 \leqslant y \leqslant 3$, $x + y \leqslant 5$.

Solution : First we draw the lines AB, CD and EF whose equations are $x = 3$, $y = 3$ and $x + y = 5$ respectively.

Line	Equation	Points on the X-axis	Points on the Y-axis	Sign	Region
AB	$x = 3$	A (3, 0)	—	$\leqslant$	origin side of the line AB
CD	$y = 3$	—	D (0, 3)	$\leqslant$	origin side of the line CD
EF	$x + y = 5$	E (5, 0)	F (0, 5)	$\leqslant$	origin side of the line EF

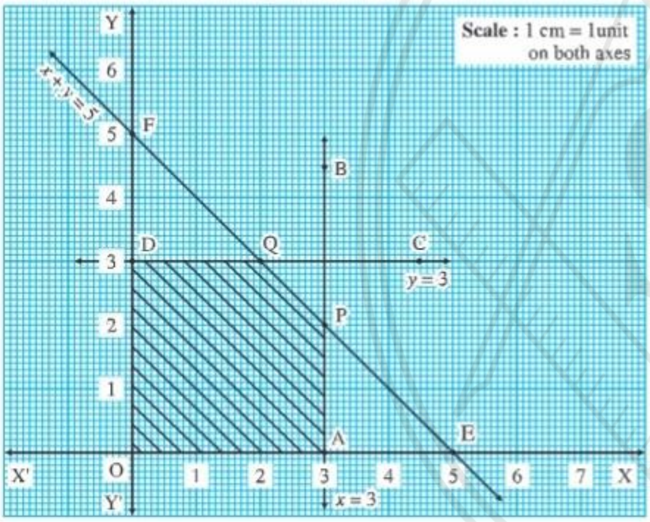


Fig. 6.12

The feasible region is OAPQDO which is shaded in the figure.

The vertices of the feasible region are O (0, 0), A (3, 0), P, Q and D (0, 3).

P is the point of intersection of the lines $x + y = 5$ and $x = 3$.

Substituting $x = 3$ in $x + y = 5$, we get

$3 + y = 5 \qquad \therefore \; y = 2$

$\therefore$ P is (3, 2)

Q is the point of intersection of the lines $x + y = 5$ and $y = 3$

Substituting $y = 3$ in $x + y = 5$, we get

$x + 3 = 5 \qquad \therefore \; x = 2 \qquad \therefore$ Q is (2, 3)

The values of the objective function $z = 10x + 25y$ at these vertices are

$z(O) = 10(0) + 25(0) = 0 + 0 = 0$

$z(A) = 10(3) + 25(0) = 30 + 0 = 30$

$z(P) = 10(3) + 25(2) = 30 + 50 = 80$

$z(Q) = 10(2) + 25(3) = 20 + 75 = 95$

$z(D) = 10(0) + 25(3) = 0 + 75 = 75$

$\therefore$ z has maximum value **95**, when $x = 2$ and $y = 3$.

5. Maximize $z = 3x + 5y$, subject to

$x + 4y \leqslant 24$, $3x + y \leqslant 21$, $x + y \leqslant 9$, $x \geqslant 0$, $y \geqslant 0$.

Solution : First we draw the lines AB, CD and EF whose equations are $x + 4y = 24$, $3x + y = 21$ and $x + y = 9$ respectively.

Line	Equation	Points on the X-axis	Points on the Y-axis	Sign	Region
AB	$x + 4y = 24$	A (24, 0)	B (0, 6)	$\leqslant$	origin side of the line AB
CD	$3x + y = 21$	C (7, 0)	D (0, 21)	$\leqslant$	origin side of the line CD
EF	$x + y = 9$	E (9, 0)	F (0, 9)	$\leqslant$	origin side of the line EF

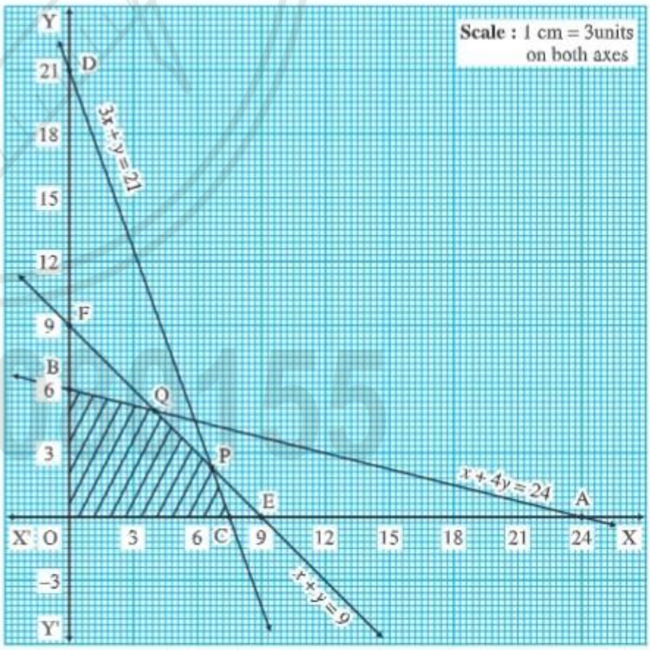


Fig. 6.13

The feasible region is OCPQBO which is shaded in the figure.

The vertices of the feasible region are O (0, 0), C (7, 0), P, Q and B (0, 6).

P is the point of intersection of the lines

$3x + y = 21$ ... (1)

and $x + y = 9$ ... (2)

On subtracting, we get

$2x = 12 \quad \therefore \; x = 6$

Substituting $x = 6$ in equation (2), we get

$6 + y = 9 \quad \therefore \; y = 3$

$\therefore \; P \equiv (6, 3)$

Q is the point of intersection of the lines

$x + 4y = 24$ ... (3)

and $x + y = 9$ ... (2)

On subtracting, we get

$3y = 15 \quad \therefore \; y = 5$

Substituting $y = 5$ in equation (2), we get

$x + 5 = 9 \quad \therefore \; x = 4$

$\therefore \; Q \equiv (4, 5)$

$\therefore$ the corner points of the feasible region are O (0, 0), C (7, 0), P (6, 3), Q (4, 5) and B (0, 6).

The values of the objective function $z = 3x + 5y$ at these corner points are

$z\,(O) = 3(0) + 5(0) = 0 + 0 = 0$

$z\,(C) = 3(7) + 5(0) = 21 + 0 = 21$

$z\,(P) = 3(6) + 5(3) = 18 + 15 = 33$

$z\,(Q) = 3(4) + 5(5) = 12 + 25 = 37$

$z\,(B) = 3(0) + 5(6) = 0 + 30 = 30$

$\therefore$ z has maximum value **37**, when $x = \mathbf{4}$ and $y = \mathbf{5}$.

---

6. Minimize $z = 7x + y$,

subject to $5x + y \geqslant 5$, $x + y \geqslant 3$, $x \geqslant 0$, $y \geqslant 0$.

Solution : First we draw the lines AB and CD whose equations are $5x + y = 5$ and $x + y = 3$ respectively.

Line	Equation	Points on the X-axis	Points on the Y-axis	Sign	Region
AB	$5x + y = 5$	A (1, 0)	B (0, 5)	$\geqslant$	non-origin side of the line AB
CD	$x + y = 3$	C (3, 0)	D (0, 3)	$\geqslant$	non-origin side of the line CD

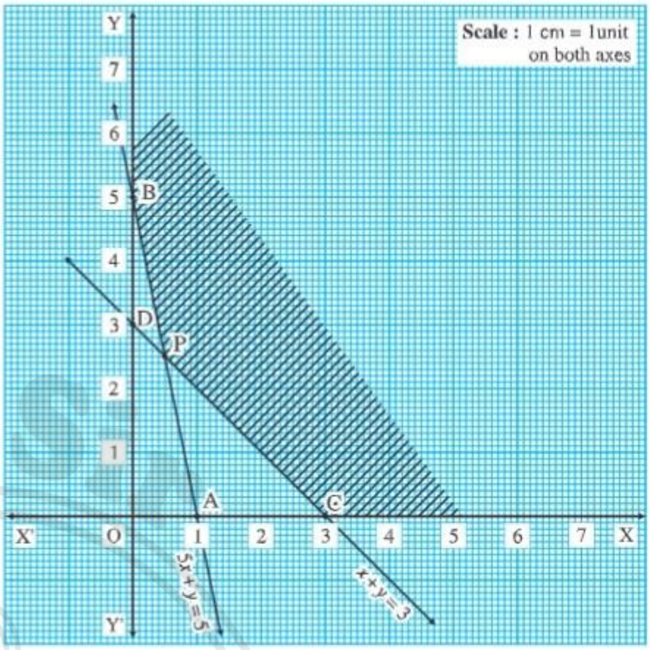


Fig. 6.14

The feasible region is XCPBY which is shaded in the figure.

The vertices of the feasible region are C (3, 0), P and B (0, 5).

P is the point of the intersection of the lines

$5x + y = 5$ and $x + y = 3$

On subtracting, we get

$4x = 2 \quad \therefore \; x = \frac{1}{2}$

Substituting $x = \frac{1}{2}$ in $x + y = 3$, we get

$\frac{1}{2} + y = 3$

$\therefore \; y = \frac{5}{2} \quad \therefore \; P = \left(\frac{1}{2}, \frac{5}{2}\right)$

The values of the objective function $z = 7x + y$ at these vertices are

$z\,(C) = 7(3) + 0 = 21$

$z\,(P) = 7\left(\frac{1}{2}\right) + \frac{5}{2} = \frac{7}{2} + \frac{5}{2} = 6$

$z\,(B) = 7(0) + 5 = 5$

$\therefore$ z has minimum value **5**, when $x = \mathbf{0}$ and $y = \mathbf{5}$.

---

7. Minimize $z = 8x + 10y$, subject to

$2x + y \geqslant 7$, $2x + 3y \geqslant 15$, $y \geqslant 2$, $x \geqslant 0$, $y \geqslant 0$.

Solution : First we draw the lines AB, CD and EF whose equations are $2x + y = 7$, $2x + 3y = 15$ and $y = 2$ respectively.

The vertices of the feasible region are O(0, 0), C(7, 0), P, Q and B(0, 6).

P is the point of intersection of the lines

$3x+y=21$ … (1)

and $x+y=9$ … (2)

On subtracting, we get

$2x=12$ $\therefore$ $x=6$

Substituting $x=6$ in equation (2), we get

$6+y=9$ $\therefore$ $y=3$

$\therefore$ P $\equiv$ (6, 3)

Q is the point of intersection of the lines

$x+4y=24$ … (3)

and $x+y=9$ … (2)

On subtracting, we get

$3y=15$ $\therefore$ $y=5$

Substituting $y=5$ in equation (2), we get

$x+5=9$ $\therefore$ $x=4$

$\therefore$ Q $\equiv$ (4, 5)

$\therefore$ the corner points of the feasible region are O(0, 0), C(7, 0), P(6, 3), Q(4, 5) and B(0, 6).

The values of the objective function $z=3x+5y$ at these corner points are

$z(\text{O})=3(0)+5(0)=0+0=0$

$z(\text{C})=3(7)+5(0)=21+0=21$

$z(\text{P})=3(6)+5(3)=18+15=33$

$z(\text{Q})=3(4)+5(5)=12+25=37$

$z(\text{B})=3(0)+5(6)=0+30=30$

$\therefore$ z has maximum value **37**, when $x=$ **4** and $y=$ **5.**

---

6. Minimize $z=7x+y$,
subject to $5x+y\geqslant 5$, $x+y\geqslant 3$, $x\geqslant 0$, $y\geqslant 0$.

Solution : First we draw the lines AB and CD whose equations are $5x+y=5$ and $x+y=3$ respectively.

Line	Equation	Points on the X-axis	Points on the Y-axis	Sign	Region
AB	$5x+y=5$	A(1, 0)	B(0, 5)	$\geqslant$	non-origin side of the line AB
CD	$x+y=3$	C(3, 0)	D(0, 3)	$\geqslant$	non-origin side of the line CD

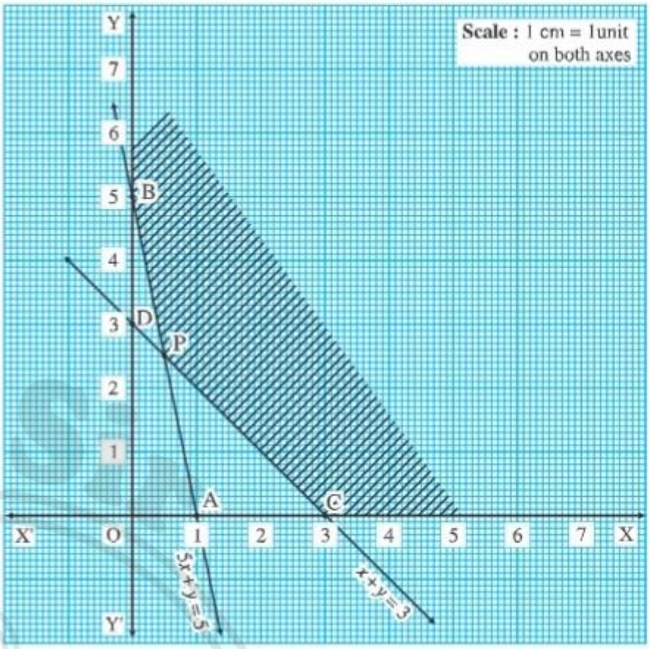


Fig. 6.14

The feasible region is XCPBY which is shaded in the figure.

The vertices of the feasible region are C(3, 0), P and B(0, 5).

P is the point of the intersection of the lines

$5x+y=5$ and $x+y=3$

On subtracting, we get

$4x=2$ $\therefore$ $x=\frac{1}{2}$

Substituting $x=\frac{1}{2}$ in $x+y=3$, we get

$\frac{1}{2}+y=3$

$\therefore$ $y=\frac{5}{2}$ $\therefore$ $\text{P}=\left(\frac{1}{2},\frac{5}{2}\right)$

The values of the objective function $z=7x+y$ at these vertices are

$z(\text{C})=7(3)+0=21$

$z(\text{P})=7\left(\frac{1}{2}\right)+\frac{5}{2}=\frac{7}{2}+\frac{5}{2}=6$

$z(\text{B})=7(0)+5=5$

$\therefore$ z has minimum value **5**, when $x=$ **0** and $y=$ **5.**

---

7. Minimize $z=8x+10y$, subject to
$2x+y\geqslant 7$, $2x+3y\geqslant 15$, $y\geqslant 2$, $x\geqslant 0$, $y\geqslant 0$.

Solution : First we draw the lines AB, CD and EF whose equations are $2x+y=7$, $2x+3y=15$ and $y=2$ respectively.

Line	Equation	Points on the X-axis	Points on the Y-axis	Sign	Region
AB	$2x+y=7$	A (3.5, 0)	B (0, 7)	$\geqslant$	non-origin side of the line AB
CD	$2x+3y=15$	C (7.5, 0)	D (0, 5)	$\geqslant$	non-origin side of the line CD
EF	$y=2$	—	F (0, 2)	$\geqslant$	non-origin side of the line EF

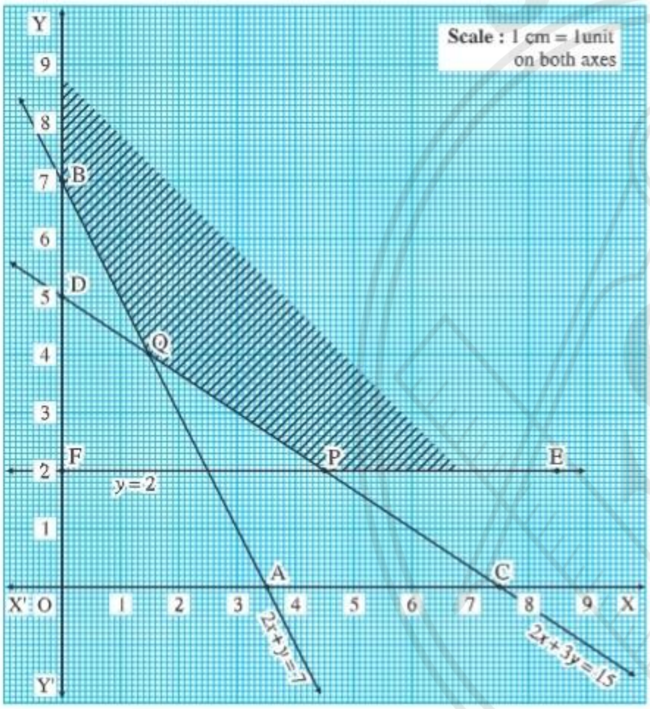


Fig. 6.15

The feasible region is EPQBY which is shaded in the figure.

The vertices of the feasible region are P, Q and B (0,7).

P is the point of intersection of the lines $2x+3y=15$ and $y=2$.

Substituting $y=2$ in $2x+3y=15$, we get

$2x+3(2)=15$

$\therefore\ 2x=9 \qquad \therefore\ x=4.5 \qquad \therefore\ \text{P}=(4.5,\ 2)$

Q is the point of intersection of the lines

$2x+3y=15 \qquad \ldots (1)$

and $2x+\ y=\ 7 \qquad \ldots (2)$

On subtracting, we get

$2y=8 \qquad \therefore\ y=4$

$\therefore$ from (2), $2x+4=7$

$\therefore\ 2x=3 \qquad \therefore\ x=1.5$

$\therefore\ \text{Q}=(1.5,\ 4)$

The values of the objective function $z=8x+10y$ at these vertices are

$z(\text{P})=8(4.5)+10(2)=36+20=56$

$z(\text{Q})=8(1.5)+10(4)=12+40=52$

$z(\text{B})=8(0)\ \ +10(7)=70$

$\therefore$ z has minimum value **52**, when $x=\mathbf{1.5}$ and $y=\mathbf{4}$

---

8. Minimize $z=6x+21y$, subject to $x+2y\geqslant 3$, $x+4y\geqslant 4$, $3x+y\geqslant 3$, $x\geqslant 0$, $y\geqslant 0$.

[*Note : Question has been modified.*]

Solution : First we draw the lines AB, CD and EF whose equations are $x+2y=3$, $x+4y=4$ and $3x+y=3$ respectively.

Line	Equation	Points on the X-axis	Points on the Y-axis	Sign	Region
AB	$x+2y=3$	A (3, 0)	B $\left(0,\ \frac{3}{2}\right)$	$\geqslant$	non-origin side of the line AB
CD	$x+4y=4$	C (4, 0)	D (0, 1)	$\geqslant$	non-origin side of the line CD
EF	$3x+y=3$	E (1, 0)	F (0, 3)	$\geqslant$	non-origin side of the line EF

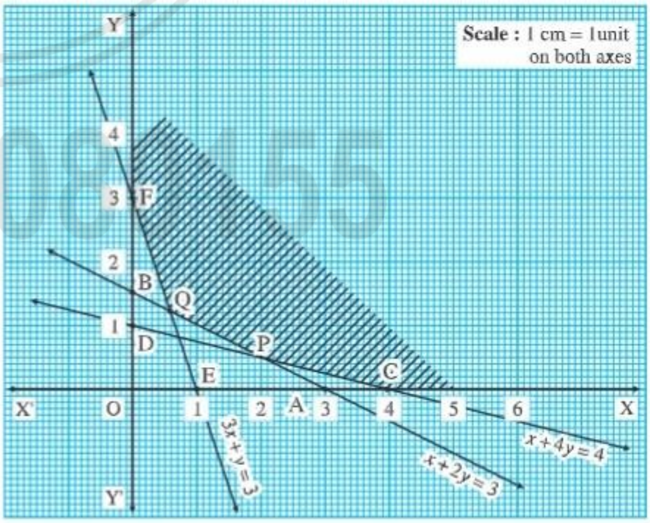


Fig. 6.16

The feasible region is XCPQFY which is shaded in the figure.

The vertices of the feasible region are C (4, 0), P, Q and F (0, 3). P is the point of intersection of the lines

$$x + 4y = 4$$

and $x + 2y = 3$

On subtracting, we get

$2y = 1 \qquad \therefore\ y = \frac{1}{2}$

Substituting $y = \frac{1}{2}$ in $x + 2y = 3$, we get

$x + 2\left(\frac{1}{2}\right) = 3$

$\therefore\ x = 2$

$\therefore\ P \equiv \left(2, \frac{1}{2}\right)$

Q is the point of intersection of the lines

$$x + 2y = 3 \qquad \dots (1)$$

and $3x + y = 3 \qquad \dots (2)$

Multiplying equation (1) by 3, we get

$3x + 6y = 9$

Subtracting equation (2) from this equation, we get

$5y = 6$

$\therefore\ y = \frac{6}{5}$

$\therefore$ from (1), $x + 2\left(\frac{6}{5}\right) = 3$

$\therefore\ x = 3 - \frac{12}{5} = \frac{3}{5}$

$\therefore\ Q \equiv \left(\frac{3}{5}, \frac{6}{5}\right)$

The values of the objective function $z = 6x + 21y$ at these vertices are

$z(C) = 6(4) + 21(0) = 24$

$z(P) = 6(2) + 21\left(\frac{1}{2}\right)$

$\quad = 12 + 10.5 = 22.5$

$z(Q) = 6\left(\frac{3}{5}\right) + 21\left(\frac{6}{5}\right)$

$\quad = \frac{18}{5} + \frac{126}{5} = \frac{144}{5} = 28.8$

$z(F) = 6(0) + 21(3) = 63$

$\therefore$ z has minimum value **22.5**, when $x = \mathbf{2}$ and $y = \mathbf{\frac{1}{2}}$.

EXAMPLES FOR PRACTICE 6.2

1. Maximize $z = 8x + 12y$, subject to $x \geqslant 0$, $y \geqslant 0$, $x + y \leqslant 9$, $x \geqslant 2$, $y \geqslant 3$, $2x + 6y \leqslant 36$.
2. Minimize $z = 4x + 3y$, subject to $x \geqslant 0$, $y \geqslant 0$, $2x + y \geqslant 40$, $x + 2y \geqslant 50$, $x + y > 35$.
3. Maximize $z = 2x_1 + x_2$, subject to $x_1 + 2x_2 \leqslant 10$, $x_1 + x_2 \leqslant 6$, $x_1 - x_2 \leqslant 2$, $x_1, x_2 \geqslant 0$.
4. Minimize $z = 2000x_1 + 1600x_2$, subject to $6x_1 + 2x_2 \geqslant 12$, $2x_1 + 2x_2 \geqslant 8$, $4x_1 + 12x_2 \geqslant 24$, $x_1 \leqslant 7$, $x_2 \leqslant 7$, $x_1, x_2 \geqslant 0$.
5. Maximize $z = 4x + y$, subject to $20x + 30y \leqslant 1050$, $x + y \leqslant 40$, $x \geqslant 0$, $y \geqslant 0$.
6. Minimize $z = x_1 + x_2$, subject to $5x_1 + 10x_2 \leqslant 50$, $x_1 + x_2 \geqslant 1$, $x_2 \leqslant 4$, $x_1, x_2 \geqslant 0$.
7. Maximize $z = 4x + 10y$, subject to $2x + 5y \leqslant 10$, $5x + 3y \leqslant 15$, $x \geqslant 0$, $y \geqslant 0$.
8. Solve the following LPP graphically.
   Minimize $z = 3x + 2y$, subject to $2x + 3y \geqslant 12$, $-x + y \leqslant 3$, $x \leqslant 4$, $y \geqslant 3$.
9. Maximize $z = 4x + 5y$, subject to $2x + y \geqslant 4$, $x + y \leqslant 5$, $0 \leqslant x \leqslant 3$, $0 \leqslant y \leqslant 3$.
10. Maximize $z = 9x + 13y$, subject to $2x + 3y \leqslant 18$, $2x + y \leqslant 10$, $x \geqslant 0$, $y \geqslant 0$.
11. Maximize $z = 100x + 70y$, subject to $2x \geqslant 4$, $y \leqslant 3$, $x + y \leqslant 8$, $x \geqslant 0$, $y \geqslant 0$.
12. Maximize $z = 5x + 3y$, subject to $x \leqslant 4$, $y \leqslant 8$, $x + y \leqslant 8$, $x \geqslant 0$, $y \geqslant 0$.
13. Minimize $z = 5x + 7y$, subject to $2x + y \geqslant 8$, $x + 2y \geqslant 10$, $x \geqslant 0$, $y \geqslant 0$.
14. Minimize $z = 3x + y$, subject to $2x + 3y \leqslant 6$, $x + y \geqslant 1$, $x \geqslant 0$, $y \geqslant 0$.
15. Minimize $z = 4x + 5y$, subject to $2x + y \geqslant 14$, $2x + 3y \geqslant 30$, $x \geqslant 0$, $y \geqslant 4$.

Answers

1. z has maximum value 84, when $x = 6$ and $y = 3$
2. z has minimum value 110, when $x = 5$ and $y = 30$
3. z has maximum value 10, when $x_1 = 4$ and $x_2 = 2$

4. z has minimum value 6800, when $x_1 = 1$ and $x_2 = 3$

5. z has maximum value 210, when $x = 15$ and $y = 25$ or $x = 0$, $y = 35$

6. z has minimum value 1, when $x_1 = 0$ and $x_2 = 1$ or $x_1 = 1$, $x_2 = 0$

7. z has maximum value 20, when $x = \frac{45}{19}$ and $y = \frac{20}{19}$

8. z has minimum value 19.5, when $x = 1.5$ and $y = 3$

9. z has maximum value 23, when $x = 2$ and $y = 3$

10. z has maximum value 79, when $x = 3$ and $y = 4$

11. z has maximum value 800, when $x = 8$ and $y = 0$

12. z has maximum value 32, when $x = 4$ and $y = 4$

13. z has minimum value 38, when $x = 2$ and $y = 4$

14. z has minimum value 1, when $x = 0$ and $y = 1$

15. z has minimum value 52, when $x = 3$ and $y = 8$

MISCELLANEOUS EXERCISE – 6

(Textbook pages 102 to 105)

I. Choose the correct alternative :

1. The value of objective function is maximize under linear constraints.
   (a) at the centre of feasible region
   (b) at (0, 0)
   (c) at a vertex of feasible region
   (d) The vertex which is at maximum distance from (0, 0).

2. Which of the following is correct?
   (a) Every LPP has an optimal solution.
   (b) Every LPP has unique optimal solution.
   (c) If LPP has two optimal solution then it has infinitely many solutions.
   (d) The set of all feasible solutions of LPP may not be a convex set.

   [**Note :** *Options (a), (b) and (c) has been modified.*]

3. Objective function of LPP is
   (a) a constraint
   (b) a function to be maximized or minimized
   (c) a relation between the decision variables
   (d) a feasible region

4. The maximum value of $z = 5x + 3y$ subject to the constraints $3x + 5y = 15$; $5x + 2y \leqslant 10$, $x, y \geqslant 0$ is
   (a) 235 (b) $\frac{235}{9}$ (c) $\frac{235}{19}$ (d) $\frac{235}{3}$

5. The maximum value of $z = 10x + 6y$, subject to the constraints $3x + y \leqslant 12$, $2x + 5y \leqslant 34$, $x \geqslant 0$, $y \geqslant 0$ is
   (a) 56 (b) 65 (c) 55 (d) 66

6. The point of which the maximum value of $z = x + y$ subject to the constraints $x + 2y \leqslant 70$, $2x + y \leqslant 95$, $x \geqslant 0$, $y \geqslant 0$ is obtained at
   (a) (36, 25) (b) (20, 35) (c) (35, 20) (d) (40, 15)

   [**Note :** *Question has been modified.*]

7. Of all the points of the feasible region the optimal value of z is obtained at a point
   (a) inside the feasible region.
   (b) at the boundary of the feasible region.
   (c) at vertex of feasible region.
   (d) on X-axis.

8. Feasible region; the set of points which satisfy
   (a) the objective function
   (b) all the given constraints
   (c) some of the given constraints
   (d) only non-negavite constraints

   [**Note :** *Option (b) has been modified.*]

9. Solution of LPP to minimize $z = 2x + 3y$ subject to $x \geqslant 0$, $y \geqslant 0$, $1 \leqslant x + 2y \leqslant 10$ is
   (a) $x = 0, y = \frac{1}{2}$ (b) $x = \frac{1}{2}, y = 0$
   (c) $x = 1, y = -2$ (d) $x = y = \frac{1}{2}$

10. The corner points of the feasible region given by the inequations $x + y \leqslant 4$, $2x + y \leqslant 7$, $x \geqslant 0$, $y \geqslant 0$ are
    (a) (0, 0), (4, 0), (3, 1), (0, 4)
    (b) (0, 0), $\left(\frac{7}{2}, 0\right)$, (3, 1), (0, 4)
    (c) (0, 0), $\left(\frac{7}{2}, 0\right)$, (3, 1), (5, 7)
    (d) (6, 0), (4, 0), (3, 1), (0, 7)

11. The corner points of the feasible region are (0, 0), (2, 0), $\left(\frac{12}{7}, \frac{3}{7}\right)$ and (0, 1) then the point of maximum $z = 6.5x + y = 13$.
    (a) (0, 0) (b) (2, 0) (c) $\left(\frac{11}{7}, \frac{3}{7}\right)$ (d) (0, 1)

12. If the corner points of the feasible region are (0, 0), (3, 0), (2, 1) and $\left(0, \frac{7}{3}\right)$ the maximum value of $z = 4x + 5y$ is

(a) 12 (b) 13 (c) $\frac{35}{2}$ (d) 0

13. If the corner points of the feasible region are (0, 10), (2, 2) and (4, 0) then the point of minimum $z = 3x + 2y$ is

(a) (2, 2) (b) (0, 10) (c) (4, 0) (d) (2, 4)

14. The half plane represented by $3x + 2y \leqslant 8$ contains the point

(a) $\left(1, \frac{5}{2}\right)$ (b) (2, 1) (c) (0, 0) (d) (5, 1)

[**Note :** *Question has been modified.*]

15. The half plane represented by $4x + 3y \geqslant 14$ contains the point

(a) (0, 0) (b) (2, 2) (c) (3, 4) (d) (1, 1)

Answers

1. (c) at a vertex of feasible region*

2. (c) If LPP has two optimal solution then it has infinitely many solutions.

3. (b) a function to be maximized or minimized

4. (c) $\frac{235}{19}$ **5.** (a) 56 **6.** (d) (40, 15)

7. (c) at vertex of feasible region

8. (b) all the given constraints

9. (a) $x = 0, y = \frac{1}{2}$

10. (b) (0, 0), $\left(\frac{7}{2}, 0\right)$, (3, 1), (0, 4)

11. (b) (2, 0) **12.** (b) 13 **13.** (a) (2, 2)

14. (c) (0, 0) **15.** (b) (2, 2)*.

[**Note :** * *Answer given in the textbook is incorrect.*]

II. Fill in the blanks :

1. Graphical solution set of the inequations $x \geqslant 0, y \geqslant 0$ is in ……… quadrant.

2. The region represented by the inequations $x \leqslant 0, y \leqslant 0$ lies in ……… quadrants.

[**Note :** *Question has been modified.*]

3. The optimal value of the objective function is attained at the ……… points of feasible region.

4. The region represented by the inequality $y \leqslant 0$ lies in ……… quadrants.

5. The constraint that a factory has to employ more women (y) than men (x) is given by ………

6. A garage employs eight men to work in its showroom and repair shop. The constraints that there must be not least 3 men in showroom and repair shop. The constraints that there must be at least 3 men in showroom and at least 2 men in repair shop are ……… and ……… repsectively.

7. A train carries at least twice as many first class passengers (y) as second class passengers (x). The constraint is given by ……… .

8. A dish washing machine holds up to 40 pieces of large crockery (x). This constraint is given by ……… .

Answers

1. I **2.** III **3.** vertex **4.** III and IV **5.** $y > x$

6. $x \geqslant 3, y \geqslant 2$ **7.** $x \geqslant 2y$ **8.** $x \leqslant 40$.

III. State whether each of the following is *True* or *False* :

1. The region represented by the inequalities $x \geqslant 0, y \geqslant 0$ lies in first quadrant.

2. The region represented by the inequalities $x \leqslant 0, y \leqslant 0$ lies in first quadrant.

3. The optimum value of the objective function of LPP occurs at the center of the feasible region.

4. Graphical solution set of $x \leqslant 0, y \geqslant 0$ in xy system lies in second quadrant.

5. Saina wants to invest at most ₹ 24000 in bonds and fixed deposits. Mathematically this constraint is written as $x + y \leqslant 24000$ where x is investment in bond and y is in fixed deposits.

6. The point (1, 2) is not a vertex of the feasible region bounded by $2x + 3y \leqslant 6$, $5x + 3y \leqslant 15$, $x \geqslant 0$, $y \geqslant 0$.

7. The feasible solution of LPP belongs to only quadrant I. The feasible region of graph $x + y \leqslant 1$ and $2x + 2y \geqslant 6$ exists.

Answers

1. True **2.** False **3.** False **4.** True **5.** True

6. True **7.** True.

IV. Solve the following problems :

1. Maximize $z = 5x_1 + 6x_2$, subject to $2x_1 + 3x_2 \leqslant 18$, $2x_1 + x_2 \leqslant 12$, $x \geqslant 0$, $y \geqslant 0$.

Solution : Given : $2x_1 + 3x_2 \leqslant 18$, $2x_1 + x_2 \leqslant 12$, $x \geqslant 0$, $y \geqslant 0$.

To draw the graph, we prepare the following table :

Inequation	Equation	Points (x_1, x_2)				Region
$2x_1 + 3x_2 \leqslant 18$	$2x_1 + 3x_2 = 18$	x_1	0	9	A(0, 6)	$2(0) + 3(0) < 18$
		x_2	6	0	B(9, 0)	∴ origin side of the line AB
$2x_1 + x_2 \leqslant 12$	$2x_1 + x_2 = 12$	x_1	0	6	C(0, 12)	$2(0) + 0 < 12$
		x_2	12	0	D(6, 0)	∴ origin side of the line CD

$x_1, x_2 \geqslant 0$

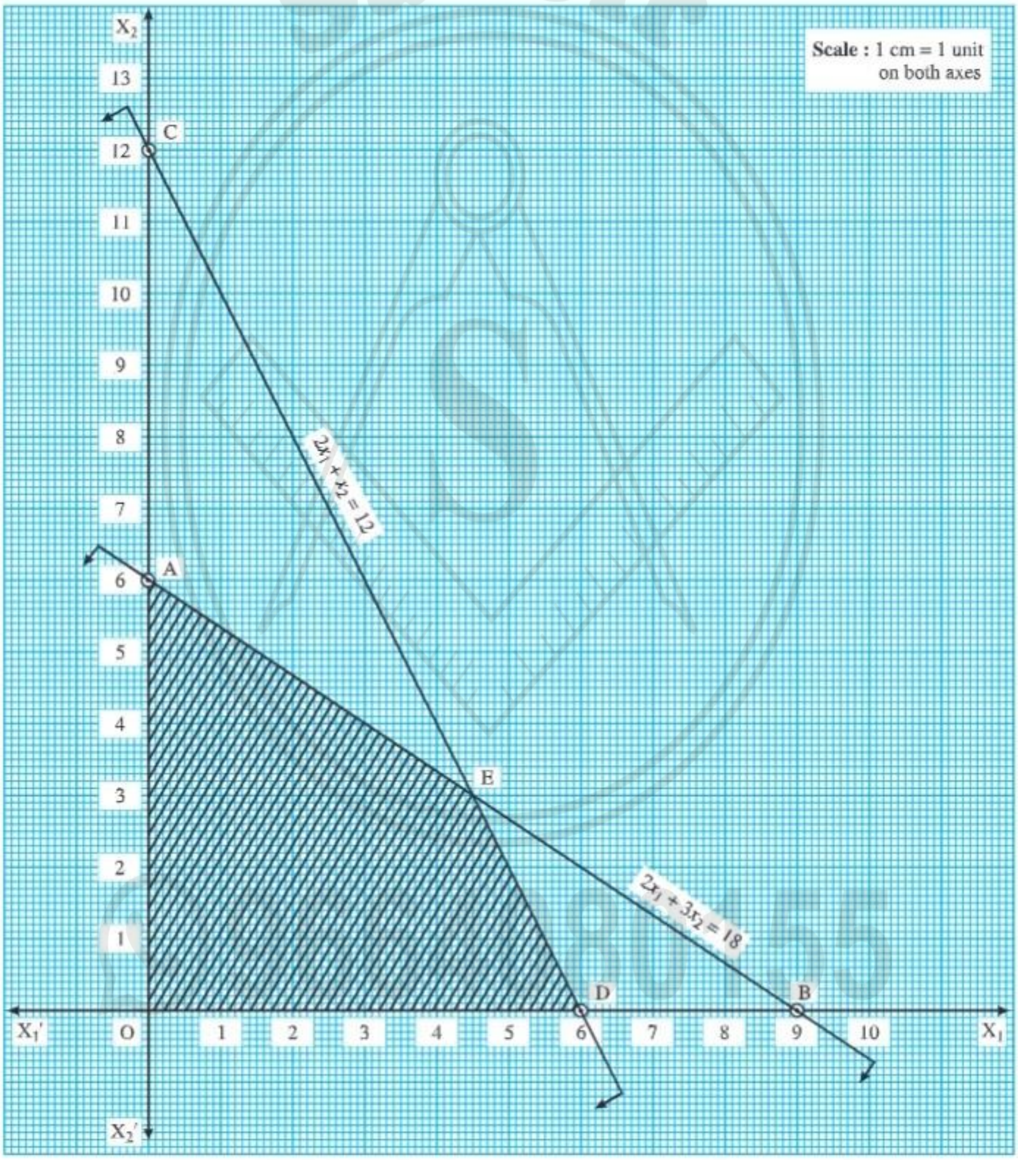


Fig. 6.17

OAED is feasible region. The vertices are O (0, 0), A (0, 6), E (4.5, 3) D (6, 0).

At least any one of the vertices the value of $z=5x_1+6x_2$ will be maximum.

At O (0, 0), $z=5\,(0)+6\,(0)=0$

A (0, 6), $z=5\,(0)+6\,(6)=0+36=36$

E (4.5, 3), $z=5\,(4.5)+6\,(3)=22.5+18=40.5$

D (6, 0), $z=5\,(6)+6\,(0)=30$

$\therefore$ at E (4.5, 3) the value of z is maximum.

Hence, z has maximum value 40.5, when $x_1=4.5$ and $x_2=3$.

[**Note :** Here, point E is the point of intersection of lines

$$2x_1+x_2=12 \quad \ldots (1) \text{ and}$$

$$2x_1+3x_2=18 \quad \ldots (2)$$

Subtract (2) from (1),

$$\begin{array}{rcll} 2x_1 + x_2 &=& 12 & \ldots (1) \\ 2x_1 + 3x_2 &=& 18 & \ldots (2) \\ - \quad - & & - & \\ \hline \therefore \; -2x_2 &=& -6 & \end{array}$$

$\therefore \quad x_2=3$

Put $x_2=3$ in (1),

$2x_1+3=12$

$\therefore \; 2x_1=12-3$

$$=x_1=\frac{9}{2}=4.5$$

Hence, E is (4.5, 3)]

2. Minimize $z=4x+2y$, subject to $3x+y \geqslant 27$, $x+y \geqslant 21$, $x \geqslant 0$, $y \geqslant 0$.

Solution : Given : Minimize $z=4x+2y$

$3x+y \geqslant 27$, $x+y \geqslant 21$, $x \geqslant 0$, $y \geqslant 0$.

To draw the graph, we prepare the following table :

Inequation	Equation	Points (x, y)				Region
$3x+y \geqslant 27$	$3x+y=27$	x	0	9	A(0, 27)	$3(0)+0 \not\geqslant 27$
		y	27	0	B(9, 0)	$\therefore$ non-origin side of the line AB
$x+y \geqslant 21$	$x+y=21$	x	0	21	C(0, 21)	$0+0 \not\geqslant 21$
		y	21	0	D(21, 0)	$\therefore$ non-origin side of the line CD
$x \geqslant 0, y \geqslant 0$	$x=0, y=0$					First quadrant

ABC is unbounded feasible region. The vertices are A (0, 27), B (3, 18) and C (21, 0). At least any one of the vertices the value of objective function $z = 4x + 2y$ will be minimum.

At A (0, 27), $z = 4\ (0) + 2\ (27) = 54$

B (3, 18), $z = 4\ (3) + 2\ (18) = 48$

C (0, 21), $z = 4\ (21) + 2\ (0) = 84$

$\therefore$ at B (3, 18) the value of z is minimum.

Hence, z has minimum value 48, when $x = 3$ and $y = 18$.

[**Note :** Here, B is the point of intersection of lines

$3x + y = 27$ ... (1)

and $x + y = 21$ ... (2)

Subtract (2) from (1),

$$\begin{array}{rcrl} 3x + y & = & 27 & \quad \dots (1) \\ x + y & = & 21 & \quad \dots (2) \\ - \quad - & & - & \\ \hline \therefore\ 2x & = & 6 & \end{array}$$

$\therefore\ x = 3$

Put $x = 3$ in (2), $3 + y = 21$ $\therefore\ y = 18$

Hence, B is (3, 18)]

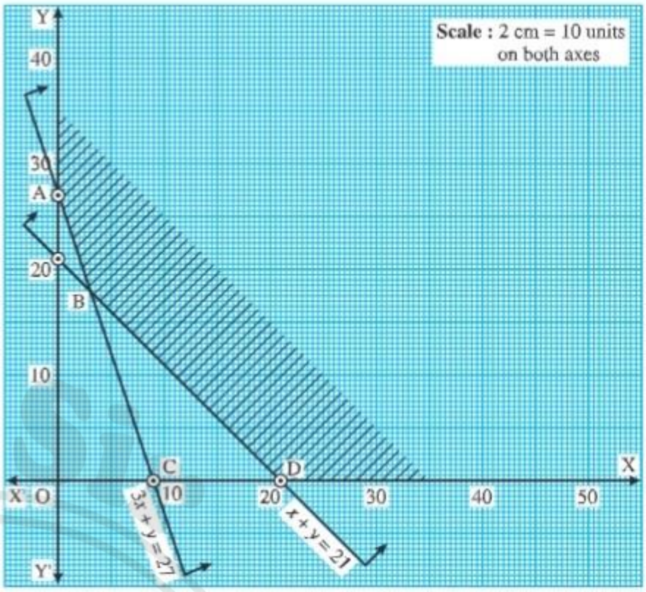


Fig. 6.18

3. Maximize $z = 6x + 10y$, subject to $3x + 5y \leqslant 10$, $5x + 3y \leqslant 15$, $x \geqslant 0$, $y \geqslant 0$.

Solution : Given : Maximize $z = 6x + 10y$. $3x + 5y \leqslant 10$, $5x + 3y \leqslant 15$, $x \geqslant 0$, $y \geqslant 0$.

To draw the graph, we prepare the following table :

Inequation	Equation	Points (x, y)				Region
$3x + 5y \leqslant 10$	$3x + 5y = 10$	x	0	$\frac{10}{3}$	A(0, 2)	$3(0) + 5(0) < 10$
		y	2	0	$B\left(\frac{10}{3}, 0\right)$	$\therefore$ origin side of the line AB
$5x + 3y \leqslant 15$	$5x + 3y = 15$	x	0	3	C (0, 5)	$5(0) + 3(0) < 15$
		y	5	0	D (3, 0)	$\therefore$ origin side of the line CD
$x \geqslant 0$, $y \geqslant 0$	$x = 0$, $y = 0$					First quadrant

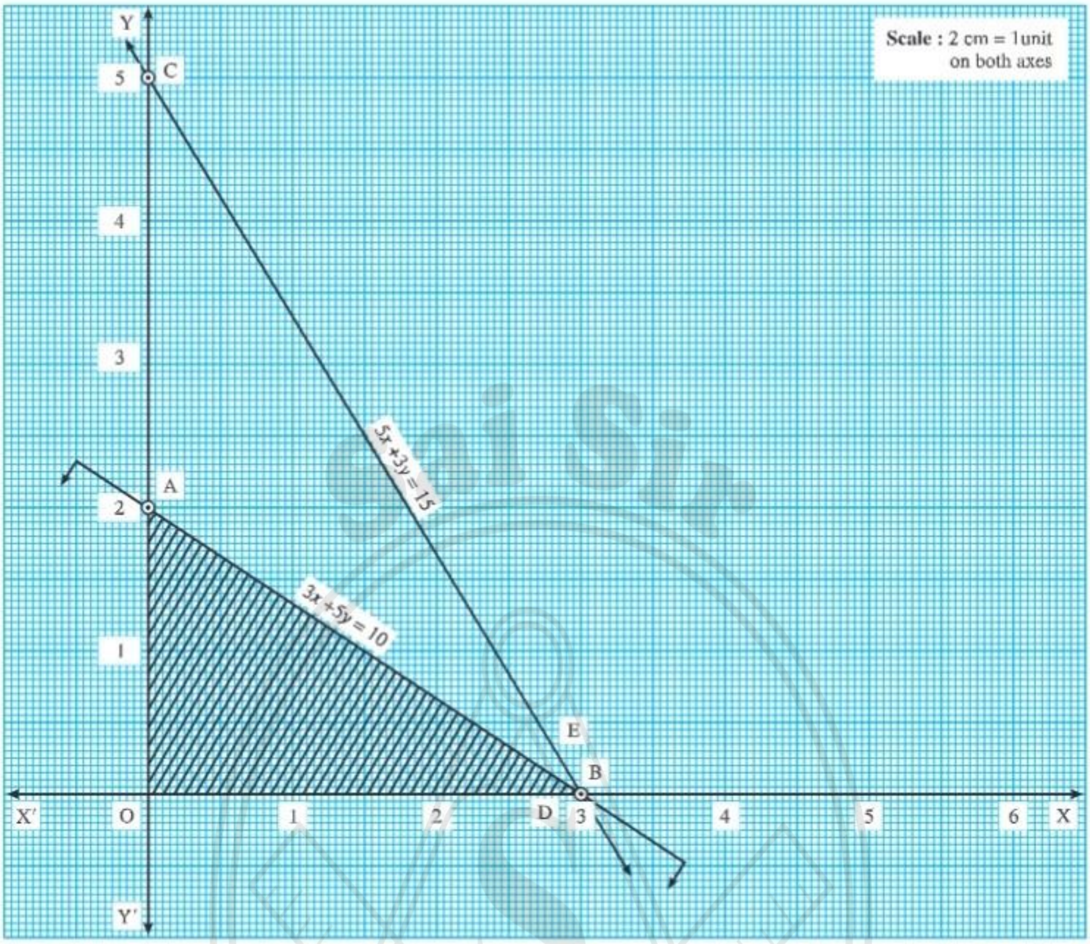


Fig. 6.19

OAD is feasible region. The vertices are (0, 0), A (0, 2), D $\left(\frac{45}{16}, \frac{5}{16}\right)$.

At least any one of the vertices the value of $z = 6x + 10y$ will be maximum.

At O (0, 0), $z = 6(0) + 10(0) = 0$

A (0, 2), $z = 6(0) + 10(2) = 20$

D $\left(\frac{45}{16}, \frac{5}{16}\right)$, $z = \frac{270}{16} + \frac{50}{16} = \frac{320}{16} = 20$

Here, we have infinite number of optimum solutions on the line $3x + 5y = 10$ between D $\left(\frac{45}{16}, \frac{5}{16}\right)$ and A (0, 2).

[**Note :** Point D is the point of intersection of lines

$3x + 5y = 10$ ... (1)

$5x + 3y = 15$ ... (2)

Multiply (1) by 5 and (2) by 3, then subtract (2) from (1)

$15x + 25y = 50$ ... (1) × 5

$15x + 9y = 45$ ... (2) × 3

$\underline{- \quad - \quad -}$

$\therefore\ 16y = 5$

$\therefore\ y = \frac{5}{16}$

Put $y = \frac{5}{16}$ in equation (1), $3x + 5 = \left(\frac{5}{16}\right) = 10$

$\therefore\ 3x = 10 - \frac{25}{16}$ $\quad\therefore\ 3x = \frac{160 - 25}{16}$ $\quad\therefore\ x = \frac{135}{16 \times 3} = \frac{45}{16}$.]

4. Minimize $z = 2x + 3y$, subject to $x - y \leqslant 1$, $x + y \geqslant 3$, $x \geqslant 0$, $y \geqslant 0$.

Solution : Given : Minimize $z = 2x + 3y$

$x - y \leqslant 1$, $x + y \geqslant 3$, $x \geqslant 0$, $y \geqslant 0$.

To draw the graph, we prepare the following table :

Inequation	Equation	Points (x, y)				Region
$x - y \leqslant 1$	$x - y = 1$	x	2	1	A(2, 1)	$0 - 0 < 1$
		y	1	0	B(1, 0)	$\therefore$ origin side of the line AB
$x + y \geqslant 3$	$x + y = 3$	x	0	3	C(0, 3)	$0 + 0 \not> 3$
		y	3	0	D(3, 0)	$\therefore$ non-origin side of the line CD
$x \geqslant 0$, $y \geqslant 0$	$x = 0$, $y = 0$					First quadrant

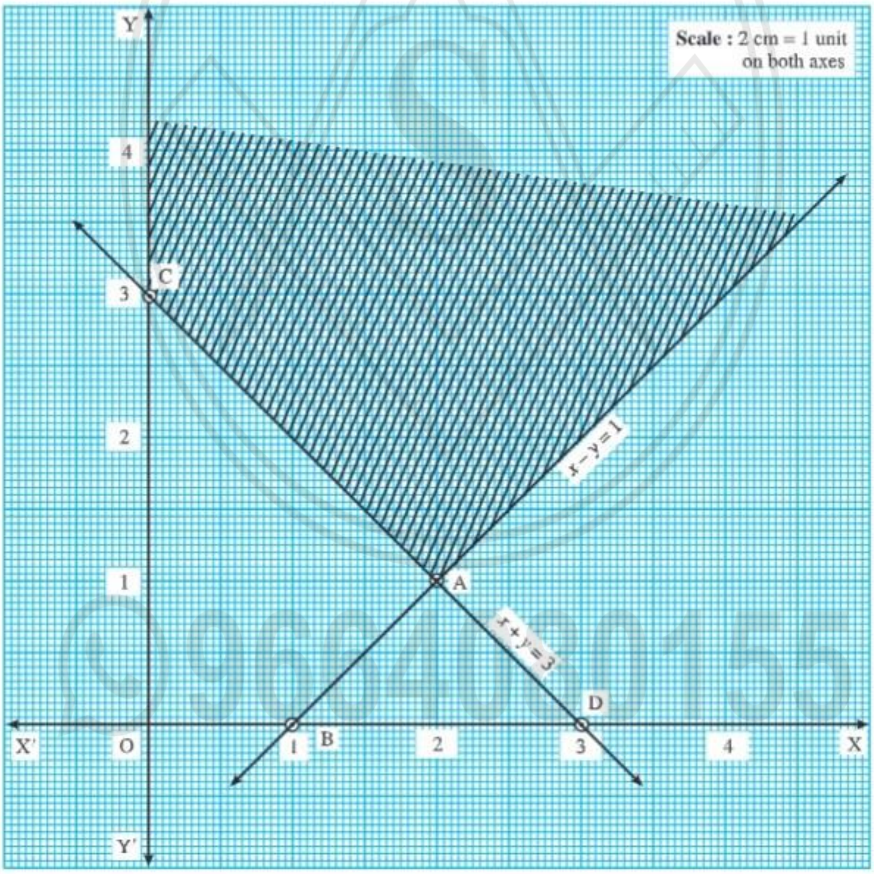


Fig. 6.20

CA is unbounded feasible region. The vertices are C(0, 3), A(2, 1).

At least any one of the vertices the value of $z = 2x + 3y$ will be minimum.

At C(0, 3), $z = 2(0) + 3(3) = 9$, A = (2, 1), $z = 2(2) + 3(1) = 7$

$\therefore$ at A(2, 1) the value of z is minimum. Hence, z has minimum value 7, when $x=2$ and $y=1$.

[**Note :** Here A is point of intersection of lines and can also be obtain by solving

$x-y=1$ ... (1)

$x+y=3$ ... (2)

Add (1) and (2)

$$\begin{array}{r} x-y=1 \\ \underline{x+y=3} \\ \therefore\ 2x \quad =4 \end{array}$$

$\therefore\ x=2$

Put $x=2$ in (1)

$2-y=1 \qquad \therefore\ y=1$]

5. Maximize $z=4x_1+3x_2$, subject to $3x_1+x_2 \leqslant 15$, $3x_1+4x_2 \leqslant 24$, $x_1 \geqslant 0$, $x_2 \geqslant 0$.

Solution : Given : Maximize $z=4x_1+3x_2$, $3x_1+x_2 \leqslant 15$, $3x_1+4x_2 \leqslant 24$, $x_1 \geqslant 0$, $x_2 \geqslant 0$.

To draw the graph, we prepare the following table :

Inequation	Equation	Points (x_1, x_2)				Region
$3x_1+x_2 \leqslant 15$	$3x_1+x_2=15$	x_1	0	5	A(0, 15)	$3(0)+0 < 15$
		x_2	15	0	B(5, 0)	$\therefore$ origin side of the line AB
$3x_1+4x_2 \leqslant 24$	$3x_1+4x_2=24$	x_1	0	8	C(0, 6)	$3(0)+4(0) < 24$
		x_2	6	0	D(8, 0)	$\therefore$ origin side of the line CD
$x \geqslant 0, y \geqslant 0$	$x=0, y=0$					First quadrant

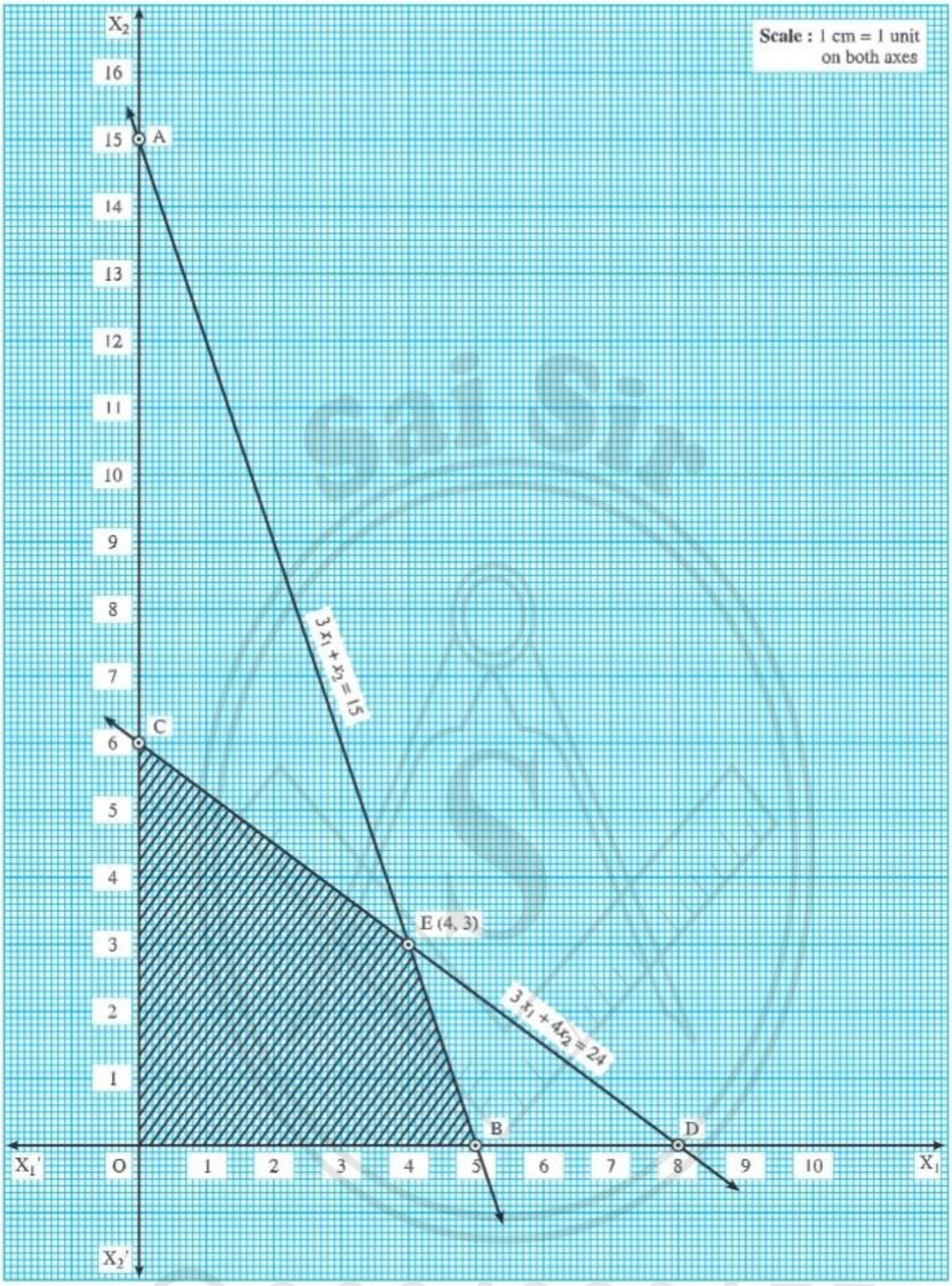


Fig. 6.21

The shaded portion OBEC in the graph represents the graphical solution of the given LPP. The vertices of OBEC are O (0, 0), B (5, 0), E (4, 3) and C (6, 0) respectively.

At any one of these vertices the maximum value of z is obtained.

Now, $z = 4x_1 + 3x_2$

$\therefore$ at O (0, 0), $z = 4(0) + 3(0) = 0$

B (5, 0), $z = 4(5) + 3(0) = 20$

E (4, 3), $z = 4(4) + 3(3) = 16 + 9 = 25$

C (0, 6), $z = 4(6) + 3(0) = 24$

The value of z is maximum at point E. Therefore the solution of the given LPP is as follows :

z has maximum value 25, when $x_1 = 4$ and $x_2 = 3$.

[**Note :** Here E is the point of intersection of the lines

$3x_1 + 4x_2 = 24$ ... (1)

$3x_1 + x_2 = 15$ ... (2)

Subtract (2) from (1)

$$\begin{array}{rl} & 3x_1 + 4x_2 = 24 \\ - & 3x_1 + x_2 = 15 \\ & - \quad - \quad - \\ \hline & 3x_2 = 9 \end{array}$$

$\therefore\ 3x_2 = 9 \qquad \therefore\ x_2 = 3$

Put $x_2 = 3$ in (1), we get $3x_1 + 4(3) = 24$

$\therefore\ 3x_1 = 24 - 12 \qquad \therefore\ 3x_1 = 12 \qquad \therefore\ x_1 = 4$

$\therefore\ x_1 = 4,\ x_2 = 3.$

Hence, E(4, 3)]

6. Maximize $z = 60x + 50y$, subject to $x + 2y \leqslant 40$, $3x + 2y \leqslant 60$, $x \geqslant 0$, $y \geqslant 0$.

Solution : Given : Maximize $z = 60x + 50y$, $x + 2y \leqslant 40$, $3x + 2y \leqslant 60$, $x \geqslant 0$, $y \geqslant 0$.

To draw the graph, we prepare the following table :

Inequation	Equation	Points (x, y)				Region
$x + 2y \leqslant 40$	$x + 2y = 40$	x	0	40	A(0, 20)	$0 + 2(0) < 40$
		y	20	0	B(40, 0)	$\therefore$ origin side of the line AB
$3x + 2y \leqslant 60$	$3x + 2y = 60$	x	0	20	C(0, 30)	$3(0) + 2(0) < 60$
		y	30	0	D(20, 0)	$\therefore$ origin side of the line CD
$x \geqslant 0,\ y \geqslant 0$	$x = 0,\ y = 0$					First quadrant

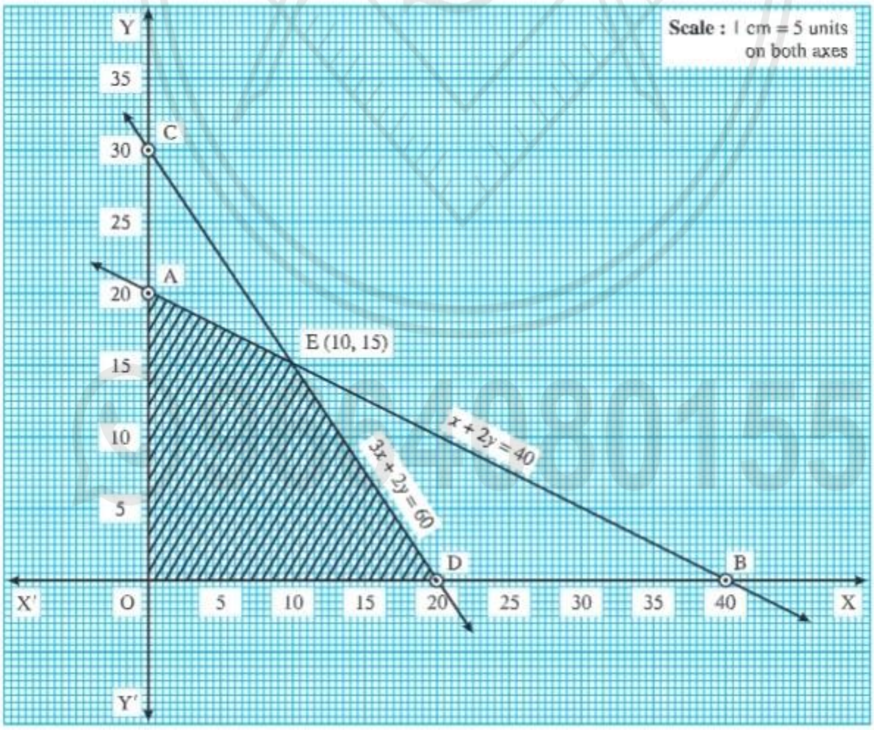


Fig. 6.22

The shaded portion ODEA in the graph represents the graphical solution of the given LPP.

The vertices of ODEA are O (0, 0), D (20, 0), E (10, 15) and A (0, 20) respectively.

At any of these vertices the maximum value of z is obtained.

Now, $z = 60x + 50y$

$\therefore$ at O (0, 0), $z = 60(0) + 50(0) = 0$

D (20, 0), $z = 60(20) + 50(0) = 1200$

E (10, 15), $z = 60(10) + 50(15) = 600 + 750 = 1350$

A (0, 20), $z = 60(0) + 50(20) = 1000$

The value of z is maximum at the point E.

Therefore the solution of the given LPP is as follows :

z has maximum value 1350, when $x = 10$ and $y = 15$.

[**Note :** Here E is the point of intersection of the lines

$x + 2y = 40$ ... (1)

$3x + 2y = 60$ ... (2)

Subtract (1) from (2), we get

$$\begin{array}{rcl} 3x + 2y &=& 60 \\ x + 2y &=& 40 \\ - \quad - & & - \\ \hline 2x \qquad &=& 20 \end{array}$$

$\therefore\ 2x = 20 \qquad \therefore\ x = 10$

Put $x = 10$ in (1)

$\therefore\ 10 + 2y = 40 \qquad \therefore\ 2y = 30 \qquad \therefore\ y = 15.$

Hence, E(10, 15)]

7. Minimize $z = 4x + 2y$, subject to $3x + y \geqslant 27$, $x + y \geqslant 21$, $x + 2y \geqslant 30$, $x \geqslant 0$, $y \geqslant 0$.

Solution : Given : Minimize $z = 4x + 2y$, $3x + y \geqslant 27$, $x + y \geqslant 21$, $x + 2y \geqslant 30$, $x \geqslant 0$, $y \geqslant 0$.

To draw the graph, we prepare the following table :

Inequation	Equation	Points (x, y)				Region
$3x + y \geqslant 27$	$3x + y = 27$	x	0	9	A (0, 27)	$3(0) + 2(0) \ngtr 1$
		y	27	0	B (9, 0)	$\therefore$ non-origin side of the line AB
$x + y \geqslant 21$	$x + y = 21$	x	0	21	C (0, 21)	$0 + 0 \ngtr 21$
		y	21	0	D (21, 0)	$\therefore$ non-origin side of the line CD
$x + 2y \geqslant 30$	$x + 2y = 30$	x	0	30	E (0, 15)	$0 + 2(0) \ngtr 30$
		y	15	0	F (30, 0)	$\therefore$ non-origin side of the line CD
$x \geqslant 0$, $y \geqslant 0$	$x = 0$, $y = 0$					First quadrant

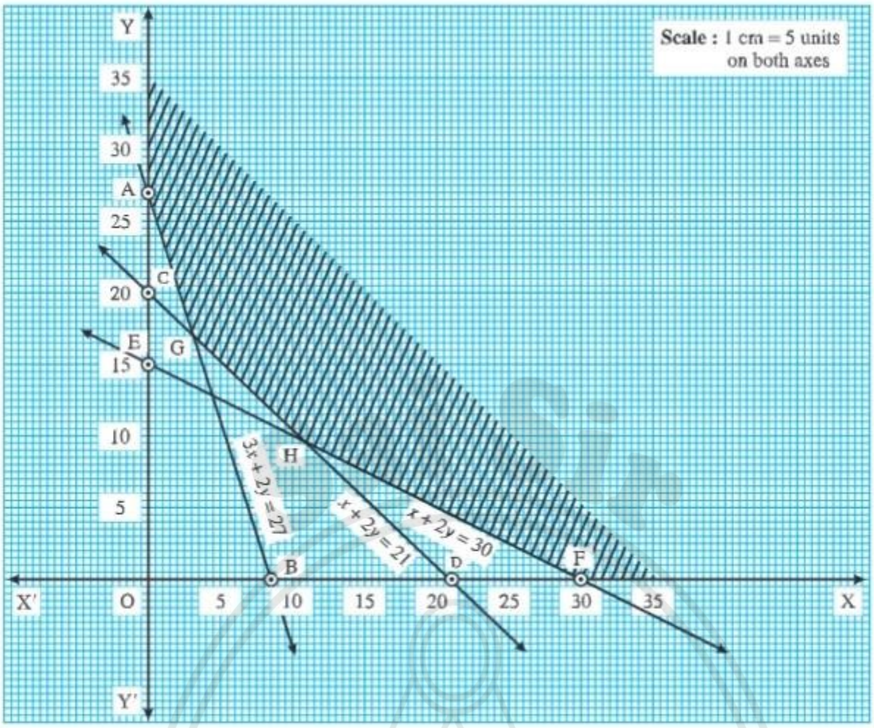


Fig. 6.23

The shaded unbounded portion AGHF in the graph represents the graphical solution of the given LPP.

The lower vertices of the unbounded region are A (0, 27), G (3, 18), H (12, 9), F (30, 0) respectively.

At any one of these vertices the minimum value of z is obtained.

Now, $z = 4x + 2y$

$\therefore$ at A (0, 27), $z = 4(0) + 2(27) = 54$

G (3, 18), $z = 4(3) + 2(18) = 12 + 36 = 48$

H (12, 9), $z = 4(12) + 2(9) = 48 + 18 = 66$

F (30, 0), $z = 4(30) + 2(0) = 120$

The minimum value of z is obtained at the point G.

Therefore the solution of the given LPP is as follows :

z has minimum value 48, when $x = 3$ and $y = 18$.

[**Note :** Here G is the pont of intersection of lines

$3x + y = 27$ … (1)

$x + y = 21$ … (2)

Subtract (2) from (1)

$$\begin{array}{rcl} 3x + y &=& 27 \\ x + y &=& 21 \\ - \quad - && - \\ \hline 2x \quad &=& 6 \end{array}$$

$\therefore$ $x = 3$

Put $x = 3$ in (1) $3(3) + y = 27$ $\therefore$ $y = 27 - 9$ $\therefore$ $y = 18$ Hence, G(3, 18).

Similarly H is the point of intersection of lines

$x + y = 21$ … (1)

$x + 2y = 30$ … (2)

Subtract (1) from (2)

$$\begin{array}{r} x+2y=30 \\ \underline{\begin{array}{r} x+\ y=21 \\ -\ \ -\quad - \end{array}} \\ y=\ 9 \end{array}$$

$\therefore\ y=9$ Put $y=9$ in (1)

$x+9=21$ $\therefore\ x=21-9=12$

$\therefore\ x=12$. Hence, H(12, 9)]

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8. A carpenter makes chairs and tables. Profits are ₹ 140 per chair and ₹ 210 per table. Both products are processed on three machines : Assembling, Finishing and Polishing. The time required for each product in hours and availability of each machine is given by following table :

Product/ Machines	**Chair (x)**	**Table (y)**	**Available time (hours)**
Assembling	3	3	36
Finishing	5	2	50
Polishing	2	6	60

Formulate and solve the following Linear programming problems using graphical method.

Solution :

Let $x=$ Number of chairs, $y=$ Number of tables.

Since the number of chairs and tables cannot be negative, $x \geqslant 0,\ y \geqslant 0$.

For assembling process, 36 machine hours are available. $\therefore\ 3x+3y \leqslant 36$

For finishing process, 50 machine hours are available. $\therefore\ 5x+2y \leqslant 50$

For polishing process, 60 machine hours are available. $\therefore\ 2x+6y \leqslant 60$

A carpenter gets profit of ₹ 140 per chair and ₹ 210 per table.

Let z denote the total profit.

$\therefore$ objective function is $z=140x+210y$

Hence, LPP is formulated as follows :

Maximise $z=140x+210y$

Subject to constraints,

$3x+3y \leqslant 36,\ 5x+2y \leqslant 50,\ 2x+6y \leqslant 60,\ x \geqslant 0,\ y \geqslant 0$

To draw the graph, we prepare the following table :

Inequation	**Equation**	**Points (x, y)**				**Region**
$3x+3y \leqslant 36$	$3x+3y=36$	x	0	12	A(0, 12)	$3(0)+3(0) < 36$
		y	12	0	B(12, 0)	$\therefore$ origin side of the line AB
$5x+2y \leqslant 50$	$5x+2y=50$	x	0	10	C(0, 25)	$5(0)+2(0) < 60$
		y	25	0	D(10, 0)	$\therefore$ origin side of the line CD
$2x+6y \leqslant 60$	$2x+6y=60$	x	0	30	E(0, 10)	$2(0)+6(0) < 60$
		y	10	0	F(30, 0)	$\therefore$ origin side of the line EF
$x \geqslant 0,\ y \geqslant 0$	$x=0,\ y=0$					First quadrant

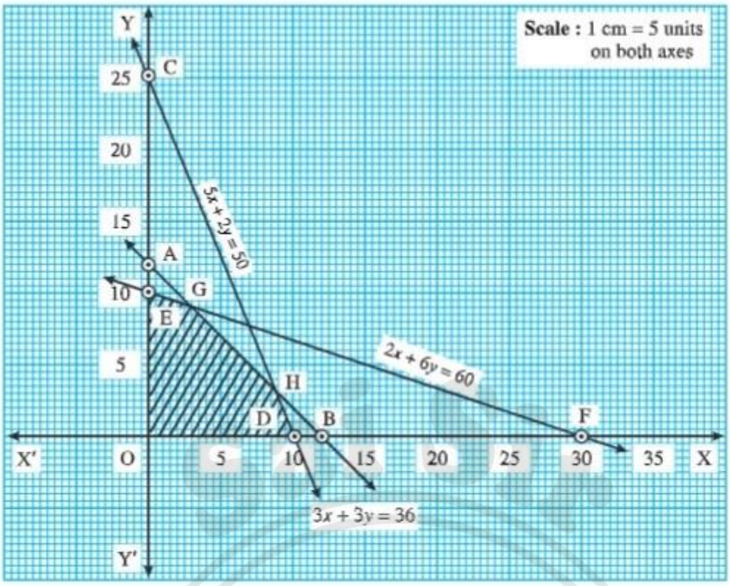


Fig. 6.24

From the graph the feasible region is OEGHD. The vertices of the feasible region are O(0, 0), E(0, 10), G(3, 9), H$\left(\frac{26}{3}, \frac{10}{3}\right)$, D(10, 0). At any one of the vertices the value of $z = 140x + 210y$ is maximum.

At O(0, 0), $z = 140 \times 0 + 210(0) = 0$

E(0, 10), $z = 140 \times 0 + 210 \times 10 = 2100$

G(3, 9), $z = 140 \times 3 + 210 \times 9 = 420 + 1890 = 2310$

H$\left(\frac{26}{3}, \frac{10}{3}\right)$, $z = 140 \times \frac{26}{3} + 210 \times \frac{10}{3} = 1213.33 + 700 = 1913.33$

D(10, 0), $z = 140 \times 10 + 210 \times 0 = 1400$

The value of z is maximum at the point G(3, 9). Hence, z has maximum value 2310, when $x = 3$ and $y = 9$.

Hence, a carpenter should make 3 chairs and 9 tables to get maximum profit of ₹ 2310.

[**Note :** Here G is the point of intersection of

$3x + 3y = 36$ … (1)

$2x + 6y = 60$ … (2)

Multiply (1) by 2 and then subtract (2) from (1)

$$\begin{array}{rcrr} 6x & + & 6y = & 72 \\ 2x & + & 6y = & 60 \\ - & & - & - \\ \hline 4x & & = & -12 \end{array} \qquad \therefore\ x = 3$$

Put $x = 3$ in (1), we get

$3(3) + 3y = 36$

$\therefore\ 3y = 36 - 9$ $\quad\therefore\ 3y = 27$ $\quad\therefore\ y = 9$

Hence, G(3, 9).

Also, H is the point of intersection of

$3x + 3y = 36$ … (1)

$5x + 2y = 50$ … (2)

Multiply (1) by 2 and (2) by 3 and then subtract (2) from (1)

$$\begin{array}{rcr} 6x+6y & = & 72 \\ 15x+6y & = & 150 \\ - \quad - & & - \\ \hline -9x & = & -78 \end{array}$$

$\therefore\ x=\frac{78}{9}=\frac{26}{3}$ Put $x=\frac{26}{3}$ in (1)

$\therefore\ 3\frac{26}{3}+3y=36$ $\therefore\ 26+3y=36$

$\therefore\ 3y=36-26$ $\therefore\ 3y=10$ $\therefore\ y=\frac{10}{3}$]

[**Note :** *Answer given in the textbook is incorrect.*]

9. A company manufactures bicycles and tricycles, each of which must be processed through two machines A and B. Maximum availability of machine A and B is respectively 120 and 180 hours. Manufacturing a bicycle requires 6 hours on machine A and 3 hours on machine B. Manufacturing a tricycle requires 4 hours on machine A and 10 hours on machine B. If profits are ₹ 180 for a bicycle and ₹ 220 for a tricycle, determine the number of bicycles and tricycles that should be manufacture in order to maximize the profit.

Solution :

Let $x=$ Number of bicycles.

$y=$ Number of tricycles.

Since, the number of bicycles and tricyles cannot be negative, $x \geqslant 0$, $y \geqslant 0$.

The given information is tabulated as follows :

Processed through	Machine hours required per unit		Available machine hours
	Bicycles (x)	Tricycles (y)	
Machine A	6	4	120
Machine B	3	10	180
Profit ₹	180	220	

From the table the inequations we get

$6x+4y \leqslant 120$

$3x+10y \leqslant 180$

Let $z=$ Total profit

$\therefore$ objective function to be maximized is

$z=180x+220y$

Thus, the LPP formulated is as follows :

Maximize $z=180x+220y$,

subject to the constraints

$6x+4y \leqslant 120$

$3x+10y \leqslant 180$

$x \geqslant 0,\ y \geqslant 0$

To draw the graph, we prepare the following table :

Inequation	Equation	Points (x, y)				Region
$6x + 4y \leqslant 120$	$6x + 4y = 120$	x	0	20	A (0, 30)	$6(0) + 4(0) < 120$
		y	30	0	B (20, 0)	$\therefore$ origin side of the line AB
$3x + 10y \leqslant 180$	$3x + 10y = 180$	x	0	60	C (0, 18)	$3(0) + 10(0) < 180$
		y	18	0	D (60, 0)	$\therefore$ origin side of the line CD
$x \geqslant 0,\ y \geqslant 0$	$x = 0,\ y = 0$					First quadrant

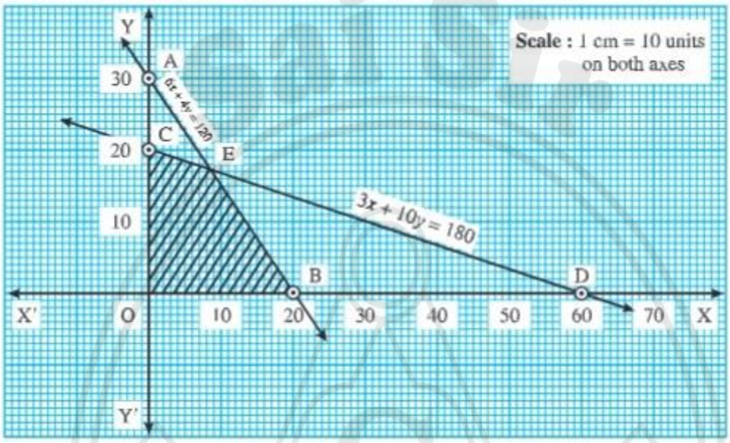


Fig. 6.25

From the graph the feasible region is OCEB. The vertices of this feasible region are O (0, 0), C (0, 18), E (10, 15), B (20, 0). At any one of these vertices, the value of z is maximum.

$z = 180x + 220y$

At O (0, 0), $z = 180 \times 0 + 220 \times 0 = 0$

C (0, 18), $z = 180 \times 0 + 220 \times 18 = 3960$

E (10, 15), $z = 180 \times 10 + 220 \times 15 = 1800 + 3300 = 5100$

B (20, 0), $z = 180 \times 20 + 220 \times 0 = 3600$

The value of z is maximum at the point E (10, 15).

Hence, z has maximum value 5100, when $x = 10$ and $y = 15$.

Hence, 10 bicycles and 15 tricycles, a company should manufacture in order to get maximum profit.

[**Note :** E is the point of intersection of

$3x + 10y = 180$ ... (1)

$6x + 4y = 120$ ... (2)

Multiply (1) by 2 and subtract (2) from (1)

$$\begin{array}{r} 6x + 20y = 360 \\ 6x + \ \ 4y = 120 \\ - \quad - \quad\ \ - \\ \hline 16y = 240 \end{array} \qquad \therefore\ y = 15$$

Put $y = 15$ in (1) $\quad \therefore\ 3x + 10(15) = 180 \quad \therefore\ 3x = 180 - 150 \quad \therefore\ x = \frac{30}{3} = 10$

Hence, E (10, 15).]

10. **A factory produced two types of chemicals A and B. The following table gives the units of ingredients P and Q (per kg) of chemicals A and B as well as minimum requirements of P and Q and also cost per kg of chemicals A and B.**

Ingredients (per kg)	Chemicals (in units)		Minimum requirements (in units)
	A (x)	B (y)	
P	1	2	80
Q	3	1	75
Cost (in ₹)	4	6	

Find the number of units of chemicals A and B should be produced so as to minimize the cost.

Solution : Let $x =$ Number of units of chemical A.

$y =$ Number of units of chemical B.

Since, the number of units cannot be negative, $x \geqslant 0$, $y \geqslant 0$.

From the given data, we get the following inequations :

$x + 2y \geqslant 80$

$3x + y \geqslant 75$

Let $z =$ Total cost

The cost of one unit of chemical A is ₹ 4 and that of chemical B is ₹ 6.

$\therefore$ the objective function to be minimized is $z = 4x + 6y$

Thus, the LPP is formulated as follows :

Minimize $z = 4x + 6y$,

subject to $x + 2y \geqslant 80$, $3x + y \geqslant 75$, $x \geqslant 0$, $y \geqslant 0$.

To draw the graph, we prepare the following table :

Inequation	Equation	Points (x, y)				Region
$x + 2y \geqslant 80$	$x + 2y = 80$	x	0	80	A(0, 40)	$0 + 2(0) \ngtr 80$
		y	40	0	B(80, 0)	$\therefore$ non-origin side of the line AB
$3x + y \geqslant 75$	$3x + y = 75$	x	0	25	C(0, 75)	$3(0) + 0 \ngtr 75$
		y	75	0	D(25, 0)	$\therefore$ non-origin side of the line CD
$x \geqslant 0$, $y \geqslant 0$	$x = 0$, $y = 0$					First quadrant

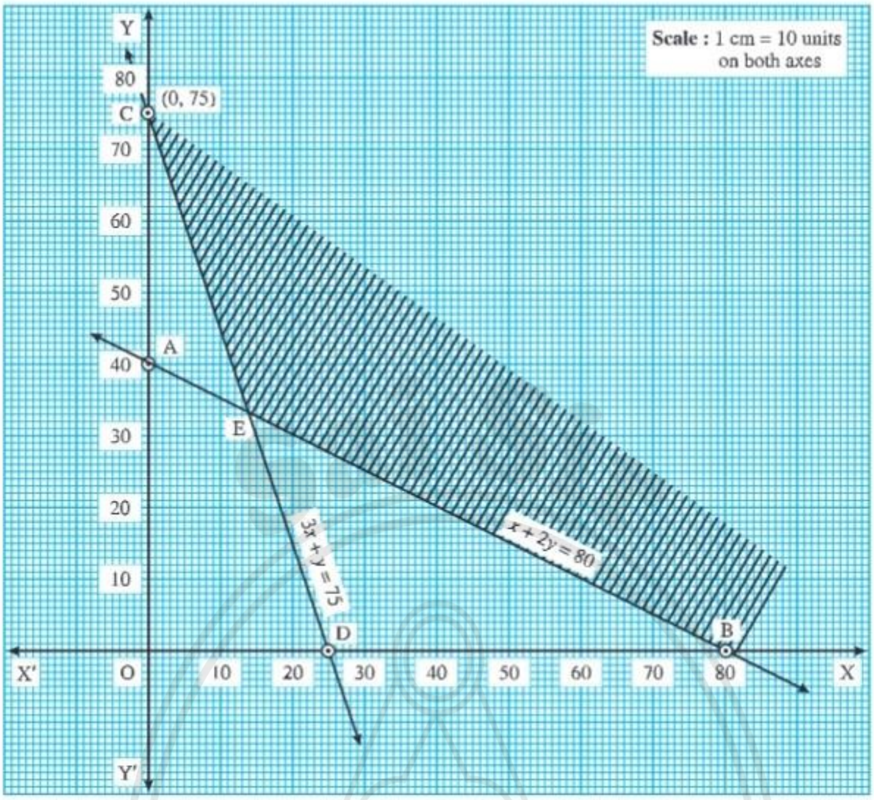


Fig. 6.26

From the graph the unbounded feasible region is CEB. The lower vertices of this region are C (0, 75), E (14, 33), B (80, 0). At any one of these vertices, the value of z is minimum.

$z = 4x + 6y$

At C (0, 75), $z = 4(0) + 6(75) = 450$

E (14, 33), $z = 4(14) + 6(33) = 56 + 198 = 254$

B (80, 0), $z = 4(80) + 6(0) = 320$

The value of z is minimum at the point E (14, 33).

Hence, z has minimum value 254, when $x = 14$ and $y = 33$.

Hence, 14 units of chemical A and 33 units of chemical B should be produced so as the cost is minimum.

[**Note :** Here E is the point of intersection of

$x + 2y = 80$ … (1)

$3x + y = 75$ … (2)

Multiply (2) by 2 and subtract from (1)

$$\begin{array}{rcr} x + 2y & = & 80 \\ \underset{-}{6x} + \underset{-}{2y} & \underset{-}{=} & 150 \\ \hline -5x & = & -\ 70 \end{array} \qquad \therefore\ x = 14$$

Put $x = 14$ in (1) $\quad \therefore\ 14 + 2y = 80$

$\therefore\ 2y = 80 - 14 \quad \therefore\ 2y = 66 \quad \therefore\ y = 33$

Hence, E(14, 33)]

11. A company produces mixers and food processors. Profit on selling one mixer and one food processor is ₹ 2000 and ₹ 3000 respectively. Both the products are processed through three machines A, B, C. The time required in hours by each product and total time available in hours per week on each machine are as follows :

Product → Machine ↓	Mixer per unit	Food processor per unit	Available time
A	3	3	36
B	5	2	50
C	2	6	60

How many mixers and food processors should be produced to maximize the profit?

Solution : Let $x =$ Number of mixers.

$y =$ Number of food processors.

Since, the number of mixers and food processors cannot be negative, $x \geqslant 0$, $y \geqslant 0$.

From the given data, we get the following inequations :

$3x + 3y \leqslant 36$

$5x + 2y \leqslant 50$

$2x + 6y \leqslant 60$

Let $z =$ Total profit

Profit on selling a mixer and a food processors is ₹ 2000 and ₹ 3000 respectively.

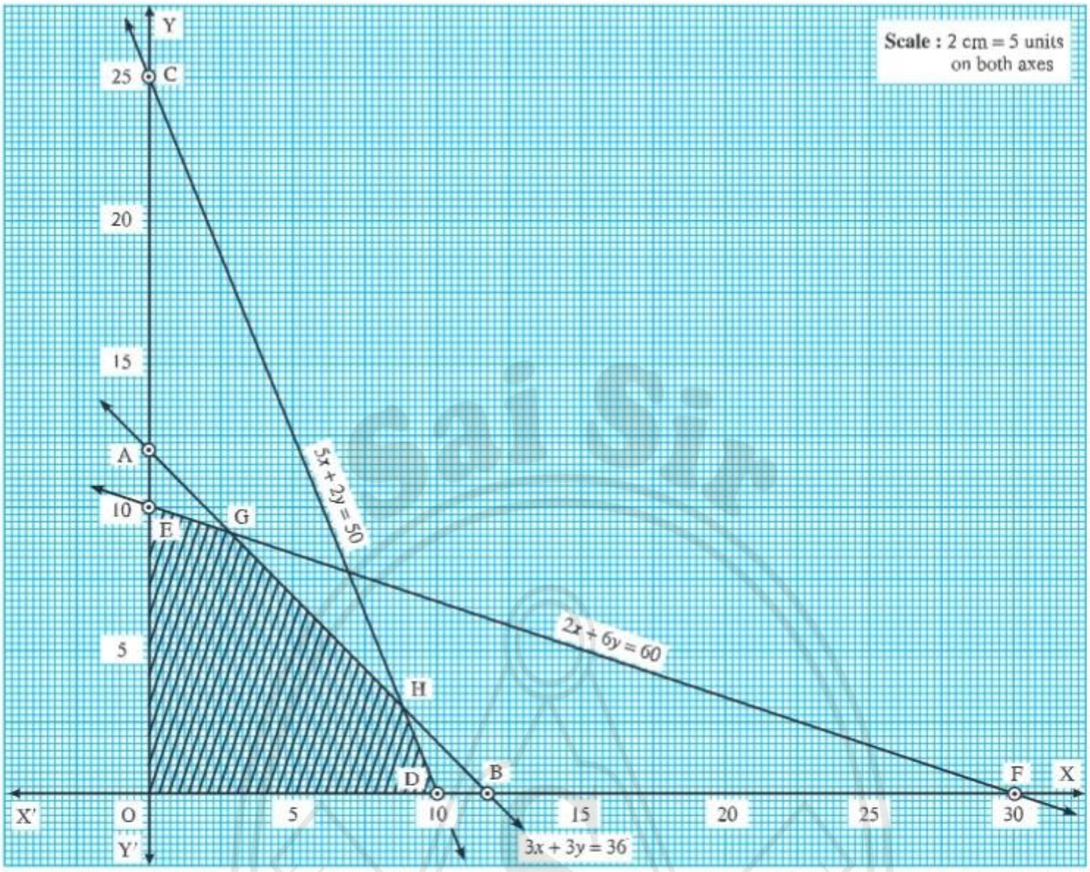
$\therefore$ the objective function to be maximized is $z = 2000x + 3000y$

Thus, the LPP formulated is maximize $z = 2000x + 3000y$,

subject to $3x + 3y \leqslant 36$, $5x + 2y \leqslant 50$, $2x + 6y \leqslant 60$, $x \geqslant 0$, $y \geqslant 0$.

To draw the graph, we prepare the following table :

Inequation	Equation	Points (x, y)				Region
$3x + 3y \leqslant 36$	$3x + 3y = 36$	x	0	12	A(0, 12)	$3(0) + 3(0) < 36$
		y	12	0	B(12, 0)	$\therefore$ origin side of the line AB
$5x + 2y \leqslant 50$	$5x + 2y = 50$	x	0	10	C(0, 25)	$5(0) + 2(0) < 50$
		y	25	0	D(10, 0)	$\therefore$ origin side of the line CD
$2x + 6y \leqslant 60$	$2x + 6y = 60$	x	0	30	E(0, 10)	$2(0) + 6(0) < 60$
		y	10	0	F(30, 0)	$\therefore$ origin side of the line EF
$x \geqslant 0$, $y \geqslant 0$	$x = 0$, $y = 0$					First quadrant



Scale : 2 cm = 5 units on both axes

Fig. 6.27

From the graph the feasible region is OEGHD. The vertices of this region are $O(0,\ 0)$, $E(0,\ 10)$, $G(3,\ 9)$, $H\left(\frac{26}{3},\ \frac{10}{3}\right)$, $D(10,\ 0)$. At any one of these vertices, the value of z is maximum.

$z = 2000x + 3000y$

At $O(0,\ 0)$, $z = 2000(0) + 3000(0) = 0$

$E(0,\ 10)$, $z = 2000(0) + 3000(10) = 30000$

$G(3,\ 9)$, $z = 2000(3) + 3000(9) = 6000 + 27000 = 33000$

$H\left(\frac{26}{3},\ \frac{10}{3}\right)$, $z = 2000\left(\frac{26}{3}\right) + 3000\left(\frac{10}{3}\right) = \frac{52000 + 30000}{3} = \frac{82000}{3} = 27333.33$

$D(10,\ 0)$, $z = 2000(10) + 3000(0) = 20000$

The value of z is maximum at the point $G(3,\ 9)$.

Hence, z has maximum value 33000, when $x = 3$ and $y = 9$.

Hence, 3 mixers and 9 food processor should be produced in order to get maximum profit.

[**Note :** Here G is the point of intersection of

$2x + 6y = 60$ ... (1)

$3x + 3y = 36$ ... (2)

Multiply (2) by 2 and subtract from (1)

$$\begin{array}{rcl} 2x + 6y & = & 60 \\ \underset{-}{6x} + \underset{-}{6y} & \underset{-}{=} & 72 \\ \hline -4x \qquad & = & -12 \end{array} \qquad \therefore\ x = 3$$

Put $x = 3$ in (1) $\quad\therefore\ 2(3) + 6y = 60 \quad\therefore\ 6y = 60 - 6 \quad\therefore\ 6y = 54 \quad\therefore\ y = 9$

Hence, coordinates of G(3, 9)

Also, it is the point of intersection of

$3x + 3y = 36 \quad \ldots (1)$

$5x + 2y = 50 \quad \ldots (2)$

Multiply (1) by 2 and (2) by 3 and subtract

$$\begin{array}{rcr} 6x + 6y & = & 72 \\ \underset{-}{}15x \underset{-}{+} 6y & = & \underset{-}{}150 \\ \hline -9x \qquad & = & -\ 78 \end{array}$$

$\therefore\ x = \dfrac{78}{9} \qquad \therefore\ x = \dfrac{26}{3}$

Put $x = \dfrac{26}{3}$ in (1) $\quad \therefore\ 3\left(\dfrac{26}{3}\right) + 3y = 36 \quad \therefore\ 3y = 36 - 26 \quad \therefore\ y = \dfrac{10}{3}$

Hence, $H\left(\dfrac{26}{3}, \dfrac{10}{3}\right)$]

12. A chemical company produces a chemical containing three basic elements A, B, C, so that it has at least 16 litres of A, 24 litres of B and 18 litres of C. This chemical is made by mixing two compounds I and II. Each unit of compound I, has 4 litres of A, 12 litres of B, 2 litres of C. Each unit of compound II has 2 litres of A, 2 litres of B and 6 litres of C. The cost per unit of compound I is ₹ 800 and that of compound II is ₹ 640. Formulate the problems as LPP and solve it to minimize the cost.

Solution :

Let x = Units of compound I.

y = Units of compound II.

Since, the number of units cannot be negative, $x \geqslant 0$, $y \geqslant 0$.

The given information is tabulated as follows :

Basic elements	Litres required		Availability (At least) Litres
	Compound I (x)	Compound II (y)	
A	4	2	16
B	12	2	24
C	2	6	18
Cost per unit ₹	800	640	

From the table, we get the following inequations :

$4x + 2y \geqslant 16$

$12x + 2y \geqslant 24$

$2x + 6y \geqslant 18$

Let z = Total cost

$\therefore$ the objective function to be minimized is

$z = 800x + 640y$

Thus, the LPP formulated is

Minimize $z = 800x + 640y$, subject to $4x + 2y \geqslant 16$, $12x + 2y \geqslant 24$, $2x + 6y \geqslant 18$, $x \geqslant 0$, $y \geqslant 0$.

To draw the graph, we prepare the following table :

Inequation	Equation	Points (x, y)				Region
$4x + 2y \geqslant 16$	$4x + 2y = 16$	x	0	4	A(0, 8)	$4(0) + 2(0) \ngeqslant 16$
		y	8	0	B(4, 0)	$\therefore$ non-origin side of the line AB
$12x + 2y \geqslant 24$	$12x + 2y = 24$	x	0	2	C(0, 12)	$12(0) + 2(0) \ngeqslant 24$
		y	12	0	D(2, 0)	$\therefore$ non-origin side of the line CD
$2x + 6y \geqslant 18$	$2x + 6y = 18$	x	0	9	E(0, 3)	$2(0) + 6(0) \ngeqslant 18$
		y	3	0	F(9, 0)	$\therefore$ non-origin side of the line EF
$x \geqslant 0, y \geqslant 0$	$x = 0, y = 0$					First quadrant

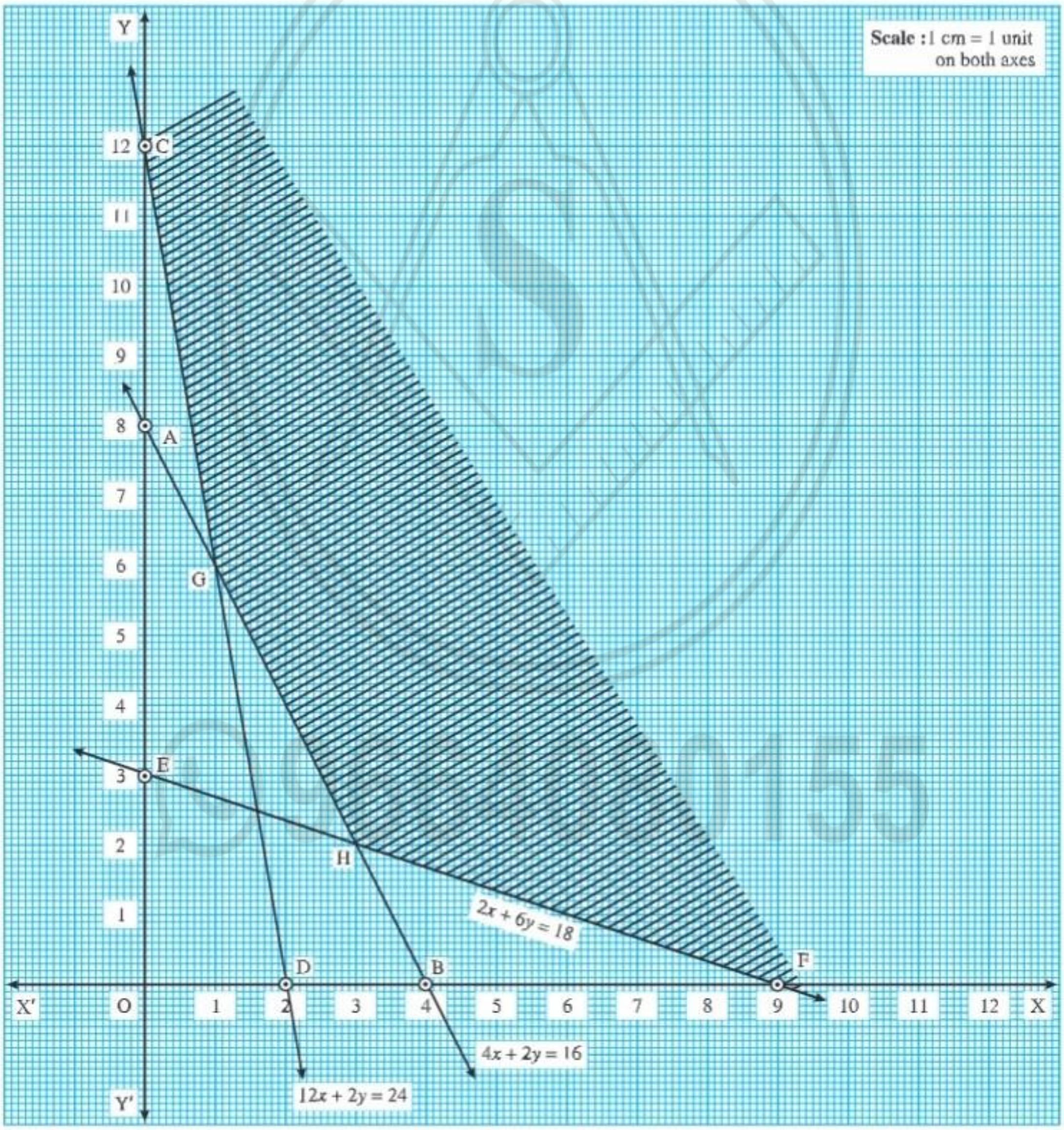


Fig. 6.28

From the graph the unbounded feasible region is CGHF. The vertices of this region are C(0, 12), G(1, 6), H(3, 2), F(9, 0). At any one of these vertices, the value of z is minimum.

$z = 800x + 640y$

At C(0, 12), $z = 800(0) + 640(12) = 7680$

G(1, 6), $z = 800(1) + 640(6) = 800 + 3840 = 4640$

H(3, 2), $z = 800(3) + 640(2) = 2400 + 1280 = 3680$

F(9, 0), $z = 800(9) + 640(0) = 7200$

The value of z is minimum at the point H(3, 2).

Hence, z has minimum value 3680, when $x = 3$ and $y = 2$.

Hence, 3 units of compound I and 2 units of compound II should be produced so that the cost is minimum.

[**Note :** Here G is the point of intersection of lines

$12x + 2y = 24$ ... (1)

$4x + 2y = 16$ ... (2)

Subtract (2) from (1)

$$\begin{array}{rcl} 12x + 2y &=& 24 \\ 4x + 2y &=& 16 \\ - \quad - && - \\ \hline 8x \qquad &=& 8 \end{array} \qquad \therefore x = 1$$

Put $x = 1$ in (1) $\quad \therefore 12(1) + 2y = 24 \quad \therefore 2y = 24 - 12 \quad \therefore 2y = 12 \quad \therefore y = 6$

Hence, G(1, 6)

Also, H is the point of intersection of lines

$2x + 6y = 18$ ... (3)

$4x + 2y = 16$ ... (4)

Multiply (3) by 2 and then subtract (4) from (3)

$$\begin{array}{rcl} 4x + 12y &=& 36 \\ 4x + \ \ 2y &=& 16 \\ - \quad - && - \\ \hline 10y &=& 20 \end{array} \qquad \therefore y = 2$$

Put $y = 2$ in (3) $\quad 2x + 6(2) = 18 \quad 2x = 18 - 12 \quad \therefore 2x = 6 \quad \therefore x = 3$

Hence, H(3, 2)]

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13. A person makes two types of gift items A and B requiring the services of a cutter and a finisher. Gift item A requires 4 hours of cutter's time and 2 hours of finisher's time. Gift item B requires 2 hours of cutter's time and 4 hours of finisher's time. The cutter and finisher have 208 hours and 152 hours available time respectively every month. The profit on one gift item of type A is ₹ 75 and on one gift item of type B is ₹ 125. Assuming that the person can sell all the items produced, determine how many gift items of each type should he make every month to obtain the best returns?

Solution :

Let x = Number of gift items A.

y = Number of gift items B.

Since, the number of gift items cannot be negative, $x \geqslant 0$, $y \geqslant 0$.

The given information is tabulated as follows :

Service	Time required (in hours)		Available time (in hours)
	Gift items A (x)	Gift items B (y)	
Cutter	4	2	208
Finisher	2	4	152
Profit (in ₹)	75	125	

From the table, we get the following inequations :

$4x + 2y \leqslant 208$

$2x + 4y \leqslant 152$

Let $z =$ Total profit

$\therefore$ the objective function to be maximized is

$z = 75x + 125y$

Thus, the LPP formulated is,

Maximize $z = 75x + 125y$, subject to, $4x + 2y \leqslant 208$, $2x + 4y \leqslant 152$, $x \geqslant 0$, $y \geqslant 0$.

To draw the graph, we prepare the following table :

Inequation	Equation	Points (x, y)				Region
$4x + 2y \leqslant 208$	$4x + 2y = 208$	x	0	52	A(0, 104)	$4(0) + 2(0) < 208$
		y	104	0	B(52, 0)	$\therefore$ origin side of the line AB
$2x + 4y \leqslant 152$	$2x + 4y = 152$	x	0	76	C(0, 38)	$2(0) + 4(0) < 152$
		y	38	0	D(76, 0)	$\therefore$ origin side of the line CD
$x \geqslant 0$, $y \geqslant 0$	$x = 0$, $y = 0$					First quadrant

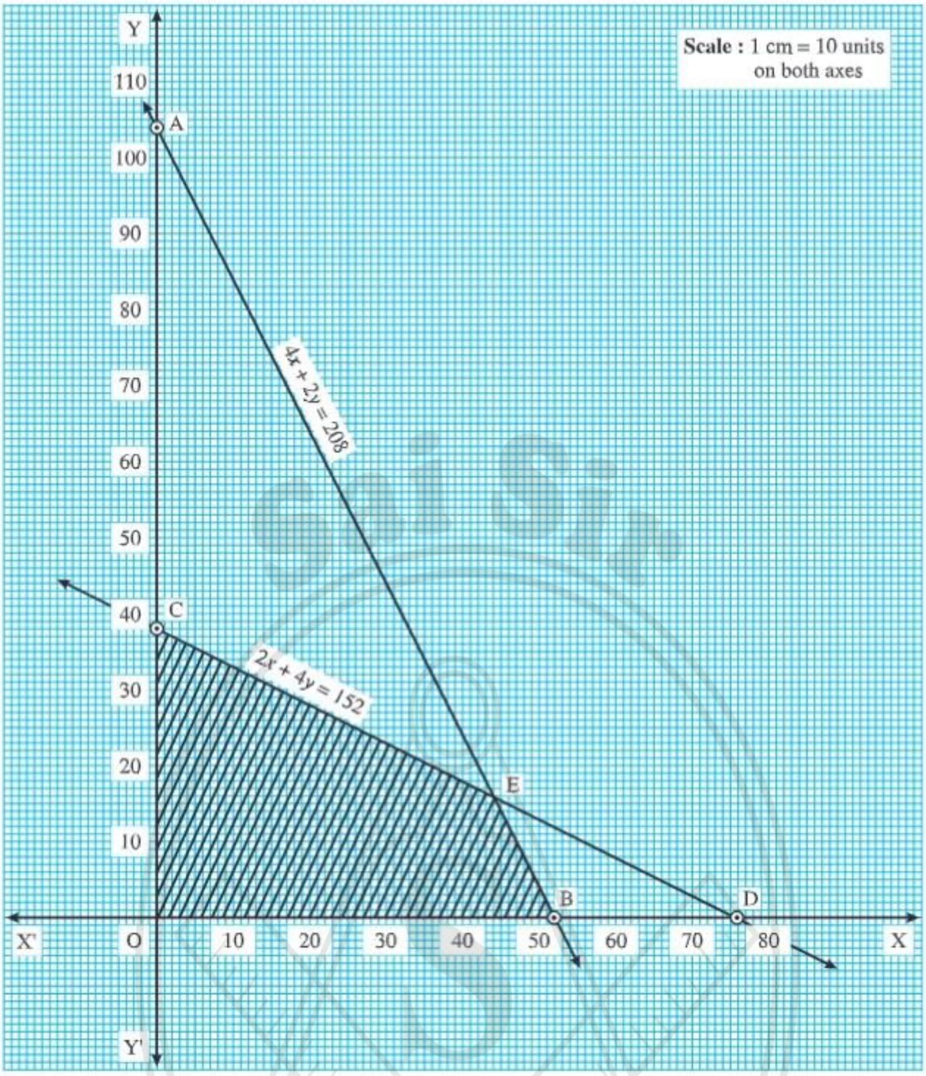

Fig. 6.29

From the graph the feasible region is OCEB. The vertices of this region are O (0, 0), C (0, 38), E (44, 16), B (50, 0). At any one of these vertices, the value of z is maximum.

$z = 75x + 125y$

At O (0, 0), $z = 75(0) + 125(0) = 0$

C (0, 38), $z = 75(0) + 125(38) = 4750$

E (44, 16), $z = 75(44) + 125(16) = 3300 + 2000 = 5300$

B (50, 0), $z = 75(50) + 125(0) = 3750$

The value of z is maximum at the point E (44, 16).

Hence, z has maximum value 5300, when $x = 44$ and $y = 16$.

Hence, 44 gift items of type A and 16 gift items of type B the person should make every month to obtain the best returns.

[**Note :** Here E is the point of intersection of

$4x + 2y = 208$ … (1)

$2x + 4y = 152$ … (2)

Multiply (2) by 2 and then subtract from (1)

$$\begin{array}{rr} 4x + 2y = & 208 \\ 4x + 8y = & 304 \\ - \quad - & - \\ \hline -6y = - & 96 \end{array} \qquad \therefore\ y = 16$$

Put $y = 16$ in (1) $\quad \therefore\ 4x + 2(16) = 208 \quad \therefore\ 4x = 208 - 32 \quad \therefore\ 4x = 176$

$x = \frac{176}{4} = 44$ Hence, E(44, 16)]

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14. A firm manufactures two products A and B on which profit earned per unit is ₹ 3 and ₹ 4 respectively. The product A requires one minute of processing time on M_1 and two minutes on M_2. B requires one minute on M_1 and one minute on M_2. Machine M_1 is available for use for 450 minutes while M_2 is available for 600 minutes during any working day. Find the number of units of product A and B to be manufactured to get the maximum profit.

Solution : Let the firm manufactures x units of product A and y units of product B.

The profit earned per unit of A is ₹ 3 and B is ₹ 4. Hence, the total profit is $z =$ ₹ $(3x + 4y)$.

This is the linear function which is to be maximized. Hence, it is the objective function.

The constraints are as per the following table :

Machine	Product A (x)	Product B (y)	Total availability of time (minutes)
M_1	1	1	450
M_2	2	1	600

From the table, the constraints are

$x + y \leqslant 450,\ x + y \leqslant 600$

Since, the number of gift items cannot be negative,

$x \geqslant 0,\ y \geqslant 0.$

$\therefore$ the mathematical formulation of LPP is

Maximize $z = 3x + 4y$, subject to $x + y \leqslant 450,\ 2x + y \leqslant 600,\ x \geqslant 0,\ y \geqslant 0.$

Now, we draw the lines AB and CD whose equations are $x + y = 450$ and $2x + y = 600$ respectively.

Line	Equation	Points on the X-axis	Points on the Y-axis	Sign	Region
AB	$x + y = 450$	A (450, 0)	B (0, 450)	$\leqslant$	origin side of the line AB
CD	$2x + y = 600$	C (300, 0)	D (0, 600)	$\leqslant$	origin side of the line CD

The feasible region is OCPBO which is shaded in the graph.

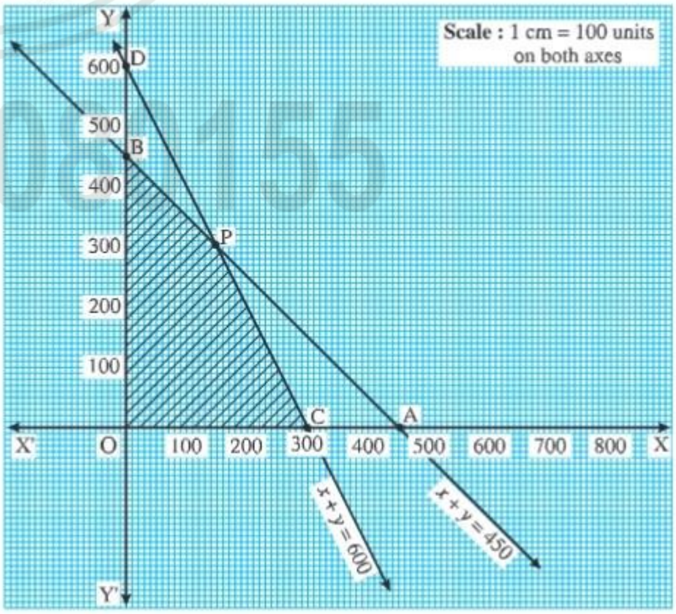

Fig. 6.30

The vertices of the feasible region are O (0, 0), C (300, 0), P and B (0, 450).

P is the point of intersection of the lines

$$2x + y = 600 \quad \dots (1)$$

and $x + y = 450 \quad \dots (2)$

On subtracting, we get

$\therefore\ x = 150$

Substituting $x = 150$ in equation (2), we get

$150 + y = 450$

$\therefore\ y = 300$

$\therefore\ P \equiv (150, 300)$

The values of the objective function $z = 3x + 4y$ at these vertices are

$z(O) = 3(0) + 4(0) = 0 + 0 = 0$

$z(C) = 3(300) + 4(0) = 900 + 0 = 900$

$z(P) = 3(150) + 4(300) = 450 + 1200 = 1650$

$z(B) = 3(0) + 4(450) = 0 + 1800 = 1800$

$\therefore$ z has the maximum value **1800**, when $x = 0$ and $y = 450$.

Hence, the firm gets maximum profit of ₹ **1800** if it manufactures **450** units of product B and **no** unit product A.

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15. A firm manufacturing two types of electrical items A and B, can make a profit of ₹ 20 per unit of A and ₹ 30 per unit of B. Both A and B make use of two essential components a motor and a transformer. Each unit of A requires 3 motors and 2 transformers and each units of B requires 2 motors and 4 transformers. The total supply of components per month is restricted to 210 motors and 300 transformers. How many units of A and B should be manufactured per month to maximize profit? How much is the maximum profit?

Solution : Let the firm manufactures x units of item A and y units of item B.

Firm can make profit of ₹ 20 per unit of A and ₹ 30 per unit of B.

Hence, the total profit is $z =$ ₹ $(20x + 30y)$.

This is the objective function which is to be maximized.

The constraints are as per the following table :

	Item A (x)	Item B (y)	Total supply
Motor	3	2	210
Transformer	2	4	300

From the table, the constraints are

$3x + 2y \leqslant 210,\ 2x + 4y \leqslant 300$

Since, number of items cannot be negative, $x \geqslant 0,\ y \geqslant 0$.

Hence, the mathematical formulation of given LPP is

Maximize $z = 20x + 30y$, subject to

$3x + 2y \leqslant 210,\ 2x + 4y \leqslant 300,\ x \geqslant 0,\ y \geqslant 0$.

We draw the lines AB and CD whose equations are $3x + 2y = 210$ and $2x + 4y = 300$ respectively.

Line	Equation	Points on the X-axis	Points on the Y-axis	Sign	Region
AB	$3x+2y=210$	A (70, 0)	B (0, 105)	$\leqslant$	origin side of the line AB
CD	$2x+4y=300$	C (150, 0)	D (0, 75)	$\leqslant$	origin side of the line CD

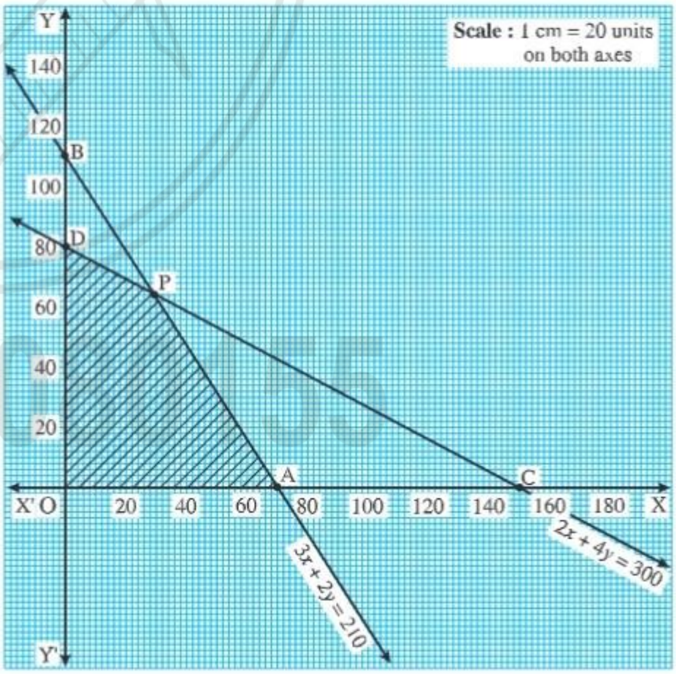


Fig. 6.31

The feasible region is OAPDO which is shaded in the graph.

The vertices of the feasible region are O (0, 0), A (70, 0), P and D (0, 75).

P is the point of intersection of the lines

$2x + 4y = 300$ ... (1)

and $3x + 2y = 210$ ... (2)

Multiplying equation (2) by 2, we get

$6x + 4y = 420$

Subtracting equation (1) from this equation, we get

$\therefore\ 4x = 120$

$\therefore\ x = 30$

Substituting $x = 30$ in (1), we get

$2(30) + 4y = 300$

$\therefore\ 4y = 240$

$\therefore\ y = 60$

$\therefore$ P is (30, 60)

The values of the objective function $z = 20x + 30y$ at these vertices are

$z(\mathrm{O}) = 20(0) + 30(0) = 0 + 0 = 0$

$z(\mathrm{A}) = 20(70) + 30(0) = 1400 + 0 = 1400$

$z(\mathrm{P}) = 20(30) + 30(60) = 600 + 1800 = 2400$

$z(\mathrm{D}) = 20(0) + 30(75) = 0 + 2250 = 2250$

$\therefore$ z has the maximum value **2400**, when $x = $ **30** and $y = $ **60**.

Hence, the firm should manufactured **30** units of item A and **60** units of item B to get the maximum profit of **₹ 2400.**

[**Note :** *Answer given in the textbook is incorrect.*]

ACTIVITIES | Textbook pages 105 to 107

(Answers are given directly.)

1. Find the graphical solution for the following system of linear inequations.

$8x + 5y \leqslant 40$, $4x + 5y \leqslant 40$, $x \geqslant 0$, $y \geqslant 0$.

Solution : To draw $8x + 5y \leqslant 40$

Draw line L_1 $8x + 5y = 40$

x	y	(x, y)	Sign	Region
$\boxed{5}$	0	A ($\boxed{5}$, 0)	$\leqslant$	On origin
0	$\boxed{8}$	B (0, $\boxed{8}$)		side of the line

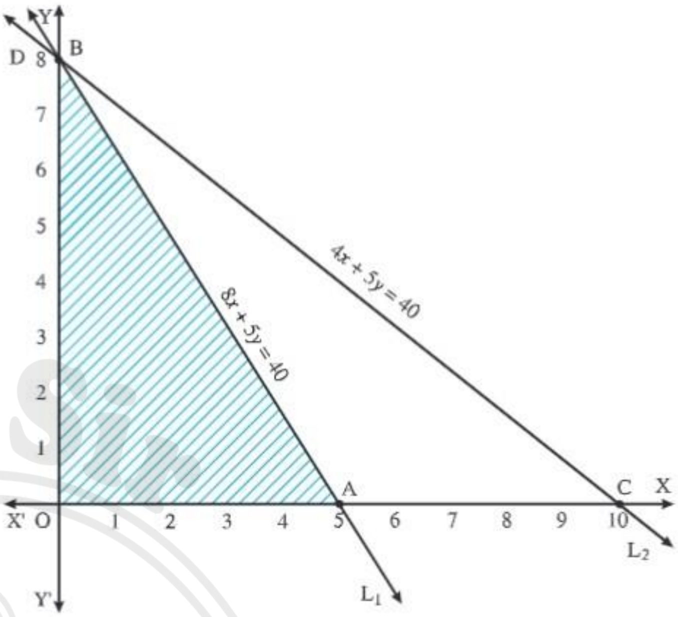


Fig. 6.32

To draw $4x + 5y \leqslant 40$

Draw line L_2 : $4x + 5y = 40$

x	y	(x, y)	Sign	Region
$\boxed{10}$	0	C ($\boxed{10}$, 0)	$\leqslant$	On origin
0	$\boxed{8}$	D (0, $\boxed{8}$)		side of the line

The common shaded region OABO is graphical solution, with vertices O ($\boxed{0}$, $\boxed{0}$), A ($\boxed{5}$, $\boxed{0}$), B ($\boxed{0}$, $\boxed{8}$).

2. Find the graphical solution for the following system of linear inequations $3x + 5y \leqslant 15$, $2x + 5y \leqslant 15$, $2x + 3y \leqslant 18$, $x \geqslant 0$, $y \geqslant 0$.

[**Note :** *This activity has been modified.*]

Solution : To draw $3x + 5y \leqslant 15$

Draw line $3x + 5y = 15$.

x	y	(x, y)	Sign	Region
$\boxed{5}$	0	A ($\boxed{5}$, 0)	$\leqslant$	On origin
0	$\boxed{3}$	B (0, $\boxed{3}$)		side of the line

Now, draw line $2x+5y \leqslant 15$

i.e. $L_2 : 2x+5y=15$

x	y	(x, y)	Sign	Region
$\boxed{7.5}$	0	C ($\boxed{7.5}$, 0)	$\leqslant$	On origin side of the line
0	$\boxed{3}$	D (0, $\boxed{3}$)		

Also draw line $2x+3y \leqslant 18$

i.e. $L_3 = 2x+3y=18$

x	y	(x, y)	Sign	Region
$\boxed{9}$	0	E ($\boxed{9}$, 0)	$\leqslant$	On origin side of the line
0	$\boxed{6}$	F (0, $\boxed{6}$)		

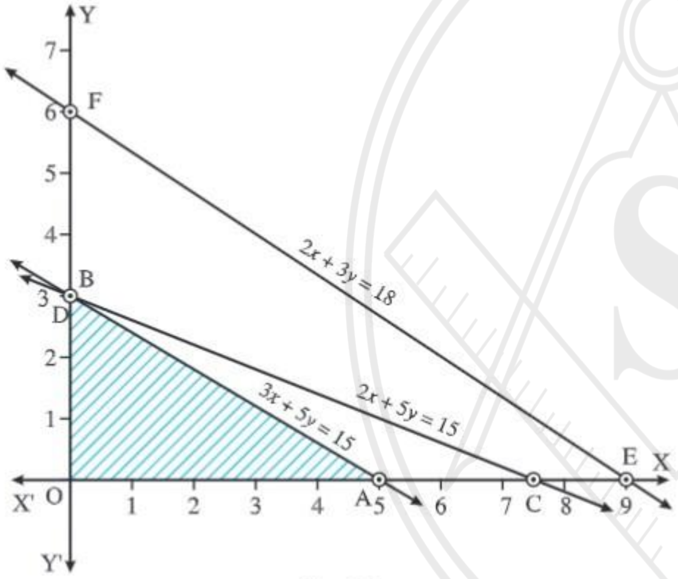


Fig. 6.33

The common shaded region OABO is graphical solution, with vertices O (0, 0), A (5, 0), B (0, 3).

---

3. Shraddha wants to invest atmost ₹ 25,000 in savings certificates and fixed deposits. She wants to invest at least ₹ 10,000 in savings certificate and at least ₹ 15,000 in fixed deposits. The rate of interest on saving certificate is 5% per annum and that on fixed deposits is 7% per annum. Formulate the above problem as LPP to determine maximum yearly income.

Solution : Let x_1 amount (in ₹) invest in saving certificate

x_2 : amount (in ₹) invest in fixed deposits,

$x_1 \geqslant 0, x_2 \geqslant 0$

From given conditions $x_1 + x_2 \boxed{\leqslant} 25{,}000$

She wants to invest at least ₹ 10000 in saving certificate

$\therefore\ x_1 \boxed{\geqslant} 10{,}000$

Shraddha want to invest at least ₹ 15,000 in fixed deposits.

$x_2 \boxed{\geqslant} 15{,}000$

$$\text{Total interest} = z = \boxed{\frac{5x_1}{100} + \frac{7x_2}{100}}$$

$$\text{Maximize } z = \boxed{\frac{5x_1}{100} + \frac{7x_2}{100}}$$

subject to $x_1 \geqslant \mathbf{10{,}000}$

$\boxed{x_2 \geqslant 15000,\ x_1 + x_2 \leqslant 25000,\ x_1 \geqslant 0,\ x_2 \geqslant 0}$.

---

4. The graphical solution of LPP is shown by following figure. Find the maximum value of $z = 3x+2y$ subject to the conditions given in grpahical solution.

Solution : From Fig. 6.34. The common shaded region OABC is feasible region with vertices 0 ($\boxed{0}$, $\boxed{0}$), A ($\boxed{6}$, $\boxed{0}$), B ($\boxed{4}$, $\boxed{3}$), C ($\boxed{0}$, $\boxed{5}$).

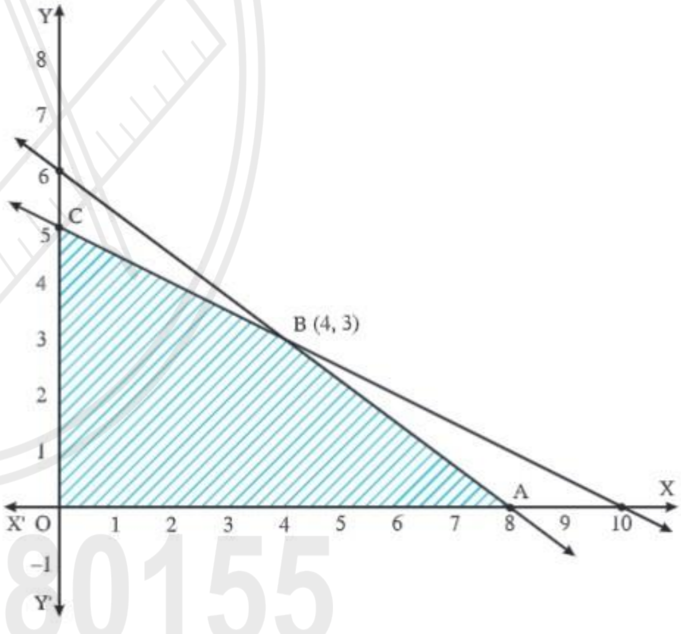


Fig. 6.34

Sr. No.	(x, y)	Value of $z = 3x+2y$ at (x, y)
1.	O (0, 0)	$z = 0$
2.	A (6, 0)	$z = \boxed{18}$
3.	B ($\boxed{4}$, $\boxed{3}$)	$z = \boxed{18}$
4.	C (0, 6)	$z = 10$

From the above table, maximum value of $z = \boxed{18}$ occurs at two point $\boxed{A}$ and $\boxed{B}$ that is when $x=\boxed{6}$, $y=\boxed{0}$ and when $x=\boxed{4}$, $y=\boxed{3}$.

5. Formulate and solve the following LPP. A company manufactures bicycles and tricycles, each of which must be processed through two machines A and B. Machine A has maximum of 120 hours available and machine B has a maximum of 180 hours available. Manufacturing a bicycle requires 6 hours on machine A and 3 hours on machine B. Manufacturing a tricycle requires 4 hours on machine A and 10 hours on machine B. If profits are ₹ 180 for a bicycle and ₹ 220 for a tricycle, determine the number of bicycles and tricycles that should be manufactured in order to maximize the profit.

[**Note :** *This activity has been modified.*]

Solution : Let x number of bicycles and y number of tricycles be manufactured.

$x \geqslant 0$, $y \geqslant 0$ ... (1)

Total profit $= z = \boxed{x(180) + y(220) = 180x + 220y}$

Maximize $z = \boxed{180x + 220y}$

The remaining conditions are

$\boxed{6x + 4y \leqslant 120}$

$\boxed{3x + 10y \leqslant 180}$

$\therefore$ LPP is maximize $z = \boxed{180x + 220y}$

subject to $\boxed{6x + 4y \leqslant 120,\ 3x + 10y \leqslant 180}$, $x \geqslant 0$, $y \geqslant 0$.

To draw $6x + 4y \leqslant 120$

Draw line $L_1 : 6x + 4y = 120$

x	y	(x, y)	Sign	Region
$\boxed{20}$	0	A$(\boxed{20}, 0)$	$\leqslant$	origin side of the line L_1
0	$\boxed{30}$	B$(0, \boxed{30})$		

To draw $3x + 10y \leqslant 180$

Draw line $L_2 : 3x + 10y = 180$

x	y	(x, y)	Sign	Region
$\boxed{60}$	0	C$(\boxed{60}, 0)$	$\leqslant$	origin side
0	$\boxed{18}$	D$(0, \boxed{18})$		of the line L_2

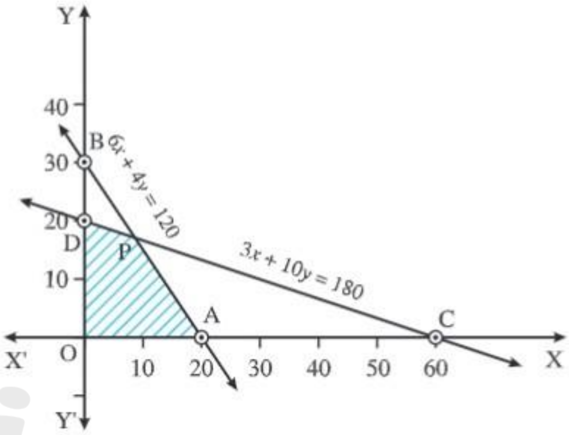


Fig. 6.35

The common shaded region OAPDO is feasible region with vertices O(0, 0), A(20, 0), P, D(0, 18).

Here P is the point of intersection of lines

$6x + 4y = 120$ ... (1)

$3x + 10y = 180$ ... (2)

Multiply (2) by 2 and subtract (1)

$$\begin{array}{rcr} 6x + 4y &=& 120 \\ 6x + 20y &=& 360 \\ - \quad - \quad && - \\ \hline -16y &=& -240 \end{array}$$

$\therefore\ 16y = 240$

$y = \dfrac{240}{16} = 15$

Put $y = 15$ in (1) $\quad \therefore\ 6x + 4(15) = 120$

$\therefore\ 6x = 120 - 60 \quad \therefore\ 6x = 60 \quad \therefore\ x = 10$

Hence, P(10, 15)

Sr. No.	(x, y)	Value of $z = 3x + 2y$
1.	O(0, 0)	$z = 0$
2.	A(20, 0)	$z = 3(20) + 2(0) = 60$
3.	P(10, 15)	$z = 3(10) + 2(15) = 60$
4.	D(0, 18)	$z = 3(0) + 2(18) = 36$

Maximum value of $z = \boxed{60}$ occurs at points A(20, 0), P(10, 15) that is when $x = 20$ and $y = 0$ OR when $x = 10$ and $y = 15$

The company gets maximum profit $z = \boxed{60}$ when $x = \boxed{20}$ bicycles and $y = 0$ (i.e. no tricycle), or when $x = 10$ bicycles and $\boxed{15}$ tricycles are manufactured.

* * *

7 ASSIGNMENT PROBLEM AND SEQUENCING

CHAPTER OUTLINE

INTRODUCTION

The Assignment problem is a special type of linear programming problem. It deals with assigning n jobs to n workers on a one-to-one basis in such a way that the objective is optimised. This means we find the best possible assignment that gives maximum efficiency and minimum cost. For example, assigning activities to students, assigning different jobs to different machines, assigning jobs to salesmen to different regions, etc. A problem of such nature is called an Assignment Problem. In such cases, assignment is made on a one-to-one basis in such a manner that cost is minimized or profit is maximized.

Since, there are equal number of asignees (say n) and assignments (say n) the problem can be represented in the form of $n \times n$ cost matrix.

IMPORTANT POINTS TO REMEMBER

1. **Assignment Problem :** Assignment problem is a particular case of Transportation problem with two characteristics :

   (i) The cost matrix is a square matrix

   (ii) Each supply and requirement are 1.

   Objective of Assignment problem is to assign the origins to the equal number of destinations so as to minimize cost or maximize profit.

(1) General Form of Assignment Problem :

Let $x_{ij} = 1$, if i^{th} person is assigned to j^{th} job

$= 0$, if i^{th} person is not assigned to j^{th} job

$c_{ij} =$ Cost (time) required by i^{th} person to complete j^{th} job,

where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$.

Minimize (total cost) $Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, x_{ij}$ subject to the constraints,

(i) each person should be assigned to one and only one job, i.e.

$$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, 2, 3, \ldots, n$$

(ii) each job should be assigned to one and only one person i.e.

$$\sum_{i=1}^{n} x_{ij} = 1, \quad j = 1, 2, 3, \ldots, n$$

(iii) $x_{ij} = 0$ or 1 for all i and j

(2) Hungarian Method of solving Assignment Problem :

- **Basic Principles :**

  (i) The objective remains unchanged if the origin is changed for the elements of the given cost matrix.

  (ii) A solution having zero total cost is an optimal solution.

- **Assumptions :**

  (i) Each facility is capable of performing each task.

  (ii) Only one task can be assigned to each facility.

  (iii) Cost matrix must be a square matrix.

(3) Special cases of Assignment Problem : The Assignment problem is generally defined as a problem of minimization.

(i) Unbalanced Assignment Problem :

Cost matrix is not a square matrix. Adding 'dummy tasks/facilities with zero costs.'

(ii) Maximization Assignment Problem :

Convert into minimization problem (a) by subtracting all the elements from the largest element of cost matrix or (b) multiplying all the elements of cost matrix by -1.

(iii) Restricted (Prohibited) Assignment Problem :

Certain combinations of tasks/facilities are restricted. Assign very high cost (say infinity) at the restricted (prohibited) combinations.

(iv) Alternate/Multiple Optimal Solution :

Assignment matrix contains more than required number of zero elements.

2. Sequencing Problem :

(1) Notations :

M_{ij} = Processing time required by the i^{th} job on the j^{th} machine. $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m)$

T = Total elapsed time (including idle time if any) for processing all jobs.

X_{ij} = Idle time on machine j from end of $(i-1)^{th}$ job to start of i^{th} job. $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m)$

(2) Assumptions :

(i) No machine can process more than one job at a time.

(ii) Each job, once started on a machine has to be completed.

(iii) Only one machine of each type is available, i.e. all machines are of different types.

(iv) Processing times are independent of processing the jobs.

(v) All jobs are completely known and are ready for processing before the period under consideration begins.

(3) Types of Sequencing Problem :

(i) Processing 'n' jobs through 'Two machines' :

Processing time for n jobs and two machines :

Job (*i*)		1	2	3	...	*i*	...	*n*
Machines	**M_1**	M_{11}	M_{12}	M_{13}	...	M_{1i}	...	M_{1n}
	M_2	M_{21}	M_{22}	M_{23}	...	M_{2i}	...	M_{2n}

Clearly the total elapsed time T is given by

$$T = \sum_{i=1}^{n} M_{i2} + \sum_{i=1}^{n} X_{i2}$$

where some of the X_{i2}'s may be zero.

The problem is to determine the sequence of jobs that minimizes T.

$\sum_{i=1}^{n} M_{i2}$ = Total time for which machine M_2 has to work.

Since, $\sum_{i=1}^{n} M_{i2}$ is fixed, the problem reduces to that of minimizing $\sum_{i=1}^{n} X_{2i}$ (Total idle time)

(ii) Processing 'n' jobs through 'Three machines' :

Let 'n' jobs, each of which is to be processed through three machines M_1, M_2, M_3 in the order $M_1M_2M_3$.

If at least one of the conditions, min. $M_1 \geqslant$ max. M_2 and min. $M_3 \geqslant$ max. M_2 is satisfied, then convert the three machines problem into two machines problem by two fictitious machines, say $G = M_1 + M_2$ and $H = M_2 + M_3$ and then determine the optimal sequence of n jobs for G and H.

7.1 : ASSIGNMENT PROBLEM

7.1.1 : Definition of Assignment Problem

The assignment problem is a special case of problem in which a number of origins (resources) are assigned to the equal number of destinations (activities) on one-to-one basis, so that the total cost is minimized (or profit is maximized). Thus, the essential characteristic of the assignment problem is as follows :

'n resources are to be assigned to n activities such that each resource is allocated to each activity and each activity is performed by one resource only'.

Conditions :

(i) Number of jobs is equal to number of machines or workers.

(ii) Each worker or machine is assigned to only one job.

(iii) Each worker or machine is independently capable of handling any job.

(iv) Objective of assignment is clearly specified (minimizing cost or maximizing profit).

7.1.2 : Assignment Model

Let n jobs be assigned to n facilities and c_{ij} be the cost associated with assignee i ($i = 1, 2, \dots, n$) performing the assignment j ($j = 1, 2, \dots, n$). The object is to determine how all the assignments should be made so that the overall return (cost or profit) is optimized (i.e. minimized or maximized.)

c_{ij} = cost of assigning the j^{th} job to i^{th} person

$x_{ij} = 1$, if i^{th} person is assigned to j^{th} job

$= 0$, if i^{th} person is not assigned to j^{th} job

where $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, n$.

Therefore, the total cost $Z = \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} c_{ij} \cdot x_{ij}$

The mathematical formulation of the assignment problem is given as follows :

Minimize $Z = \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} c_{ij} \cdot x_{ij}$

subject to the constraints

(i) $x_{i1} + x_{i2} + \dots + x_{in} = 1,\ i = 1, 2, \dots, n$

i.e. $\sum\limits_{j=1}^{n} x_{ij} = 1$ for $i = 1, 2, \dots, n$

(ii) $x_{1j} + x_{2j} + \dots + x_{nj} = 1,\ j = 1, 2, \dots, n$

i.e. $\sum\limits_{i=1}^{n} x_{ij} = 1$ for $j = 1, 2, \dots, n$

(iii) $x_{ij} = 0$ or 1 for all i and j

The assignment problem can be represented in the form of $n \times n$ matrix as follows :

Origins (Persons)	**Destinations (Jobs)**						**Supply**
	J_1	J_2	…	J_j	…	J_n	
P_1	c_{11} x_{11}	c_{12} x_{12}	…	c_{1j} x_{1j}	…	c_{1n} x_{1n}	1
P_2	c_{21} x_{21}	c_{22} x_{22}	…	c_{2j} x_{2j}	…	c_{2n} x_{2n}	1
⋮ ⋮							⋮ ⋮
P_i	c_{i1} x_{i1}	c_{i2} x_{i2}	…	c_{ij} x_{ij}	…	c_{in} x_{1n}	1
⋮							⋮
P_n	c_{n1} x_{n1}	c_{n2} x_{n2}	…	c_{nj} x_{nj}	…	c_{nn} x_{nn}	1
Requirement	1	1	…	1	…	1	

Thus, an assignment problem can be represented by $n \times n$ matrix which covers $n!$ possible ways of making assignments.

7.1.3 : Hungarian Method of Solving Assignment Problem

A Hungarian Mathematician D. Konia developed the most effective method for solving an assignment problem.

Basic principles of this method are as follows :

(i) The objective of the assignment problem remains unchanged if we shift the origin of elements of any row or column in the given cost matrix.

(ii) A solution having zero total cost is an optimum solution.

Assumptions :

(i) Each facility is capable of performing each task.

(ii) Only one task can be assigned to each facility.

(iii) The assignment matrix must be of $n \times n$ size (i.e. a square matrix of size n).

Algorithm of Hungarian Method :

Minimization case :

Step 1 : Select the smallest element in each row and subtract it from every element of that row.

Step 2 : In the reduced matrix obtained from Step 1, select the smallest element in each column and subtract it from every element of that column.

Step 3 : In the reduced matrix obtained from Step 2, select a row (or column) with exactly one zero and make an assignment by enclosing this zero in a box (☐) and cross (×) all other zeros appearing in the corresponding column (or row). Proceed in this way until all the rows (or columns) have been examined.

Step 4 : If all the zeros in rows/columns are either marked (☐) or crossed (×) and there is exactly one assignment in each row and in each column (i.e. the number of assigned cells (☐) is equal to the number of rows/columns), then optimal solution is obtained for the given problem. If the number of assigned cells (☐) is less the number of rows/columns (i.e. order of matrix), then proceed to next Step 5.

Step 5 : Draw the minimum number of vertical and horizontal lines [which should be exactly equal to the number of assigned cells (☐)] to cover all zeros in the reduced matrix obtained from Step 3.

Step 6 : If the number of straight lines drawn in Step 5 is equal to the number of rows or columns, then it is an optimal solution, otherwise go to Step 7.

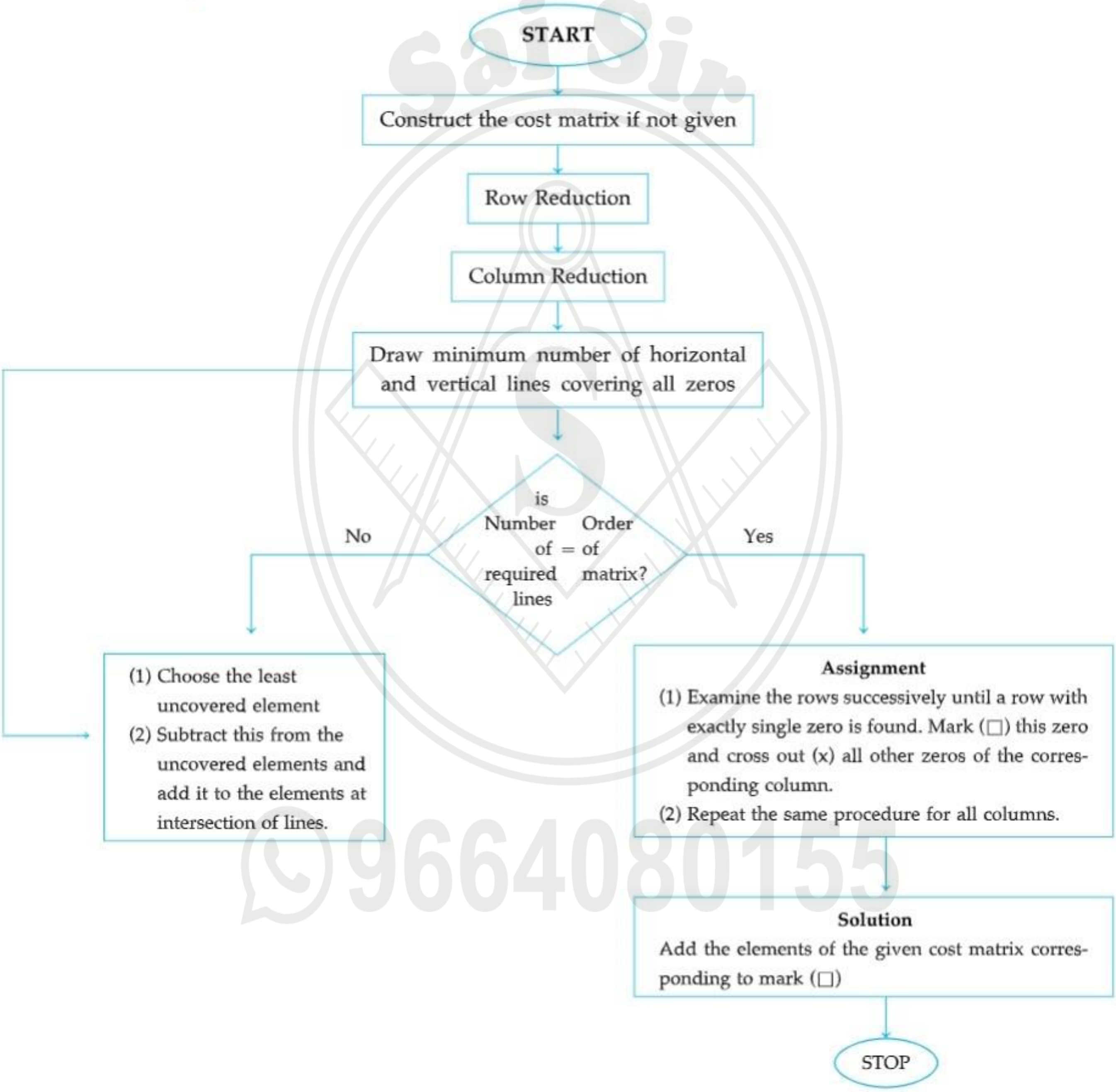
Step 7 : Select the smallest element among all the uncovered elements. Subtract this smallest element from all the uncovered elements and add it to the element which lies at the intersection of two lines. Thus, we obtain another reduced matrix for fresh assignment.

Step 8 : Go to Step 3 and repeat the procedure until the number of assignments become equal to the number of rows/columns.

Step 9 : The optimal assignment schedule corresponding to assigned zeros obtained is noted down.

Step 10 : Optimal cost can be obtained by adding the cost at the assigned points in the cost matrix.

Flow Chart of Hungarian Method :



7.1.4 : Special cases of Assignment Problem

1. **Maximization case :** To solve the problem of maximization objective, all the elements in the matrix are subtracted from the largest element in the matrix. Then solve the problem with the reduced matrix as per the method of minimization case.
2. **Unbalanced Assignment Problem :** An assignment problem in which the number of tasks and facilities are not equal to each other (i.e. pay off matrix of the problem is not a square matrix) is called an unbalanced assignment problem. It can be balanced by adding 'dummy tasks/facilities with zero costs'. Then solve the problem with reduced matrix as per the method of minimization case.
3. **Restricted (Prohibited) Assignments :** An assignment problem which involves restrictions in allocation is referred as the problem with restricted (prohibited) assignment. Such problem can be solved by assigning very high cost (say infinity) at the restricted (prohibited) combinations.
4. **Alternate (Multiple) Optimal Solution :** An assignment problem in which the final assignment matrix contains more than the required number of zero element is referred as the assignment problem with alternate (multiple) optimal solution.

EXERCISE 7.1 | **Textbook pages 118 and 119**

1. A job production unit has four jobs P, Q, R, S which can be manufactured on each of the four machines I, II, III and IV. The processing cost of each job for each machine is given in the following table :

Jobs	Processing cost (in ₹)			
	Machines			
	I	II	III	IV
P	31	25	33	29
Q	25	24	23	21
R	19	21	23	24
S	38	36	34	40

Find the optimal assignment to minimize the total processing cost.

[**Note :** *Question is modified.*]

Solution :

Step 1 : Subtract the smallest element in each row from every element of it. New assignment matrix is obtained as follows :

Jobs	Processing cost (in ₹)			
	Machines			
	I	II	III	IV
P	6	0	8	4
Q	4	3	2	0
R	0	2	4	5
S	4	2	0	6

Step 2 : Subtract the smallest element in each column from every element of it. New assignment matrix is obtained as above, because each column in it contains one zero.

Step 3 : Cover all zeros by minimum number of horizontal and vertical lines.

Jobs	Processing cost (in ₹)			
	Machines			
	I	II	III	IV
P	~~6~~	~~0~~	~~8~~	~~4~~
Q	~~4~~	~~3~~	~~2~~	~~0~~
R	~~0~~	~~2~~	~~4~~	~~5~~
S	~~4~~	~~2~~	~~0~~	~~6~~

As the minimum number of straight lines required to cover all zeros in the assignment matrix equals the number of rows/columns. Optimal solution has reached.

Step 4 : Examine the rows one by one starting with the first row with exactly one zero is found. Mark the zero by enclosing it in (☐), indicating assignment of the job. Cross all the zeros in the same column.

This step is shown in the following table :

Jobs	Processing cost (in ₹)			
	Machines			
	I	II	III	IV
P	6	$\boxed{0}$	8	4
Q	4	3	2	$\boxed{0}$
R	$\boxed{0}$	2	4	5
S	4	2	$\boxed{0}$	6

It is observed that all the zeros are assigned and each row and each column contains exactly one assignment. Hence, the optimal (minimum) assignment schedule is :

Jobs	Machines	Processing cost (in ₹)
P	II	25
Q	IV	21
R	I	19
S	III	34

Hence, total (minimum) processing cost = 25 + 21 + 19 + 34 = ₹ 99.

2. Five wagons are available at stations 1, 2, 3, 4 and 5. These are required at 5 stations I, II, III, IV and V. The mileage between various stations are given in the table below. How should the wagons be transported so as to minimize the mileage covered?

	I	II	III	IV	V
1	10	5	9	18	11
2	13	9	6	12	14
3	7*	2	4	4	5
4	18	9	12	17	15
5	11	6	14	19	10

[* *Data has been modified.*]

Solution :

Step 1 : Subtract the smallest element in each row from every element of that row.

Wagons	Mileage of Stations				
	I	II	III	IV	V
1	5	0	4	13	6
2	7	3	0	6	8
3	5	0	2	2	3
4	9	0	3	8	6
5	5	0	8	13	4

Step 2 : Subtract the smallest element of each column from every element of that column.

Wagons	Mileage of stations				
	I	II	III	IV	V
1	0	0	4	11	3
2	2	3	0	4	5
3	0	0	2	0	0
4	4	0	3	6	3
5	0	0	8	11	1

The number of lines covering all zeros (4) is not equal to order of matrix (5). So solution has not reached.

Step 3 : Therefore, subtract the smallest uncovered element (1) from all uncovered elements and add it to all elements which lie at the intersection of two lines. All other elements on the line remain unchanged.

Wagons	Mileage of stations				
	I	II	III	IV	V
1	0	0	4	10	2
2	2	3	0	3	4
3	1	1	3	0	0
4	4	0	3	5	2
5	0	0	8	10	0

The number of lines covering all zeros is equal to order of matrix.

Step 4 : Hence, optimal solution has reached. Therefore, the optimal assignment is made as follows :

Wagons	Mileage of stations				
	I	II	III	IV	V
1	$\boxed{0}$	$\cancel{0}$	4	10	2
2	2	3	$\boxed{0}$	3	4
3	1	1	3	$\boxed{0}$	$\cancel{0}$
4	4	$\boxed{0}$	3	5	2
5	$\cancel{0}$	$\cancel{0}$	8	2	$\boxed{0}$

The optimal assignment is shown as follows :

Wagon	Station	Miles
1	I	10
2	III	6
3	IV	4
4	II	9
5	V	10

The minimum mileage covered = 10 + 6 + 4 + 9 + 10 = 39 miles.

3. Five different machines can do any of the five required jobs, with different profits resulting from each assignment as shown below :

Job	Machines Profit (in ₹)				
	A	B	C	D	E
1	30	37	40	28	40
2	40	24	27	21	36
3	40	32	33	30	35
4	25	38	40	36	36
5	29	62	41	34	39

Find the optimal assignment schedule.

Solution :

Step 1 : Since, it is a maximization problem, subtract each of the elements in the given matrix from the largest element, which is **62** here. The assignment matrix is obtained as follows :

Jobs	Profit (in ₹)				
	Machines				
	A	B	C	D	E
1	32	25	22	34	22
2	22	38	35	41	26
3	22	30	29	32	27
4	37	24	22	26	26
5	33	0	21	28	23

Step 2 : Subtract the smallest element in each row from the every element of that row. We get

Jobs	Profit (in ₹)				
	Machines				
	A	B	C	D	E
1	10	3	0	12	0
2	0	16	13	19	4
3	0	8	7	10	5
4	15	2	0	4	4
5	33	0	21	28	23

Step 3 : Subtract the smallest element in each column from the every element of that column. We get

Jobs	Profit (in ₹)				
	Machines				
	A	B	C	D	E
1	10	3	0	8	0
2	0	16	13	15	4
3	0	8	7	6	5
4	15	2	0	0	4
5	33	0	21	24	23

Step 4 : Cover zero elements with minimum number of straight lines. We get

Jobs	Profit (in ₹)				
	Machines				
	A	B	C	D	E
1	10	3	0	8	0
2	0	16	13	15	4
3	0	8	7	6	5
4	15	2	0	0	4
5	33	0	21	24	23

Since, number of straight lines covering all zeros is not equal to number of rows/columns, optimum solution has not reached.

Step 5 : Select the smallest element among the uncovered elements, which is **4** here. Subtract it from each element of the uncovered elements and add it to the elements at the intersection of two lines. We get

Jobs	Profit (in ₹)				
	Machines				
	A	B	C	D	E
1	14	3	0	8	0
2	0	12	9	11	0
3	0	4	3	2	1
4	19	2	0	0	4
5	37	0	21	24	23

Since, number of straight lines covering all zeros is equal to number of rows / columns. Optimum solution has reached.

Optimum assigment can be made as follows :

Jobs	Profit (in ₹)				
	Machines				
	A	B	C	D	E
1	14	3	$\boxed{0}$	8	$\cancel{0}$
2	$\cancel{0}$	12	9	11	$\boxed{0}$
3	$\boxed{0}$	4	3	2	1
4	19	2	$\cancel{0}$	$\boxed{0}$	4
5	37	$\boxed{0}$	21	24	23

Optimum solution is shown as follows :

Jobs	Machines	Profit (in ₹)
1	C	40
2	E	36
3	A	40
4	D	36
5	B	62

Hence, total (maximum) profit

$= 40 + 36 + 40 + 36 + 62$

$=$ ₹ 214.*

[Note : *Answer given in the textbook is incorrect.]

4. Four new machines M_1, M_2, M_3 and M_4 are to be installed in a machine shop. There are five vacant places A, B, C, D and E available. Because of limited space, machine M_2 cannot be placed at C and M_3 cannot be placed at A. The cost matrix is given below :

Machines	Places				
	A	B	C	D	E
M_1	4	6	10	5	6
M_2	7	4	–	5	4
M_3	–	6	9	6	2
M_4	9	3	7	2	3

Find the optimal assignment schedule.

Solution :

As the number of machines is less than the number of places, the problem is unbalanced. It is balanced by introducing a dummy machine M_5 with zero cost.

As M_2 cannot be placed at C and M_3 cannot be placed at A, a very high cost say ∞ is assigned to the corresponding element.

Machines	Places				
	A	B	C	D	E
M_1	4	6	10	5	6
M_2	7	4	∞	5	4
M_3	∞	6	9	6	2
M_4	9	3	7	2	3
M_5	0	0	0	0	0

Step 1 : Minimum element of each row is subtracted from every element of that row.

Machines	Places				
	A	B	C	D	E
M_1	0	2	6	1	2
M_2	3	0	∞	1	0
M_3	∞	4	7	4	0
M_4	7	1	5	0	1
M_5	0	0	0	0	0

Step 2 : Minimum element of each row is subtracted from every element of that row.

Machines	Places				
	A	B	C	D	E
M_1	0	2	6	1	2
M_2	3	0	∞	1	0
M_3	∞	4	7	4	0
M_4	7	1	5	0	1
M_5	0	0	0	0	0

Step 3 : Since, the number of lines covering zeros is 5 and is equal to order of matrix 5, the optimal solution has reached. Optimal assignment can be made as follows :

Machines	Places				
	A	B	C	D	E
M_1	$\boxed{0}$	2	6	1	2
M_2	3	$\boxed{0}$	∞	1	$\cancel{0}$
M_3	∞	4	7	4	$\boxed{0}$
M_4	7	1	5	$\boxed{0}$	1
M_5	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$

The following optimal solution is obtained :

Machines	Places	Cost (in ₹)
M_1	A	4
M_2	B	4
M_3	E	2
M_4	D	2
M_5	C	0

Total cost = ₹ 12.

5. A company has a team of four salesmen and there are four districts where the company wants to start its business. After taking into account the capabilities of salesmen and the nature of districts, the company estimates that the profit per day in rupees for each salesman in each district is as below :

Salesmen	Districts			
	1	2	3	4
A	16	10	12	11
B	12	13	15	15
C	15	15	11	14
D	13	14	14	15

Find the assignment of salesman to various districts which will yield maximum profit.

Solution :

Since, it is a maximization problem, subtract each of the elements in the matrix from the largest element of the matrix which is 16 here.

Salesmen	Districts			
	1	2	3	4
A	0	6	4	5
B	4	3	1	1
C	1	1	5	2
D	3	2	2	1

Step 1 : Subtract the minimum (smallest) element of each row from the elements of that row.

Salesmen	Districts			
	1	2	3	4
A	0	6	4	5
B	3	2	0	0
C	0	0	4	1
D	2	1	1	0

Step 2 : Subtract the smallest element of each column from the elements of that column.

Salesmen	Districts			
	1	2	3	4
A	0	6	4	5
B	3	2	0	0
C	0	0	4	1
D	2	1	1	0

Step 3 : Since, the number of lines covering zeros is 4 equal to the order of matrix 4. The optimal solution has reached.

Salesmen	Districts			
	1	2	3	4
A	$\boxed{0}$	6	4	5
B	3	2	$\boxed{0}$	$\cancel{0}$
C	$\cancel{0}$	$\boxed{0}$	4	1
D	2	1	1	$\boxed{0}$

The following optimal solution is obtained :

Salesmen	Districts	Profit (₹)
A	1	16
B	3	15
C	2	15
D	4	15

Total profit = ₹ 61.

6. In the modification of a plant layout of a factory four new machines M_1, M_2, M_3 and M_4 are to be installed in a machine shop. There are five vacant places A, B, C, D and E available. Because of limited space, machine M_2 cannot be placed at C and machine M_3 cannot be placed at A. The cost of locating a machine at a place (in hundred rupees) is as follows :

Machines	Locations				
	A	B	C	D	E
M_1	9	11	15	10	11
M_2	12	9	–	10	9
M_3	–	11	14	11	7
M_4	14	8	12	7	8

Find the optimal assignment schedule.

Solution :

As the number of machines is less than the number of vacant places, the problem is unbalanced. It is balanced by introduction of dummy machine M_5 with zero cost.

Also machine M_2 cannot be placed at C and machine M_3 cannot be placed at A, a very high cost say ∞ is assigned to the corresponding elements. We get

Machines	Locations				
	A	B	C	D	E
M_1	9	11	15	10	11
M_2	12	9	∞	10	9
M_3	∞	11	14	11	7
M_4	14	8	12	7	8
M_5	0	0	0	0	0

Subtract the smallest element of each row from every element in that row. We get

Machines	Locations				
	A	B	C	D	E
M_1	0	2	6	1	2
M_2	3	0	∞	1	0
M_3	∞	4	7	4	0
M_4	7	1	5	0	1
M_5	0	0	0	0	0

Since, the smallest element in each column is zero, the resultant matrix is as given in the above table.

Since, number of straight lines covering all zeros is equal to number of rows/columns, the optimal solution has reached. The optimal assignment can be made as follows :

Machines	Locations				
	A	B	C	D	E
M_1	$\boxed{0}$	2	6	1	2
M_2	3	$\boxed{0}$	∞	1	$\cancel{0}$
M_3	∞	4	7	4	$\boxed{0}$
M_4	7	1	5	$\boxed{0}$	1
M_5	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$

The following is the optimum solution obtained :

Machines	Locations	Cost (₹)
M_1	A	9
M_2	B	9
M_3	E	7
M_4	D	7

Total cost = ₹ 32.

EXAMPLES FOR PRACTICE 7.1

1. Four freelancers P, Q, R, S can do four types of jobs. The corresponding cost matrix is given below. If each person is to be assigned exactly one job, solve the problem for minimizing the cost.

Jobs	Persons (Freelancers)			
	P	Q	R	S
J_1	0	18	9	3
J_2	10	25	1	23
J_3	24	5	4	1
J_4	9	16	14	0

2. A team of 4 horses and 4 riders has entered the jumping show contest. The number of penalty points to be expected when each rider rides each horse is shown below. How should the horses be assigned to the riders so as to minimize the expected loss ? Also, find the minimum expected loss.

Riders	Horses			
	H_1	H_2	H_3	H_4
R_1	12	3	3	2
R_2	1	11	4	13
R_3	11	10	6	11
R_4	5	8	1	7

3. The cost (in hundreds of ₹) of sending material to 'five' terminals by 'four' trucks, incurred by a company is as given below. Find the assignment of trucks to terminals which will minimize the cost. ['One' truck is assigned to only 'one' terminal.] Which terminal will 'not' receive material from the truck company ? What is the minimum cost ?

Terminals	Trucks			
	A	B	C	D
T_1	3	6	2	6
T_2	7	1	4	4
T_3	3	8	5	8
T_4	5	2	4	3
T_5	5	7	6	2

4. A company has 4 machines on which to do 3 jobs. Each job can be assigned to one and only one machine. The cost of each machine is given in the following table :

Jobs	Cost (in ₹)			
	Machines			
	A	B	C	D
I	18	24	28	32
II	8	13	17	19
III	10	15	19	22

What are the job assignments which will minimize the cost?

5. A company has a group of 4 salesmen. There are 4 districts where the company is interested in starting its business. Following table gives profit per day in rupees for different districts due to different salesmen.

Salesmen	Districts			
	1	2	3	4
A	600	100	400	100
B	400	100	500	500
C	500	500	300	200
D	300	200	400	500

Make an assignment which will yield maximum profit.

6. The owner of a small machine shop has 'four' machinists available to assign jobs for the day. 'Five' jobs are offered to be done on the day. The expected profits for each job done by each machinist are given below. Find the assignment of jobs to the machinists that will result in maximum profit. Also find the maximum profit.

[One machinist can be assigned only 'one' job.]

Machinists	Jobs				
	A	B	C	D	E
M_1	62	78	50	101	82
M_2	71	84	61	73	59
M_3	87	92	111	71	81
M_4	48	61	87	77	80

7. A pharmaceutical company has four branches, one each at city A, B, C, D. A branch manager is to be appointed one at each city, out of four candidates P, Q, R and S. The monthly business depending upon the city and effectiveness of the branch manager in that city is given below :

Candidates	Monthly business (in lakh ₹)			
	City			
	A	B	C	D
P	10	10	8	8
Q	12	15	10	9
R	11	16	12	7
S	15	13	15	11

Which manager should be appointed at which city so as to get maximum total monthly business?

8. A company has five salesmen and five sales areas. The average monthly sale (in '000 ₹) by different salesmen in each of the areas are given below. Solve this assignment problem so as to maximize total sales.

Areas	Average monthly sale (in '000 ₹)				
	Salesmen				
	P	Q	R	S	T
A	38	43	45	35	45
B	44	29	35	26	41
C	45	33	39	36	42
D	48	43	46	41	41
E	33	38	45	40	44

9. The time required by each worker to complete each job is given below, where '–' means the particular job cannot be assigned to the particular worker. Find the assignment of jobs to workers which will minimize the total time required to complete all the jobs.

Jobs	Workers		
	A	B	C
I	12	10	8
II	8	9	11
III	11	–	12

10. A Chartered Accountants' firm has accepted 'five' new cases. The estimated number of days required by each of their 'five' employees for each case are given below, where '–' means that the particular employee cannot be assigned the particular case. Determine the optimal assignment of cases to the employees so that the total number of days required to complete these 'five' cases will be minimum. Also, find the minimum number of days.

Employees	Cases				
	I	II	III	IV	V
E_1	5	2	4	2	6
E_2	3	4	–	5	7
E_3	6	3	4	1	2
E_4	4	2	2	3	5
E_5	3	6	4	7	3

Answers

1. $J_1 \to P$, $J_2 \to R$, $J_3 \to Q$, $J_4 \to S$. Minimum cost = 6 units

2. $H_1 \to R_2$, $H_2 \to R_3$, $H_3 \to R_4$, $H_4 \to R_1$
Minimum expected loss = 1 + 10 + 1 + 2 = 14.

3. $A \to T_3$, $B \to T_2$, $C \to T_1$, $D \to T_5$, $E \to T_4$
Minimum cost = 3 + 1 + 2 + 2 = 8 (in '00 ₹) = ₹ 800.

4. I → A, II → B, III → C **OR** I → A, II → C, III → B
Minimum cost = 18 + 13 + 19 = ₹ 50 or
18 + 17 + 15 = ₹ 50.

5. A → 1, B → 3, C → 2, D → 4
Maximum profit = 600 + 500 + 500 + 500 = ₹ 2100.

6. $B \to M_2$, $C \to M_3$, $D \to M_1$, $E \to M_4$
Maximum profit = 84 + 111 + 101 + 80 = ₹ 376.

7. P → D, Q → A, R → B, S → C ₹ 51 lakh

8. A → Q, B → P, C → T, D → R, E → S
Total maximum sales = 43 + 44 + 42 + 46 + 40 = 215.
[*Alternative solutions exist :*
A → Q, B → P, C → T, D → S, E → R
A → Q, B → T, C → P, D → R, E → S
A → Q, B → T, C → P, D → S, E → R]

9. I → C, II → B, III → A. Minimum time = 28 units.

10. I → E_2, II → E_1, III → E_4, IV → E_3, V → E_5
Minimum number of days required
= 3 + 2 + 2 + 1 + 3 = 11 days.

7.2 : SEQUENCING PROBLEM

Introduction to Sequencing Problem : Sequencing is a process of selecting a path which each part of the product will follow before being transformed from raw materials into the finished product. For example, a publisher wishes to publish ten different books. He wants to bring out all the books in the market in a short period of time. Each book (job) has to be processed through the following in the order (a) Composing (b) Printing (c) Binding and (d) Finishing. The approximate time taken by each of the processes (composing, printing, etc.) for each of the ten books is known. The problem of the publisher is to decide which book is to be processed first, which is to be processed next and so on. He has to determine a sequence (order) of giving the manuscripts to the press in such a way that the total elapsed time (i.e. the time from the start of the first book till the completion of the last book) and the total idle time of the machines is minimized. This is an example of 'ten jobs and four machines sequencing problem.'

Thus, sequencing problem is to determine the sequence (order) for a series of jobs to be done on a finite number of service facilities, in some pre-assigned order, so as to optimize the total time involved.

7.2.1 : General Sequencing Problem

Let in an industry there be n jobs each of which is to be performed one at a time at each of the m different machines. We are given the order in which these machines are to be used for processing each job and also the actual or expected processing time taken by each job on the machines, then the General Sequencing Problem is to determine the sequence out of $(n!)^m$ possible combinations, that minimizes the total elapsed time. (i.e. the time from the start of the first job up to the completion of the last job).

1. Conditions :

(i) No machine can process more than one job at a time.

(ii) Each job, once started on a machine has to be completed.

(iii) The time involved in moving jobs from one machine to another is negligible.

(iv) Processing times M_{ij}'s ($i = 1, 2, \ldots, n; j = 1, 2, \ldots, m$) are independent of order of processing the jobs.

(v) All machines are of different types.

(vi) All jobs are completely known and are ready for processing before the period under consideration begins.

(vii) A job is processed as soon as possible, but only in the order specified.

2. Terminology :

(1) Total Elapsed Time : It is the time required to complete all the jobs i.e. the entire task.

Thus, total elapsed time is the time between the beginning of the first job on first machine till the completion of the last job on the last machine.

(2) Idle Time : Idle time is the time when a machine is available but not being used. Thus, it is the time that the machine is available but is waiting for a job to be processed.

3. Notations : Following notations are used in the sequencing problem.

M_{ij} = Processing time required by the i^{th} job on the j^{th} machine. ($i = 1, 2, \ldots, n; j = 1, 2, \ldots, m$)

T = Total elapsed time (including idle time, if any) for processing all the jobs.

X_{ij} = Idle time on machine j from end of $(i-1)^{th}$ job to the start of i^{th} job. ($i = 1, 2, \ldots, n; j = 1, 2, \ldots, m$)

7.2.2 : Types of sequencing problem

1. Sequencing 'n' jobs on Two Machines : The n job–2 machines sequencing problem is described as follows :

(i) Only two machines M_1 and M_2 are involved.

(ii) Each job is processed in the order M_1M_2.

(iii) The actual or expected processing times $M_{11}, M_{12}, \ldots, M_{1n}$ and $M_{21}, M_{22}, \ldots, M_{2n}$ are known.

The problem is to determine the sequence (order) of jobs that minimizes T. The total elapsed time from the start of the first job to the completion of the last job.

Total elapsed time T is given by,

$$T = \sum_{i=1}^{n} M_{i2} + \sum_{i=1}^{n} X_{i2}$$

where $\sum_{i=1}^{n} M_{i2}$ = Total time for which machine M_2 has to work.

$\sum_{i=1}^{n} X_{i2}$ = Total idle time for which machine M_2 remains idle.

(Some X_{i2}'s may be zero)

Optimal Sequence Algorithm :

Step 1 : Examine M_{i1}'s and M_{i2}'s for $i = 1, 2, \ldots, n$ and find Min. $\{M_{i1}, M_{i2}\}$

Step 2 :

(i) If this minimum is M_{k1} for some $i = k$, process (or do) the k^{th} job first of all.

(ii) If this minimum is M_{r2} for some $i = r$, process (or do) the r^{th} job last of all.

Step 3 :

(i) If there is a tie, i.e. if $M_{k1} = M_{r2}$, process k^{th} job first of all and r^{th} job in the last.

(ii) If there is a tie, i.e. $M_{k1} = M_{kr}$, process k^{th} job either next to the first ones or next to the last ones in the sequence.

(iii) If the tie for the minimum occurs among the M_{i1}'s, select any job and process it first of all.

(iv) If the tie for the minimum occurs among the M_{i2}'s, select any job and process it in the last. Go to the next step.

Step 4 : Cross off the jobs already assigned and repeat steps 1 to 3, placing the jobs next to first or next to last, until all the jobs have been assigned.

2. Sequencing 'n' jobs on Three Machines : The n job-3 machines sequencing problem is described as follows :

(i) Only three machines are involved, say M_1, M_2 and M_3.

(ii) Each job is processed in the prescribed order $M_1M_2M_3$.

(iii) No passing of jobs is permitted (i.e. the same order over each machine is maintained).

(iv) The actual or expected processing times on three machines are known.

The problem is to find the optimum sequence of jobs which minimize the total elapsed time.

Algorithm to obtain an optimal sequence of jobs :

Step 1 : Find Min. (M_1), Min. (M_3) and Max. (M_2)

Step 2 : See whether

(i) Min. $(M_1) \geqslant$ Max. (M_2)

(ii) Min. $(M_3) \geqslant$ Max. (M_2)

Step 3 : If no equality of step 2 is satisfied, the method fails. Otherwise move to step 4.

Step 4 : Convert the three machines problem into two machine problem by introducing two fictitious machines, say G and H, such that $G = M_1 + M_2$; $H = M_2 + M_3$.

Step 5 : Determine the optimal sequence of *n* jobs for two machines G and H, using the iterative procedure for *n* job and 2 machine problem.

EXERCISE 7.2 | Textbook pages 125 and 126

1. A machine operator has to perform two operations, turning and threading on 6 different jobs. The time required to perform these operations (in minutes) for each job is known. Determine the order in which the jobs should be processed in order to minimize the total time required to complete all the jobs. Also find the total processing time and idle times for turning and threading operations.

Jobs	1	2	3	4	5	6
Time for turning	3	12	5	2	9	11
Time for threading	8	10	9	6	3	1

Solution :

Jobs	Time required (in minutes)	
	For turning	For threading
1	3	8
2	12	10
3	5	9
4	2	6
5	9	3
6	11	1

Step 1 : Here, Min. $(M_{i1}, M_{i2}) = 1$, which corresponds to threading. Therefore, job 6 is operated at last.

					6

The problem now reduces to five jobs 1, 2, 3, 4, 5.

Here, Min. $(M_{i1}, M_{i2}) = 2$, which corresponds to turning. Therefore, job 4 is operated first of all for turning.

4					6

The problem now reduces to four jobs 1, 2, 3, 5.

Here, Min. $(M_{i1}, M_{i2}) = 3$, which corresponds to both turning and threading.

Therefore, job 1 is operted first next to job 4 and job 5 is operated at last next to job 6.

4	1			5	6

The problem now reduces to two jobs 2 and 3.

Here, Min. $(M_{i1}, M_{i2}) = 5$, which corresponds to turning. Therefore, job 3 is operated next to job 1.

4	1	3		5	6

Now, remaining job 2 is operated next to job 3. Thus, the optimal sequence of jobs is obtained as follows :

4	1	3	2	5	6

The minimum elapsed time can be computed as follows :

Jobs Sequence	For turning		For threading		Idle time for threading
	Time in	Time out	Time in	Time out	
4	0	2	2	8	2
1	2	5	8	16	0
3	5	10	16	25	0
2	10	22	25	35	0
5	22	31	35	38	4
6	31	42	42	43	0
Total idle time for threading					06

From the above table,

The minimum (optimum) total elapsed time $T = 43$ minutes.

Idle time for turning

$$= T - \begin{Bmatrix} \text{Sum of the processing time} \\ \text{of all six jobs on turning} \end{Bmatrix}$$

$$= 43 - 42 = 1 \text{ minute}$$

Idle time for threading = 6 minutes.

2. A company has three jobs on hand. Each of these must be processed through two departments, in the order AB where

Department A : Press shop and

Department B : Finishing

The table below gives the number of days required by each job in each department :

Jobs	I	II	III
Department A	8	6	5
Department B	8	3	4

Find the sequence in which the three jobs should be processed so as to take minimum time to finish all the three jobs. Also find idle time for both the departments.

Solution :

Jobs	Department	
	A	B
I	8	8
II	6	3
III	5	4

Step I :

Min. $(M_{i1}, M_{i2}) = 3$, which corresponds to department B.

Therefore, job II is processed in the last.

		II

The problem now reduces to two jobs I and III.

Here, Min. $(M_{i1}, M_{i2}) = 4$, which corresponds to the department B.

Therefore, job III is processed in the last next to job III and then job I is processed.

Thus, the optimal sequence of jobs is obtained as follows :

I	III	II

Minimum elapsed time can be computed as follows :

Job Sequence	Department A		Department B		Idle time for B
	Time in	Time out	Time in	Time out	
I	0	8	8	16	8
III	8	13	16	20	0
II	13	19	20	23	0
			Total idle time for B		8

From the above table,

Minimum total elapsed time to finish all three jobs $T = 23$ days

Idle time for the department A

$$= T - \left\{ \begin{matrix} \text{Sum of the processing time} \\ \text{to finish all jobs in A} \end{matrix} \right\}$$

$= 23 - 19 = 4$ days.

Idle time for the department B $= 8$ days.

3. An insurance company receives three types of policy application bundles daily from its head office for data entry and tiling. The time (in minutes) required for each type for these two operations is given in the following table :

Policy	1	2	3
Data Entry	90	120	180
Filing	140	110	100

Find the sequence that minimizes the total time required to complete the entrie task. Also find the total elapsed time and idle times for each operation.

Solution :

Policy	Time required (in minutes)	
	Data Entry	Filing
1	90	140
2	120	110
3	180	100

Min. $(M_{i1}, M_{i2}) = 90$, which corresponds to job data entry. Therefore, Policy 1 is processed first in sequence.

1		

The problem now reduced to two positions 2 and 3. Here, Min. $(M_{i1}, M_{i2}) = 100$, which corresponds to job filing. Therefore, Policy 2 is placed last.

1		3

Now, Min. $(M_{i1}, M_{i2}) = 110$, which corresponds to job filing. Therefore, Policy 2 is placed next to policy 1.

1	2	3

Hence, the optimal sequence is $1 \to 2 \to 3$

Total elapsed time is obtained as follows :

Policy Sequence	Data Entry		Filing		Idle time for Filing
	Time in	Time out	Time in	Time out	
1	0	90	90	230	90
2	90	210	230	340	0
3	210	390	390	490	50
			Total idle time for Filing		140

Total elapsed time $T = 490$ minutes

Idle time for data entry

$= T - \text{Total time of data entry}$

$= 490 - 390 = 100$ minutes

Idle time for filing $= 90 + 50$

$= 140$ minutes.

[**Note :** *Answer given in the textbook is incorrect.*]

4. There are five jobs, each of which must go through two machines in the order XY. Processing times (in hours) are given below. Determine the sequence for the jobs that will minimize the total elapsed time. Also find the total elapsed time and idle time for each machine.

Jobs	A	B	C	D	E
Machine X	10	2	18	6	20
Machine Y	4	12	14	16	8

Solution :

Jobs	**Processing time (in hours)**	
	Machine X	**Machine Y**
A	10	4
B	2	12
C	18	14
D	6	16
E	20	8

Min. $(M_{i1}, M_{i2}) = 2$, which corresponds to Machine X. Therefore, job B is processed first.

B				

The problem now reduces to jobs A, C, D, E.

Here, Min. $(M_{i1}, M_{i2}) = 4$, which corresponds to Machine Y. Therefore, job A is processed last.

B				A

The problem now reduces to jobs C, D, E.

Here, Min. $(M_{i1}, M_{i2}) = 6$, which corresponds to Machine X. Therefore, job D is processed next to job B.

B	D			A

The problem now reduces to jobs C and E.

Here, Min. $(M_{i1}, M_{i2}) = 8$, which corresponds to Machine Y. Therefore, job E is processed last next to job A and job C is processed next to job D.

B	D	C	E	A

Total elapsed time is obtained as follows :

Job Sequence	**Machine X**		**Machine Y**		**Idle time for Machine Y**
	Time in	**Time out**	**Time in**	**Time out**	
B	0	2	2	14	2
D	2	8	14	30	0
C	8	26	30	44	0
E	26	46	46	54	2
A	46	56	56	60	2
Total idle time for Machine Y					6

Total elapsed time $T = 60$ hours

Idle time for machine $Y = 6$ hours

Idle time for machine X

$= T - \text{Total processing time of X}$

$= 60 - 56$

$= 4$ hours.

5. Find the sequence that minimizes the total elapsed time to complete the following jobs in the order AB. Find the total elapsed time and idle times for both the machines.

Jobs	I	II	III	IV	V	VI	VII
Machine A	7	16	19	10	14	15	5
Machine B	12	14	14	10	16	5	7

Solution :

Jobs	**Time**	
	Machine A	**Machine B**
I	7	12
II	16	14
III	19	14
IV	10	10
V	14	16
VI	15	5
VII	5	7

Here, Min. $(M_{i1}, M_{i2}) = 5$, which corresponds to both machines A and B.

Therefore, job VII is processed first and job VI is processed last.

VII						VI

The problem now reduces to jobs I, II, III, IV and V.

Here, Min. $(M_{i1}, M_{i2}) = 7$, which corresponds to machine A. Therefore, job I is processed next to job VII.

VII	I					VI

The problem now reduces to jobs II, III, IV and V.

Here, Min. $(M_{i1}, M_{i2}) = 10$, which corresponds to both machines A and B.

Therefore, job IV is processed next to job I.

VII	I	IV				VI

The problem now reduces to jobs II, III and V.

Here, Min. $(M_{i1}, M_{i2}) = 14$, which corresponds to both machines A and B.

Therefore, job V is processed next to job IV.

VII	I	IV	V			VI

The problem now reduces to jobs II and III.

Here, Min. $(M_{i1}, M_{i2}) = 14$, which corresponds to machine B.

Therefore, job II is processed in the last next to job VI and job III is processed last next to job II.

VII	I	IV	V	III	II	VI

Total elapsed time is obtained as follows :

Job Sequence	Machine A		Machine B		Idle time for Machine B
	Time in	Time out	Time in	Time out	
VII	0	5	5	12	5
I	5	12	12	24	0
IV	12	22	24	34	0
V	22	36	36	52	2
III	36	55	55	69	3
II	55	71	71	85	2
VI	71	86	86	91	1
				Total idle time for Machine B	13

Optimal sequence of jobs is

$VII \rightarrow I \rightarrow IV \rightarrow V \rightarrow III \rightarrow II \rightarrow VI$

Total elapsed time $T = 91$ units

Idle time for Machine B = 13 units

Idle time for machine A

$= T -$ Processing time of A

$= 91 - 86$

$= 5$ units

---

6. Find the optimal sequence that minimizes total time required to complete the following jobs in the order ABC. The processing times are given in hours.

(i)

Jobs	I	II	III	IV	V	VI	VII
Machine A	6	7	5	11	6	7	12
Machine B	4	3	2	5	1	5	3
Machine C	3	8	7	4	9	8	7

Solution :

Here, Min. (A) = 5, Min. (C) = 3 and Max. (B) = 5.

Since, Min. (A) ⩾ Max. (B) is satisfied, the problem can be converted into 7 jobs and 2 machine problem.

Now, the two fictitious machines are such that

$G = A + B$ and $H = B + C$

Then the problem can be written as

Jobs	Machines	
	G = A + B	H = B + C
~~I~~	~~10~~	~~7~~
II	10	11
~~III~~	~~7~~	~~9~~
~~IV~~	~~16~~	~~9~~
~~V~~	~~7~~	~~10~~
VI	12	13
VII	15	10

Here, Min. $(G_i, H_i) = 7$, which corresponds to both machines G and H.

Therefore, job III is processed first and job V is process second and job I is processed at the last.

III	V					I

OR

V	III					I

The problem now reduces to jobs II, IV, VI and VII.

Here, Min. $(G_i, H_i) = 9$, which corresponds to machine H.

Therefore, job IV is processed in the last next to job I.

III	V				IV	I

OR

V	III				IV	I

The problem now reduces to jobs II, VI and VII.

Here, Min. $(G_i, H_i) = 10$, which corresponds to both machines G and H.

Therefore, job II is processed next to job V and job VII is processed in the last next to job IV.

III	V	II		VII	IV	I

OR

V	III	II		VII	IV	I

The problem now reduces to only one job VI. It is processed next to job II.

∴ the following optimal sequence is obtained :

III	V	II	VI	VII	IV	I

OR

V	III	II	VI	VII	IV	I

Total elapsed time is obtained as follows :

Jobs Sequence	**Machine A**		**Machine B**		**Machine C**		**Idle time for Machine C**
	Time in	**Time out**	**Time in**	**Time out**	**Time in**	**Time out**	
III	0	5	5	7	7	14	7
V	5	11	11	12	14	23	0
II	11	18	18	21	23	31	0
VI	18	25	25	30	31	39	0
VII	25	37	37	40	40	47	1
IV	37	48	48	53	53	57	6
I	48	54	54	58	58	61	1
Total idle time for Machine C							15

Total elapsed time T = 61 hours

Idle time for Machine A

= T − Total processing time of Machine A

= 61 − 54 = 7 hours

Idle time for Machine B

= T − Total processing time of Machine B

= 61 − 23 = 38 hours

Idle time for Machine C = 15 hours.

(ii)

Jobs	1	2	3	4	5
Machine A	5	7	6	9	5
Machine B	2	1	4	5	3
Machine C	3	7	5	6	7

Solution :

Here, Min. (A) = 5, Min. (C) = 3, Max. (B) = 5.

Since, Min. (A) ⩾ Max. (B) is satisfied, the problem can be converted into 5 jobs and 2 machine problem.

Now, the two fictitious machines are such that

G = A + B and H = B + C

The problem can be written as the following 5 jobs and 2 machine problem.

Jobs	**Machine**	
	G = A + B	**H = B + C**
1	7	5
2	8	8
3	10	9
4	14	11
5	8	10

Min. $(G_{i1}, H_{i2}) = 5$, which corresponds to H.

Therefore, job 1 is processed last.

				1

The problem now reduces to four jobs 2, 3, 4, 5. Here, Min. $(G_{i1}, H_{i2}) = 8$, which corresponds to G and H both.

Therefore, job 2 is processed first of all and then job 5 is processed.

2	5			1

or 5, 2, …, …, 1

The problem now reduces to two jobs 3 and 4. Here, Min. $(G_{i1}, H_{i2}) = 9$, which corresponds to H. Therefore, job 3 is processed in the last next to job 1.

2	5		3	1

or 5, 2, …, 3, 1

Now, the remaining job 4 must be processed next to job 5. Thus, the optimal sequence of jobs is obtained as follows :

2	5	4	3	1

or 5, 2, 4, 3, 1

The minimum elapsed time can be computed as follows :

Jobs Sequence	**Machine A**		**Machine B**		**Machine C**		**Idle time for Machine C**
	Time in	**Time out**	**Time in**	**Time out**	**Time in**	**Time out**	
2	0	7	7	8	8	15	8
5	7	12	12	15	15	22	0
4	12	21	21	26	26	32	4
3	21	27	27	31	32	37	0
1	27	32	32	34	37	40	0
Total idle time for Machine C							12

From the above table :

The minimum (optimum) total elapsed time $T=40$ hours.

Idle time for machine A

$$=T-\left\{\begin{matrix}\text{Sum of processing time}\\ \text{of five jobs on A}\end{matrix}\right\}$$

$=40-32=8$ hours

Idle time for machine B

$$=T-\left\{\begin{matrix}\text{Sum of processing time}\\ \text{of five jobs on B}\end{matrix}\right\}$$

$=40-\{2+1+4+5+3\}$

$=40-15=25$ hours

Idle time for machine $C=12$ hours.

---

7. A publisher produces 5 books on Mathematics. The books have to go through composing, printing and binding done by 3 machines P, Q, R. The time schedule for the entire task in proper unit is as follows :

Books	A	B	C	D	E
Machine P	4	9	8	6	5
Machine Q	5	6	2	3	4
Machine R	8	10	6	7	11

Determine the optimum time required to finish the entire task.

Solution :

Here, Min. $(P)=4$, Min. $(R)=6$ and Max. $(Q)=6$.

Since, Min. $(R)\geqslant$ Max. (Q) is satisfied, the problem can be converted into 5 jobs and 2 machine problem and two fictious machines are $G=P+Q$ and $H=Q+R$

The problem can be written as follows :

Books	Machines	
	$G=P+Q$	$H=Q+R$
A	9	13
B	15	16
C	10	8
D	9	10
E	9	15

Min. $(G_{i1}, H_{i2})=8$, which corresponds to H.

Therefore, Book C is processed at the last.

				C

The problem now reduces to four jobs A, B, D and E.

Here, Min. $(G_{i1}, H_{i2})=9$, which corresponds to G.

Therefore, either of the books A or D or E is processed first of all and the remaining next to book A.

A	D	E		C

OR

$A\rightarrow E\rightarrow D$, $D\rightarrow A\rightarrow E$, $E\rightarrow A\rightarrow D$, $D\rightarrow E\rightarrow A$, $E\rightarrow D\rightarrow A$.

Now, the remaining book B is processed next to book E.

Thus, the optimal sequence of jobs is obtained as follows :

A	D	E	B	C

OR

$A\rightarrow E\rightarrow D\rightarrow B\rightarrow C$ ***OR*** $D\rightarrow A\rightarrow E\Rightarrow B\rightarrow C$ ***OR*** $E\rightarrow A\rightarrow D\rightarrow B\rightarrow C$ ***OR*** $D\rightarrow E\rightarrow A\rightarrow B\rightarrow C$ ***OR*** $E\rightarrow D\rightarrow A\rightarrow B\rightarrow C$.

Considering the first sequence of jobs, the minimum elapsed time can be computed as follows :

Books Sequence	Machine P		Machine Q		Machine R		Idle time for Machine R
	Time in	Time out	Time in	Time out	Time in	Time out	
A	0	4	4	9	9	17	9
D	4	10	10	13	17	24	0
E	10	15	15	19	24	35	0
B	15	24	24	30	35	45	0
C	24	32	32	34	45	51	0
Total idle time for Machine R							9

From the above table :

The minimum (optimum) total elapsed time $T=51$ hours.

Idle time for machine P

$$=T-\left\{\begin{matrix}\text{Sum of processing time}\\ \text{of five jobs on P}\end{matrix}\right\}$$

$=51-32=19$ hours

Idle time for machine Q

$$=T-\left\{\begin{matrix}\text{Sum of processing time}\\ \text{of five jobs on Q}\end{matrix}\right\}$$

$=51-\{5+6+2+3+4\}$

$=51-20=31$ hours

Idle time for machine R

$$=T-\left\{\begin{matrix}\text{Sum of processing time}\\ \text{of five jobs on R}\end{matrix}\right\}$$

$=51-\{8+10+6+7+11\}=51-42$

$=9$ hours.

EXAMPLES FOR PRACTICE 7.2

1. From the following time schedule for a certain task, find the possible sequences which minimize the total time required to complete all the seven jobs on the two machines M_1 and M_2 in the order M_1 M_2. Find the total elapsed time and the idle times for the machines M_1 and M_2 for any one of the sequences.

Machines	Jobs						
	A	B	C	D	E	F	G
M_1	3	4	2	6	2	5	3
M_2	4	1	5	3	5	4	2

2. Suppose that there are 5 jobs each of which must go through the two machines A and B in the order AB. Processing times in hours are given in the following table :

Machines	Jobs				
	1	2	3	4	5
A	6	2	10	4	11
B	3	7	8	9	5

Determine the sequence for the five jobs that will minimize total elapsed time. Also find idle time for machines A and B.

3. A readymade garments manufacturer has to process 7 items through two stage of production, namely cutting and sewing. The time taken in hours for each of these items in different stages are given below :

Items	1	2	3	4	5	6	7
Time for Cutting	5	7	3	4	6	7	12
Time for Sewing	2	6	7	5	9	5	8

Find the sequence in which these items are to be processed through these stages so as to minimize the total processing time.

4. Three jobs X, Y and Z have to be processed on the machines M_1 and M_2 in the order M_1, M_2. Determine the sequence that will minimize the processing time and find the total elapsed time using the following table which shows the time taken by each job on each machine :

Machines	Jobs		
	X	Y	Z
M_1	6	2	4
M_2	3	7	9

5. Six jobs go first over the machine M and then over the machine N, one at a time. The time schedule for the task is given below. Determine the sequence of the jobs, which will minimize the processing time. Also find the total elapsed time and idle times for both the machines :

Machines	Jobs					
	A	B	C	D	E	F
M	5	9	4	7	8	6
N	7	4	8	3	9	5

6. Five jobs are to be processed on three machines X, Y, Z in the order YXZ. The time schedule for the entire task is as follows :

Machines	Jobs				
	P	Q	R	S	T
X	5	6	2	3	4
Y	4	9	8	6	5
Z	8	10	6	7	11

Determine the optimal sequence, state the total elapsed time and obtain the idle time for the machine Y.

7. Five jobs have to go through three machines A, B and C in the same order of the machines. The time schedule is given as follows :

Machines	Jobs				
	1	2	3	4	5
A	8	10	6	7	11
B	5	6	2	3	4
C	4	9	8	6	5

Find the sequence of jobs that minimizes the total elapsed time.

8. Five jobs have to go through the machines A, B, C in the order ABC. Following tables shows the processing times in hours for the five jobs :

Machines	Jobs				
	J_1	J_2	J_3	J_4	J_5
A	5	7	6	9	5
B	2	2	4	5	3
C	3	6	5	6	7

Determine the sequence of jobs, which will minimise the total elapsed time.

9. Find the optimal sequence that minimizes total time required to complete five jobs, which have to be processed on each of the three machines M_1, M_2, M_3 in the order M_1, M_2, M_3 if the processing time for each job on each machine is given below. Also, find the total elapsed time and idle time for M_2 :

Machines	Jobs				
	P	Q	R	S	T
M_1	6	7	5	11	5
M_2	4	3	2	5	1
M_3	3	8	7	4	9

10. Five different books are processed through three operations–printing, binding and finishing. The time required to perform these operations for each book is known. Determine the order in which the books should be processed in order to minimize the total time required to process all books :

Books	1	2	3	4	5
Printing time	40	90	80	60	50
Binding time	50	60	20	30	40
Finishing time	80	100	60	70	110

Find also the minimum total elapsed time and idle time for each operation.

Answers

1. Two possible optimal sequences are :
C→E→A→F→D→G→B ... (1) **OR**
E→C→A→F→D→G→B ... (2)
For both the sequences, Total elapsed time = 26,
Idle time for $M_1 = 1$, Idle time for $M_2 = 2$.

2. 2→4→3→5→1. Total elapsed time = 36 hours, Idle time for machine A = 3 hours, Idle time for machine B = 4 hours.

3. 3→4→5→7→2→6→1.

4. Y→Z→X. Total elapsed time = 21.

5. C→A→E→F→B→D. Total elapsed time = 42, Idle time for machine M = 3, Idle time for machine N = 6.

6. P→S→T→Q→R
Total elapsed time = 51
Idle time for Y = 51 − 32 = 19
[The other possible sequences are :
P→T→S→Q→R, T→P→S→Q→R,
T→S→P→Q→R, S→P→T→Q→R,
S→T→P→Q→R
The total elapsed time and idle time for Y will remain the same for any sequence.]

7. 3→2→5→1→4

8. $J_5 \to J_4 \to J_3 \to J_2 \to J_1$

9. T→R→Q→S→P
Total elapsed time = 41
Idle time for $M_2 = 23$

10. 4→1→5→2→3
Total elapsed time = 510 minutes
Idle time for printing = 190 minutes
Idle time for binding = 310 minutes
Idle time for finishing = 90 minutes.

MISCELLANEOUS EXERCISE – 7

(Textbook pages 126 to 130)

I. Choose the correct alternative :

1. In sequencing, an optimal path is one that minimizes ....
(a) Elapsed time (b) Idle time
(c) Both (a) and (b) (d) Ready time

2. If job A to D have processing times as 5, 6, 8, 4 on first machine and 4, 7, 9, 10 on second machine then the optimal sequence is :
(a) CDAB (b) DBCA
(c) BCDA (d) ABCD

3. The objective of sequencing problem is
(a) to find the order in which jobs are to be made
(b) to find the time required for completing all the jobs on hand
(c) to find the sequence in which jobs on hand are to be processed to minimize the total time required for processing the jobs
(d) to maximize the cost

4. If there are n jobs and m machines, then there will be .... sequences of doing the jobs.
   (a) mn (b) $m(n!)$
   (c) n^m (d) $(n!)^m$

5. The Assignment Problem is solved by
   (a) Simplex method
   (b) Hungarian method
   (c) Vector method
   (d) Graphical method

6. In solving 2 machines and n jobs sequencing problem, the following assumption is wrong
   (a) No passing is allowed
   (b) Processing times are known
   (c) Handling time is negligible
   (d) The time of passing depends on the order of machining

7. To use the Hungarian method, a profit maximization assignment problem requires
   (a) Converting all profits to opportunity losses
   (b) A dummy person or job
   (c) Matrix expansion
   (d) Finding the maximum number of lines to cover all the zeros in the reduced matrix

8. Using Hungarian method the optimal assignment obtained for the following assignment problem to minimize the total cost is :

Agents	Jobs			
	A	B	C	D
1	10	12	15	25
2	14	11	19	32
3	18	21	23	29
4	15	20	26	28

   (a) 1–C, 2–B, 3–D, 4–A
   (b) 1–B, 2–C, 3–A, 4–D
   (c) 1–A, 2–B, 3–C, 4–D
   (d) 1–D, 2–A, 3–B, 4–C

9. The assignment problem is said to be unbalanced if
   (a) Number of rows is greater than number of columns
   (b) Number of rows is lesser than number of columns
   (c) Number of rows is equal to number of columns
   (d) Both (a) and (b)

10. The assignment problem is said to be balanced if
   (a) Number of rows is greater than number of columns
   (b) Number of rows is lesser than number of columns
   (c) Number of rows is equal to number of columns
   (d) If the entry of row is zero

11. The assignment problem is said to be balanced if it is a
   (a) Square matrix (b) Rectangular matrix
   (c) Unit matrix (d) Triangular matrix

12. In an assignment problem if number of rows is greater than number of columns, then
   (a) Dummy column is added
   (b) Dummy row is added
   (c) Row with cost 1 is added
   (d) Column with cost 1 is added

13. In a 3 machine and 5 jobs problem, the least of processing times on machine A, B and C are 5, 1 and 3 hours and the highest porocessing times are 9, 5 and 7 respectively, then it can be converted to a 2 machine problem if order of the machines is :
   (a) B–A–C (b) A–B–C
   (c) C–B–A (d) Any order

14. The objective of an assignment problem is to assign
   (a) Number of jobs to equal number of persons at maximum cost
   (b) Number of jobs to equal number of persons at minimum cost
   (c) Only the maximize cost
   (d) Only to minimize cost

Answers

1. (c) Both (a) and (b)
2. (b) DBCA
3. (c) to find the sequence in which jobs on hand are to be processed to minimize the total time required for processing the jobs
4. (d) $(n!)^m$ 5. (b) Hungarian method
6. (d) The time of passing depends on the order of machining
7. (a) Converting all profits to opportunity losses

8. (a) 1–C, 2–B, 3–D, 4–A
9. (d) Both (a) and (b)
10. (c) Number of rows is equal to number of columns
11. (a) Square matrix
12. (a) Dummy column is added
13. (b) A–B–C
14. (b) Number of jobs to equal number of persons at minimum cost

II. Fill in the blanks :

1. An assignment problem is said to be unbalanced when ....
2. When the number of rows is equal to the number of columns, then the problem is said to be .... assignment problem.
3. For solving an assignment problem the matrix should be a .... matrix.
4. If the given matrix is not a .... matrix, the assignment problem is called an unbalanced problem.
5. A dummy row(s) or column(s) with the cost elements as .... the matrix of an unbalanced assignment problem as a square matrix.
6. The time interval between starting the first job and completing the last job including the idle time (if any) in a particular order by the given set of machines is called ....
7. The time for which a machine j does not have a job to process to the start of job i is called ....
8. Maximization assignment problem is transformed to minimization problem by subtracting each entry in the table from the .... value in the table.
9. When an assignment problem has more than one solution, then it is .... optimal solution.
10. The time required for printing of four books A, B, C and D is 5, 8, 10 and 7 hours. While its data entry requires 7, 4, 3 and 6 hours respectively. The sequence that minimizes total elapsed time is ....

Answers

1. Number of rows is not equal to number of columns
2. balanced 3. square 4. square 5. zero
6. Total elapsed time 7. Idle time 8. Maximum
9. Multiple 10. A–D–B–C.

III. State whether each of the following is *True* or *False*.

1. One machine–one job is not an assumption in solving sequencing problems.
2. If there are two least processing times for machine A and machine B, priority is given for the processing time which has lowest time of the adjacent machine.
3. To convert the assignment problem into a maximization problem, the smallest element in the matrix is deducted from all other elements.
4. The Hungarian method operates on the principle of matrix reduction, whereby the cost table is reduced to a set of opportunity costs.
5. In a sequencing problem, the processing times are dependent of order of processing the jobs on machines.
6. Optimal assignments are made in the Hungarian method to cells in the reduced matrix that contain a zero.
7. Using the Hungarian method, the optimal solution to an assignment problem is found when the minimum number of lines required to cover the zero cells in the reduced matrix equals the no of persons.
8. In an assignment problem, if number of columns is greater than number of rows, then a dummy column is added.
9. The purpose of dummy row or column in an assignment problem is to obtain balance between total number of activities and total number of resources.
10. One of the assumptions made while sequencing n jobs on 2 machines is : two jobs must be loaded at a time on any machine.

Answers

1. False 2. True 3. False 4. True 5. False
6. True 7. True 8. False 9. True 10. False

PART–I

IV. Solve the following problems :

1. **A plant manager has four subordinates and four tasks to be performed. The subordinates differ in efficiency and the tasks differ in their intrinsic difficulty. This estimate of the times each man would take to perform each task is given in the following effectiveness matrix below.**

	I	II	III	IV
A	7	25	26	10
B	12	27	3	25
C	37	18	17	14
D	18	25	23	9

How should the tasks be allocated, one to a man, as to minimize the total man hours?

Solution :

Subtract the smallest element in each row from every element in that row.

Subordinates	Time for Tasks (in hours)			
	I	II	III	IV
A	0	18	19	3
B	9	24	0	22
C	23	4	3	0
D	9	16	14	0

Subtract the smallest element of each column from the element of that column.

Subordinates	Time for Tasks (in hours)			
	I	II	III	IV
A	0	14	19	3
B	9	20	0	22
C	23	0	3	0
D	9	12	14	0

The lines covering all zeros is equal to the order of matrix (4).

The assignment is made as follows :

Subordinates	Time for Tasks (in hours)			
	I	II	III	IV
A	$\boxed{0}$	14	9	3
B	9	20	$\boxed{0}$	22
C	23	$\boxed{0}$	3	$\cancel{0}$
D	9	12	14	$\boxed{0}$

Optimum solution is shown as follows :

Subordinates	Tasks	Time (in hours)
A	I	7
B	III	3
C	II	18
D	IV	9

Minimum total men hours = 7 + 3 + 18 + 9 = 37 hours.

2. A dairy plant has five milk tankers. I, II, III, IV and V. Three milk tankers are to be used on five delivery routes A, B, C, D and E. The distances (in km) between the dairy plant and the delivery routes are given in the following distance matrix.

	I	II	III	IV	V
A	150	120	175	180	200
B	125	110	120	150	165
C	130	100	145	160	170*
D	40	40	70	70	100
E	45	25	60	70	95

How should the milk tankers be assigned to the chilling center so as to minimize the distance travelled?

[***Note :** *Question has been modified.*]

Solution :

Step 1 : Subtract the smallest element in each row from every element in that row. We get

Subordinates	Time (in hours)				
	Tasks				
	I	II	III	IV	V
A	30	0	55	60	80
B	15	0	10	40	55
C	30	0	45	60	70
D	0	0	30	30	60
E	20	0	35	45	70

Step 2 : Subtract the smallest element in each column from every element in that column. We get

Subordinates	Time (in hours)				
	Tasks				
	I	II	III	IV	V
A	30	0	45	30	25
B	15	0	0	10	0
C	30	0	35	30	15
D	0	0	20	0	5
E	20	0	25	15	15

Step 3 : Since, the number of straight lines covering all zeros is not equal to the number of rows/columns, the optimal solution has not reached.

Step 4 : Select the smallest element among the uncovered elements, which is **15** here. Subtract it from each element of the uncovered elements and add it to the elements at the intersection of the lines. We get

Subordinates	Time (in hours)				
	Tasks				
	I	II	III	IV	V
A	15	0	20	15	0
B	15	15	0	10	0
C	15	0	20	15	0
D	0	15	20	0	5
E	5	0	10	0	0

Step 5 : Since the number of straight lines covering all zero is equal to number of rows/columns, the optimal solution has reached. The optimal assignment can be made as follows :

Subordinat	Time (in hours)				
	Tasks				
	I	II	III	IV	V
A	15	$\boxed{0}$	30	15	10
B	15	15	$\boxed{0}$	10	$\cancel{0}$
C	15	$\cancel{0}$	20	15	$\boxed{0}$
D	$\boxed{0}$	15	20	$\cancel{0}$	5
E	5	$\cancel{0}$	10	$\boxed{0}$	$\cancel{0}$

Hence, the optimal assignment schedule is obtained as follows :

Subordinates	Tasks	Time (in hours)
A	II	120
B	III	120
C	V	170
D	I	40
E	IV	70

Minimum distance travelled is 520 km.

[**Note :** *Answer given in the textbook is incorrect.*]

3. Solve the following assignment problem to maximize sales :

Salesmen	Territories				
	I	II	III	IV	V
A	11	16	18	15	14*
B	7	19	11	13	17
C	9	6	14	14	7
D	13	12	17	11	13

[*Note : *Question has been modified.*]

Solution : The problem is unbalanced. So add dummy salesman E with zero sale.

Salesmen	Sales of territories				
	I	II	III	IV	V
A	11	16	18	15	14
B	7	19	11	13	17
C	9	6	14	14	7
D	13	12	17	11	13
E	0	0	0	0	0

Since assignment problem is of maximization, subtract all the elements of matrix form the maximum element 19 of the matrix.

Salesmen	Sales of territories				
	I	II	III	IV	V
A	8	3	1	4	3
B	12	0	8	6	2
C	10	13	5	5	12
D	6	7	2	8	6
E	19	19	19	19	19

Subtract the smallest element of each row from the elements of that row.

Salesmen	Sales of territories				
	I	II	III	IV	V
A	7	2	0	3	2
B	12	0	8	6	2
C	5	8	0	0	7
D	4	5	0	6	4
E	0	0	0	0	0

Since the smallest element of each column is zero, matrix remains unchanged.

The number of lines covering all zeros that is 4 is not equal to the order i.e. 5 of matrix.

Therefore, subtract the smallest uncovered element 2 from the uncovered elements by lines and add to the cross elements of two lines.

Salesmen	Sales of territories				
	I	II	III	IV	V
A	5	2	0	3	0
B	10	0	8	6	2
C	3	8	0	0	7
D	2	5	0	6	4
E	0	2	2	2	0

Since the lines covering all zeros is 5 which is equal to order of matrix 5.

The assignment is made as follows :

Salesmen	Sales of territories				
	I	II	III	IV	V
A	5	2	$\cancel{0}$	3	$\boxed{0}$
B	10	$\boxed{0}$	8	6	$\cancel{0}$
C	3	8	$\cancel{0}$	$\boxed{0}$	7
D	2	5	$\boxed{0}$	6	4
E	$\boxed{0}$	2	2	2	$\cancel{0}$

The optimal assignment is shown as follows :

Salesmen	Territories	Sales
A	V	14
B	II	19
C	IV	14
D	III	17
E	I	0

Maximum sales $= 14 + 19 + 14 + 17$

$= 64$ units

[**Note :** *Answer given in the textbook is incorrect.*]

4. The estimated sales (tons) per month in four different cities by five different managers are given below :

Manager	Cities			
	P	Q	R	S
I	34	36	33	35
II	33	35	31	33
III	37	39	35	35
IV	36	36	34	34
V	35	36	35	33

Find out the assignment of managers to cities in order to maximize sales.

Solution :

Since, this is an unbalanced assignment add dummy city T with zero sales.

Manager	Sales of cities (in tons)				
	P	Q	R	S	T
I	34	36	33	35	0
II	33	35	31	33	0
III	37	39	35	35	0
IV	36	36	34	34	0
V	35	36	35	33	0

To convert the maximization problem into minimization problem, subtract all the elements of given matrix from the maximum element 39 of the matrix.

Manager	Sales of cities (in tons)				
	P	Q	R	S	T
I	5	3	6	4	39
II	6	4	8	6	39
III	2	0	4	4	39
IV	3	3	5	5	39
V	4	3	4	6	39

Subtract the lowest element of each row from the elements of that row.

Manager	Sales of cities (in tons)				
	P	Q	R	S	T
I	2	0	3	1	36
II	2	0	4	2	35
III	2	0	4	4	39
IV	0	0	2	2	36
V	1	0	1	3	36

Subtract the lowest element of each column from the elements of that column.

Manager	Sales of cities (in tons)				
	P	Q	R	S	T
I	2	0	2	0	1
II	2	0	3	1	0
III	2	0	3	3	4
IV	0	0	1	1	1
V	1	0	0	2	1

The number of lines covering all zeros is the same as the order of matrix.

Therefore, assignment is made as follows :

Manager	Sales of cities (in tons)				
	P	Q	R	S	T
I	2	$\cancel{0}$	2	$\boxed{0}$	1
II	2	$\cancel{0}$	3	1	$\boxed{0}$
III	2	$\boxed{0}$	3	3	4
IV	$\boxed{0}$	$\cancel{0}$	1	1	1
V	1	$\cancel{0}$	$\boxed{0}$	2	1

The optimal assignment is shown as follows :

Manager	City	Sales (in tons)
I	S	35
II	T	0
III	Q	39
IV	P	36
V	R	35

5. Consider the problem of assigning five operators to five machines. The assignment costs are given in following table :

Operators	Machines				
	1	2	3	4	5
A	6	6	–	3	7
B	8	5	3	4	5
C	10	4	6	–	4
D	8	3	7	8	3
E	6*	6	8	10	2

Operator A cannot be assigned to machine 3 and operator C cannot be assigned to machine 4. Find the optimal assignment schedule.

[***Note :** *Question has been modified.*]

Solution : Since this is prohibited assignment problem, assign highest cost say ∞ to operators A and C corresponding machine 3 and machine 4 respectively.

Operators	Assignment cost of Machines				
	1	2	3	4	5
A	6	6	∞	3	7
B	8	5	3	4	5
C	10	4	6	∞	4
D	8	3	7	8	3
E	6	6	8	10	2

Subtract the lowest element of each row from the elements of that row.

Operators	Assignment cost of Machines				
	1	2	3	4	5
A	3	3	∞	0	4
B	5	2	0	1	2
C	6	0	2	∞	0
D	5	0	4	5	0
E	4	4	6	8	0

Subtract the lowest element of each column from the elements of that column.

Operators	Assignment cost of Machines				
	1	2	3	4	5
A	0	3	∞	0	4
B	2	2	0	1	2
C	3	0	2	∞	0
D	2	0	4	5	0
E	1	4	6	8	0

The number of lines covering all zeros is not equal to the order of matrix.

Therefore, subtract the lowest element 1 of all uncovered elements and add it to the intersection of two lines.

Operators	Assignment cost of Machines				
	1	2	3	4	5
A	0	4	∞	0	5
B	1	2	0	0	2
C	2	0	2	∞	0
D	1	0	4	4	0
E	0	4	6	7	0

The number of lines covering all zeros is equal to the order of matrix.

Therefore, assignment is made as follows :

Operators	Assignment cost of Machines				
	1	2	3	4	5
A	$\cancel{0}$	4	∞	$\boxed{0}$	5
B	1	2	$\boxed{0}$	$\cancel{0}$	2
C	2	$\boxed{0}$	2	∞	$\cancel{0}$
D	1	$\cancel{0}$	4	4	$\boxed{0}$
E	$\boxed{0}$	4	6	7	$\cancel{0}$

Hence, optimal assignment schedule is obtained as follows :

Operator	Machine
A	4 or 4
B	3 or 3
C	2 or 5
D	5 or 2
E	1 or 1

---

6. A chartered accountant's firm has accepted five new cases. The estimated number of days required by each of their five employees for each case are given below, where-means that the particular employee cannot be assigned the particular case. Determine the optimal assignment of cases of the employees so that the total number of days required to complete these five cases will be minimum. Also, find the minimum number of days.

Employee	Cases				
	I	II	III	IV	V
E_1	5*	4	5	7	8
E_2	7	–	8	6	9
E_3	8	6	7	9	10
E_4	5	7	–	4	6
E_5	9	5	3	10	–

[*Note : *Question has been modified.*]

Solution :

This is prohibited assignment problem. Therefore, we assign very high days say infinity ∞ to employees E_2 for case II. E_4 for case III and E_5 for case V.

Employee	Cases				
	I	II	III	IV	V
E_1	5	4	5	7	8
E_2	7	∞	8	6	9
E_3	8	6	7	9	10
E_4	5	7	∞	4	6
E_5	9	5	3	10	∞

Subtracting the lowest element of each row from the element of that row.

Employee	Cases				
	I	II	III	IV	V
E_1	1	0	1	3	4
E_2	1	∞	2	0	3
E_3	2	0	1	3	4
E_4	1	3	∞	0	2
E_5	6	2	0	7	∞

Subtracting the lowest element of each column from the elements of that column.

Employee	Cases				
	I	II	III	IV	V
E_1	0	0	1	3	2
E_2	0	∞	2	0	1
E_3	1	0	1	3	2
E_4	0	3	∞	0	0
E_5	5	2	0	7	∞

The number of lines covering all zeros is equal to the order of matrix.

Therefore, optimal assignment is made as follows :

Employee	Cases				
	I	II	III	IV	V
E_1	$\boxed{0}$	$\cancel{0}$	1	3	2
E_2	$\cancel{0}$	∞	2	$\boxed{0}$	1
E_3	2	$\boxed{0}$	1	3	2
E_4	$\cancel{0}$	3	∞	$\cancel{0}$	$\boxed{0}$
E_5	5	2	$\boxed{0}$	7	∞

The optimal assignment is shown as follows :

Employee	Cases	Number of days
E_1	I	5
E_2	IV	6
E_3	II	6
E_4	V	6
E_5	III	3

Total number of minimum days $= 5 + 6 + 6 + 6 + 3$
$= 26$ days

[Note : *Answer given in the textbook is incorrect.*]

PART – II

1. A readymade garments manufacturer has to process 7 items through two stages of production, namely cutting and sewing. The time taken in hours for each of these items in different stages are given below :

Items	1	2	3	4	5	6	7
Time for Cutting	5	7	3	4	6	7	12
Time for Sewing	2	6	7	5	9	5	8

Find the sequence in which these items are to be processed through these stages so as to minimize the total processing time. Also, find the idle time of each machine.

Solution :

Items	Time (in hours)	
	Cutting (A)	Sewing (B)
1	5	2
2	7	6
3	3	7
4	4	5
5	6	9
6	7	5
7	12	8

Here, Min. (A, B) = 2, which corresponds to B.

Therefore, item 1 is processed at the last.

						1

The problem now reduces to items 2 to 7.

Here, Min. (A, B) = 3, which corresponds to A.

Therefore, item 3 is processed at first.

3						1

The problem now reduces to items 2, and 4 to 7.

Here, Min. (A, B) = 4, which corresponds to A.

Therefore, item 4 is processed next to item 3.

3	4					1

The problem now reduces to items 2, 5, 6, 7.

Here, Min. (A, B) = 5, which corresponds to B.

Therefore, item 6 is processed at last next to item 1.

3	4				6	1

The problem now reduces to items 2, 5, 7.

Here, Min. (A, B) = 6, which corresponds to both A and B.

Therefore, item 5 is processed next to item 4 and item 2 is processed at the last next to item 6.

3	4	5		2	6	1

Now item 7 is processed at the last next to item 2 and the optimal sequence is obtained as follows :

3	4	5	7	2	6	1

Total elapsed time is obtained as follows :

Items Sequence	Cutting (A)		Sewing (B)		Idle time for (B)
	Time in	Time out	Time in	Time out	
3	0	3	3	10	3
4	3	7	10	15	0
5	7	13	15	24	0
7	13	25	25	33	1
2	25	32	33	39	0
6	32	39	39	44	0
1	39	44	44	46	0
				Total idle time for (B)	4

Total elapsed time T = 46 hours

Idle time for cutting = T – total time for cutting

= 46 – 44

= 2 hours

Idle time for sewing = 4 hours.

---

2. Five jobs must pass through a lathe and a surface grinder, in that order. The processing times in hours are shown below. Determine the optimal sequence of the jobs. Also, find the idle time of each machine.

Jobs	I	II	III	IV	V
Lathe	4	1	5	2	5
Surface grinder	3	2	4	3	6

Solution :

Jobs	Time (in hours)	
	Lathe (A)	Surface grinder (B)
I	4	3
II	1	2
III	5	4
IV	2	3
V	5	6

Here, Min. (A, B) = 1, which corresponds to A. Therefore, job II is processed first.

II				

The problem now reduces to jobs I, III, IV, V.
Here, Min. (A, B) = 2, which corresponds to A.
Therefore, job IV is processed next to job II.

II	IV			

The problem now reduces to jobs I, III, V.
Here, Min. (A, B) = 3, which corresponds to B.
Therefore, job I is processed at the last.

II	IV			I

The problem now reduces to jobs III and V.
Here, Min. (A, B) = 4, which corresponds to B.
Therefore, job III is processed at the last next to job I.

II	IV		III	I

Now, job V is processed next to job IV and we get the optional sequence of jobs as follows :

II	IV	V	III	I

Total elapsed time is obtained as follows :

Job Sequence	Lathe (A)		Surface grinding (B)		Idle time for (B)
	Time in	Time out	Time in	Time out	
II	0	1	1	3	1
IV	1	3	3	6	0
V	3	8	8	14	2
III	8	13	14	18	0
I	13	17	18	21	0
				Total idle time for (B)	3

Total elapsed time T = 21 hours

Idle time for lathe = T − Total time for lathe

= 21 − 17 = 4 hours

Idle time for surface grinding = 3 hours

3. Find the sequence that minimizes the total elapsed time to complete the following jobs. Each job is processed in order AB :

Machines	Jobs (Processing times in minutes)						
	I	II	III	IV	V	VI	VII
Machine A	12	6	5	11	5	7	6
Machine B	7	8	9	4	7	8	3

Determine the sequence for the jobs so as to minimize the processing time. Find the total elapsed time and the idle times for both the machines.

Solution :

Jobs	Processing time (in minutes)	
	Machine A	Machine B
I	12	7
II	6	8
III	5	9
IV	11	4
V	5	7
VI	7	8
VII	6	3

Here, Min. (A, B) = 3, which corresponds to Machine B. Therefore, job VII is processed at the last.

						VII

Now, problem reduces to jobs I to VI.
Here, Min. (A, B) = 4, which corresponds to Machine B.
Therefore, job IV is processed at the last next to job VII.

					IV	VII

The problem now reduces to jobs I to III, V and VI.
Here, Min. (A, B) = 5, which corresponds to Machine A.
Therefore, job III is processed at the first and job V is processed next to job III.

III	V				IV	VII

The problem now reduces to jobs I, II and VI.
Here, Min. (A, B) = 6, which corresponds to machine A.
Therefore, job II is processed at first next to job V.

III	V	II			IV	VII

The problem now reduces to job I and job VI.
Here, Min. (A, B) = 7, which corresponds to both machines A and B.

Therefore, job VI is processed at first next to job II and job I is processed next to job VI and we get the optimal sequence of jobs as follows :

III	V	II	VI	I	IV	VII

Total elapsed time is obtained as follows :

Job Sequence	Machine A		Machine B		Idle time for Machine B
	Time in	Time out	Time in	Time out	
III	0	5	5	14	5
V	5	10	14	21	0
II	10	16	21	29	0
VI	16	23	29	33	0
I	23	35	37	44	0
IV	35	46	46	50	2
VII	46	52	52	55	2
Total idle time for Machine B					9

Total elapsed time T = 55 minutes

Idle time for machine A

= T − Total processing time for machine A

= 55 − 52

= 3 minutes

Idle time for machine B = 9 minutes

4. A toy manufacturing company produces five types of toys. Each toy has to go through three machines A, B, C in the order ABC. The time required in hours for each process is given in the following table :

Type	1	2	3	4	5
Machine A	16	20	12	14	22
Machine B	10	12	4	6	8
Machine C	8	18	16	12	10

Solve the problem for minimizing the total elapsed time.

Solution :

Here Min. (A) = 12, Min. (C) = 8 and Max. (B) = 12.

Since, Min. (A) ⩾ Max. (B) is satisfied, the problem can be converted into 5 type. 2 machine problem and two fictitious machines are,

G = A + B and H = B + C

The problem now can be written as follows :

Types of toys	Processing time (in hours)	
	G = A + B	H = B + C
1	26	18
2	32	30
3	16	20
4	20	18
5	30	18

Here, Min. (G, H) = 16, which corresponds to G.

Therefore, type 3 toy is processed at first.

3				

The problem now reduces to type 1, 2, 4, 5 toys.

Here, Min. (G, H) = 18, which corresponds to H.

Therefore, type 1 toy is processed in the last, type 4 toy is processed at the last next to type 1 toy and type 5 toy is processed at last next to type 4 toy.

3		5	4	1

Now, type 2 toy is processed at last next to type 5 toy and the optimal sequence is obtained as follows :

3	2	5	4	1

Total elapsed time is obtained as follows :

Sequence of type of toy	Machine A		Machine B		Machine C		Idle time for Machine C
	Time in	Time out	Time in	Time out	Time in	Time out	
3	0	12	12	16	16	32	16
2	12	32	32	44	44	62	12
5	32	54	54	62	62	72	0
4	54	68	68	74	74	86	2
1	68	84	84	94	94	102	8
Total idle time for Machine C							38

Total elapsed time T = 102 hours

Idle time for machine A

= T − Sum of processing time for all jobs on machine A

= 102 − 84 = 18 hours

Idle time for machine B

= T − Sum of processing time for all jobs on machine B

= 102 − 40 = 62 hours

Idle time for machine C = 38 hours

5. A foreman wants to process 4 different jobs on three machines : a shaping machine, a drilling machine and a tapping machine, the sequence of operations being shaping-drilling-tapping. Decide the optimal sequence for the four jobs to minimize the total elapsed time. Also find the total elapsed time and the idle time for every machine.

Job	Shaping (Minutes)	Drilling (Minutes)	Tapping (Minutes)
1	13	3	18
2	18	8	4
3	8	6	13
4	23	6	8

Solution :

Here, Min. (Shaping) = 8 minutes, Min. (Tapping) = 4 minutes, Max. (Drilling) = 8 minutes.

Since, Min. (Shaping) $\geqslant$ Max. (Drilling) is satisfied, the problem can be converted into 4 job 2 machine problem and two fictitious machines are,

G = Shaping + Drilling and

H = Drilling + Tapping.

The problem now can be written as follows :

Job	Processing time (in minutes)	
	G	H
1	16	21
2	26	12
3	14	19
4	29	14

Here, Min. $(G_{i1}, H_{i2}) = 12$, which corresponds to H. Therefore, Job 2 is processed in the last.

			2

The problem now reduces to three jobs 1, 3 and 4. Here, Min. $(G_{i1}, H_{i2}) = 14$, which corresponds to G and H both.

Therefore, job 3 is processed first of all and then job 4 is processed in the last next to job 2.

3		4	2

Now, the remaining job 1 must be processed next to job 3. Thus, the optimal sequence of jobs is obtained as follows :

3	1	4	2

The minimum elapsed time can be computed as follows :

Job Sequence	M_1		M_2		M_3		Idle time for M_3
	Time in	Time out	Time in	Time out	Time in	Time out	
3	0	8	8	14	14	27	14
1	8	21	21	24	27	45	0
4	21	44	44	50	50	58	5
2	44	62	62	70	70	74	12
Total idle time for M_3							31

From the above table :

Idle time for machine M_1

$= T - \{\text{sum of processing time for all jobs on machine } M_1\}$

$= 74 - 62 = 12$ minutes.

Idle time for machine M_2

$= T - \{\text{sum of processing time for all jobs on machine } M_2\}$

$= 74 - 23 = 51$ minutes.

Idle time for machine M_3

$= T - \{\text{sum of processing time for all jobs on machine } M_3\}$

$= 74 - 43 = 31$ minutes.

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ASSIGNMENT PROBLEMS

1. Given below the costs of assigning 3 jobs to 3 workers. Find all possible assignments by trial and error method.

Workers	Jobs		
	X	Y	Z
A	11	16	21
B	20	13	17
C	13	15	12

Among these assignments, find the optimal assignment that minimizes the total cost.

Solution :

Possible Assignments : A → X, B → Y, C → Z
A → Y, B → Z, C → X
A → Z, B → X, C → Y

Among these assignments, the optimal assignment is as follows :

Workers	Job assigned	Cost
A	X	11
B	Y	13
C	Z	12

Minimum cost = 11 + 13 + 12
= 36 units

---

2. Show that the optimal solution of an assignment problem is unchanged if we add or subtract the same constant to the entries of any row or column of the cost matrix.

Solution : Consider the following maximization assignment problem.

Salesman	Profits (in '000 ₹)			
	I	II	III	
A	56	59	60	
B	57	58	58	… (P)
C	58	56	57	

Subtract each of the elements in the pay off matrix from the largest element 60. The assignment problem is obtained as follows :

Salesman	Profits (in '000 ₹)		
	I	II	III
A	4	1	0
B	3	2	2
C	2	4	3

Smallest element of each row subtracted from the element of that row.

Salesman	Profits (in '000 ₹)			
	I	II	III	
A	4	1	0	
B	1	0	0	… (Q)
C	0	2	1	

Since the smallest element in each column is zero, columns will remain unchanged.

Since, the numbers of rows and columns covering zeros is equal to the order of matrix, the optimal solution has reached. Optimal assignment can be made as follows :

Salesman	Profits (in '000 ₹)		
	I	II	III
A	4	1	$\boxed{0}$
B	1	$\boxed{0}$	$\cancel{0}$
C	$\boxed{0}$	2	1

Hence, the optimal assignment is

Salesman	Work assigned	Profit (in '000 ₹)
A	III	60
B	II	58
C	I	58

Hence, the maximum profit = ₹ (60 + 58 + 58)
= ₹ 176 thousand

Now, subtract 55 from each element of pay off matrix given in (P)

Salesman	Profits (in '000 ₹)		
	I	II	III
A	1	4	5
B	2	3	3
C	3	1	2

Subtract each element in the pay off matrix from the largest element 5. The assignment problem is obtained as follows :

Salesman	Profits (in '000 ₹)		
	I	II	III
A	4	1	0
B	3	2	2
C	2	4	3

Smallest element of each row is subtracted from the element of that row and then from the matrix obtained, subtract smallest element of each column from the elements of that column.

We get the same pay off matrix as (Q).

Salesman	Profits (in '000 ₹)		
	I	II	III
A	4	1	0
B	2	0	0
C	0	2	1

Assigning the work is same and optimal assignments do not change.

Similarly, if we add the same constant to the entries of any row or column of the pay off matrix the optimal solution remains unchanged.

3. Construct a 3 × 3 cost matrix by taking the costs as the first 9 natural numbers and arranging them row wise in ascending order. Find all possible assignments that will minimize the total sum.

Solution :

3 × 3 cost matrix :

$$\begin{array}{c|c|c|c|} \cline{2-4} P & 1 & 2 & 3 \\ \cline{2-4} Q & 4 & 5 & 6 \\ \cline{2-4} R & 7 & 8 & 9 \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c}{A} & \multicolumn{1}{c}{B} & \multicolumn{1}{c}{C} \end{array}$$

All possible assignments

P → A, Q → B, R → C

Total sum : $1 + 5 + 9 = 15$

P → B, Q → C, R → A

Total sum : $2 + 6 + 7 = 15$

P → C, Q → A, R → B

Total sum : $3 + 4 + 8 = 15$

4. Given below the costs (in hundred rupees) of assigning 3 operators to 3 different machines. Find the assignment that will minimize the total cost. Also, find the minimum cost.

Operators	Machines		
	I	II	III
A	$3i + 4j$	$2j^2 + 5i$	$5j - 3i$
B	$i^3 + 8j$	$7i + j^2$	$4i + j$
C	$2i^2 - 1$	$3i + 5j$	$i^3 - 4j$

Where, i stands for number of rows and j stands for number of columns.

Solution :

Cost elements :

$3i + 4j$ is in (1, 1) $\quad \therefore\ 3i + 4j = 3 \times 1 + 4 \times 1 = 7$

$2j^2 + 5i$ is in (1, 2) $\quad \therefore\ 2j^2 + 5i = 2 \times 2^2 + 5 \times 1 = 8 + 5 = 13$

$5j - 3i$ is in (1, 3) $\quad \therefore\ 5j - 3i = 5 \times 3 - 3 \times 1 = 15 - 3 = 12$

$i^3 + 8j$ is in (2, 1) $\quad \therefore\ i^3 + 8j = 2^3 + 8 \times 1 = 8 + 8 = 16$

$7i + j^2$ is in (2, 2) $\quad \therefore\ 7i + j^2 = 7 \times 2 + 2^2 = 14 + 4 = 18$

$4i + j$ is in (2, 3) $\quad \therefore\ 4i + j = 4 \times 2 + 3 = 8 + 3 = 11$

$2i^2 - 1$ is in (3, 1) $\quad \therefore\ 2i^2 - 1 = 2 \times 3^2 - 1 = 18 - 1 = 17$

$3i + 5j$ is in (3, 2) $\quad \therefore\ 3i + 5j = 3 \times 3 + 5 \times 2 = 9 + 10 = 19$

$i^3 - 4j$ is in (3, 3) $\quad \therefore\ i^3 - 4j = 3^3 - 4 \times 3 = 27 - 12 = 15$

Now, the problem can be written as follows :

Operators	Cost (in '00 ₹)		
	Machine I	Machine II	Machine III
A	7	13	12
B	16	18	11
C	17	19	15

Subtract smallest element of each row from the elements of that row.

Operators	Cost (in '00 ₹)		
	Machine I	Machine II	Machine III
A	0	6	5
B	5	7	0
C	2	4	0

Subtract smallest element of each column from the elements of that column.

Operators	Cost (in '00 ₹)		
	Machine I	Machine II	Machine III
A	0	2	5
B	5	3	0
C	2	0	0

The number of horizontal and verticle lines covering all zeros is 3, which is equal to the order of matrix. Therefore, optimal solution has reached.

Assignment is made as follows :

Operators	Cost (in '00 ₹)		
	Machine I	Machine II	Machine III
A	$\boxed{0}$	2	5
B	5	3	$\boxed{0}$
C	2	$\boxed{0}$	$\cancel{0}$

The optimal solution of the problem is as follows :

Operator	Machine	Cost (in '00 ₹)
A	I	7
B	III	11
C	II	19

Total minimum cost = 7 + 11 + 19 = ₹ 37 hundred.

5. A firm marketing a product has four salesmen S_1, S_2, S_3 and S_4. There are three customers C_1, C_2 and C_3. The probability of making a sale to a customer depends upon the salesman – customer support. The table below represents the probability with which each of the salesmen can sell to each of the customers.

Customers	Salesmen			
	S_1	S_2	S_3	S_4
C_1	0.7	0.4	0.5	0.8
C_2	0.5	0.8	0.6	0.7
C_3	0.3	0.9	0.6	0.2

If only one salesman is to be assigned to one customer, what combination of salesmen and customers shall be optimal? Profit obtained by selling one unit to C_1 is ₹ 500 to C_2 is ₹ 450 and to C_3 is ₹ 540. What is the total expected profit.

Solution :

As this is an unbalanced assignment problem, we add dummy customers C_4 with 0 probability.

Customer	Probability of profit			
	S_1	S_2	S_3	S_4
C_1	0.7	0.4	0.5	0.8
C_2	0.5	0.8	0.6	0.7
C_3	0.3	0.9	0.6	0.2
C_4	0	0	0	0

Subtract smallest element of each row from the elements of that row.

Customer	Probability of profit			
	S_1	S_2	S_3	S_4
C_1	0.3	0	0.1	0.4
C_2	0	0.3	0.1	0.2
C_3	0.1	0.7	0.4	0
C_4	0	0	0	0

Since the smallest element of each column is zero, the profit matrix remains unchanged.

Since the number of horizontal/vertical lines covering all zeros (4) is equal to the order of matrix (4), the optimal solution has reached.

The assignment is made as follows :

Customers	Probability of profit			
	S_1	S_2	S_3	S_4
C_1	0.3	$\boxed{0}$	0.1	0.4
C_2	$\boxed{0}$	0.3	0.1	0.2
C_3	0.1	0.7	0.4	$\boxed{0}$
C_4	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$

The optimal solution is as follows :

Customer	Salesman	Probability of Profit	Profit
C_1	S_2	0.4	$0.4 \times 500 = 200$
C_2	S_1	0.5	$0.5 \times 450 = 225$
C_3	S_4	0.2	$0.2 \times 540 = 108$
C_4	S_3	0	0

Total profit $= 200 + 225 + 108 =$ ₹ 533

SEQUENCING PROBLEMS

1. Give two different examples of sequencing problems from your daily life.

Examples :

(1) Books on Statistics and Economics are to be printed by processing in two sections : composing and printing.

(2) Sequence of 3 type of toys are to be manufactured using the process of moulding, colouring and printing.

2. Let there be five jobs I, II, III, IV and V to be processed on two machines A and B in the order AB. Take the first 5 composite numbers as the processing times on machine A for jobs I, II, III, IV, V respectively and the first five odd numbers as the processing times on machine B for jobs V, IV, III, II, I respectively. Find the sequence that minimizes the total

elapsed time. Also, find the total elapsed time and idle times on both the machines.

Solution :

Jobs	Time	
	Machine A	Machine B
I	$1\frac{1}{2}$	1
II	$2\frac{1}{3}$	3
III	$3\frac{1}{5}$	5
IV	$4\frac{1}{8}$	7
V	$5\frac{1}{2}$	9

Here, Min. (A, B) = 1, which corresponds to machine B. Therefore, job I is processed at the last.

				I

The problem now reduces to job II to job V.

Here, Min. (A, B) = $2\frac{1}{3}$, which corresponds to machine A.

Therefore, job II is processed at the first.

II				I

The problem now reduces to jobs III, IV and V.

Here, Min. (A, B) = $3\frac{1}{5}$, which corresponds to machine A.

Therefore, job III is processed at the first next to job II.

II	III			I

Now, problem reduces to jobs IV and V.

Here, Min. (A, B) = $4\frac{1}{8}$, which corresponds to machine A.

Therefore, job IV is processed next to job III.

II	III	IV		I

Now, job V is processed next to job IV and we get optimal sequence as follows :

II	III	IV	V	I

Total elapsed time is obtained as follows :

Job Sequence	Machine A		Machine B		Idle time for Machine B
	Time in	Time out	Time in	Time out	
II	0	2.33	2.33	5.33	2.33
III	2.33	5.53	5.53	10.53	0
IV	5.53	9.66	10.53	17.53	0
V	9.66	15.16	17.53	26.53	0
I	15.16	16.66	26.53	27.53	0
Total idle time for Machine B					2.33

Total elapsed time T = 27.53 units

Idle time for machine A

= T − Total processing time of machine A

= 27.53 − 16.66

= 10.87 units

Idle time for machine B = 2.33 units

---

3. Determine the optimal sequence of jobs that minimizes the total elapsed time. Processing times are given in hours. Also find total elapsed time and idle times for the machines.

Jobs	I	II	III	IV	V	VI
Machine A	$2\frac{1}{2}$	$\frac{5}{2}$	$\frac{1}{2}$	$2\frac{3}{4}$	$3\frac{1}{4}$	$\frac{9}{4}$
Machine B	$3\frac{1}{2}$	$4\frac{1}{2}$	$\frac{7}{4}$	$2\frac{1}{4}$	$\frac{5}{4}$	$1\frac{1}{2}$

Jobs	Processing Time (in hours)	
	Machine A	Machine B
I	2.5	3.5
II	2.5	4.5
III	0.5	1.75
IV	2.75	2.25
V	3.25	1.25
VI	2.25	1.5

Here, Min. (A, B) = 0.5, which corresponds to machine A. Therefore, job III is processed at the first.

III					

The problem now reduces to jobs I, II, IV, V, VI.

Here, Min. (A, B) = 1.25, which corresponds to machine B. Therefore, job V is processed at the last.

III					V

The problem now reduces to jobs I, II, IV and VI.

Here, Min. (A, B) = 1.5, which corresponds to machine B.

Therefore, job VI is processed at the last next to job V.

III				VI	V

The problem now reduces to jobs I, II, IV.

Here, Min. (A, B) = 2.25, which corresponds to machine B.

Therefore, job IV is processed at the last next to job VI.

III			IV	VI	V

The problem now reduces to jobs I and II.

Here, Min. (A, B) = 2.5, which corresponds to machine A.

Therefore, job I is processed next to job III and job II is processed next to job I.

Thus the optimal sequence is obtained as follows :

III	I	II	IV	VI	V

Total elapsed time is obtained as follows :

Job Sequence	Machine A		Machine B		Idle time for Machine B
	Time in	Time out	Time in	Time out	
III	0	0.5	0.5	2.25	0.5
I	0.5	3.0	3.0	6.5	0.75
II	3.0	5.5	6.5	11.0	0
IV	5.5	8.25	11.0	13.25	0
VI	8.25	10.5	13.25	14.75	0
V	10.5	13.75	14.75	16.00	0
Total idle time for Machine B					1.25

Total elapsed time T = 16 hours

Idle time for machine A

= T − Total processing of machine A

= 16 − 13.75 = 2.25 hours

Idle time for Machine B = 1.25 hours

---

4. Consider 4 jobs to be processed on 3 machines A, B and C on the order ABC. Assign processing times to jobs and find the optimal sequence that minimizes the total processing time. Also find the elapsed time and idle times for all the three machines.

Consider the following problem :

Jobs	I	II	III	IV
A	5	8	7	3
B	6	7	2	5
C	7	8	10	9

Solution :

Here, Min. (A) = 3, Min. (C) = 7, Max. (B) = 7.

Since, Min. (C) ⩾ Max. (B) is satisfied, the problem can be converted into 4 job 2 machine problem and two fictitious machines are,

G = A + B and H = B + C.

The problem now can be written as follows :

Jobs	Processing time (in hours)	
	G = A + B	H = B + C
I	11	13
II	15	15
III	9	12
IV	8	14

Here, Min. $(G_{i1}, H_{i2}) = 8$, which corresponds to G.

Therefore, job IV is processed first of all.

IV			

The problem now reduces to three jobs I, II and III.

Here, Min. $(G_{i1}, H_{i2}) = 9$, which corresponds to G.

Therefore, job III is processed next to job IV.

IV	III		

The problem now reduces to two jobs I and II.

Here, Min. $(G_{i1}, H_{i2}) = 11$, which corresponds to G.

Therefore, I is processed next to job III.

IV	III	I	

Now, the remaining job II must be processed in the last. Thus, the optimal sequence of jobs is obtained as follows :

IV	III	I	II

The minimum elapsed time can be computed as follows :

Job Sequence	A		B		C		Idle time for C
	Time in	Time out	Time in	Time out	Time in	Time out	
IV	0	3	3	8	8	17	8
III	3	10	10	12	17	27	0
I	10	15	15	21	27	34	0
II	15	23	23	30	34	42	0
Total idle time for C							8

From the above table :

The minimum total elapsed time T = 42 hours.

Idle time for machine A

= T − 23

= 42 − 23 = 19 hours.

Idle time for machine B

= T − 20

= 42 − 20 = 22 hours.

Idle time for machine C

= T − 34

= 42 − 34 = 8 hours.

5.

(a) Determine the optimal sequence of jobs that minimizes the total elapsed time based on the following information. Processing time on machines is given in hours and passing is not allowed. Find total elapsed time and idle times for the machines.

Job	A	B	C	D	E	F	G
Machine X	13	18	17	14	19	18	17
Machine Y	14	13	12	13*	11	14	13
Machine Z	16	17	14	21	14	16	22

(b) What happens if we change the processing times on machine Z corresponding to jobs C and E and take them as 15 instead of 14 ?

[* **Note :** *Activity has been modified.*]

(a) Here, Min.(X) = 13, Min.(Z) = 14, Max.(Y) = 14.

Since, Min. (Z) ⩾ Max. (Y) is satisfied, the problem can be converted into 7 job 2 machine problem and two fictitious machines are,

G = X + Y and H = Y + Z.

The problem now can be written as follows :

Job	Processing time (in hours)	
	G = X + Y	H = Y + Z
A	27	30
B	31	30
C	29	26 (27)
D	27	34
E	30	25 (26)
F	32	30
G	30	35

Here, Min.(G, H) = 25, which corresponds to H.

Therefore, job E is processed at the last.

						E

The problem now reduces to jobs A, B, C, D, F and G.

Here, Min.(G, H) = 26, which corresponds to G.

Therefore, job C is processed at the last, next to job E.

					C	E

The problem now reduces to two jobs A, B, D, F and G.

Here, Min.(G, H) = 27, which corresponds to H.

Therefore, job A is processed at the first and job D is processed next to job A.

A	D				C	E

The problem now reduces to jobs B, F and G.

Here, Min.(G, H) = 30, which corresponds to both G and H.

Therefore, job G is processed at first next to job D and job B is processed at the last next to job C and job F is processed next to job B. Thus, the optimal sequence is obtained as follows :

A	D	G	F	B	C	E

Total elapsed time is obtained as follows :

Job Sequence	Machine X		Machine Y		Machine Z		Idle time for Machine Z
	Time in	Time out	Time in	Time out	Time in	Time out	
A	0	13	13	27	27	43	27
D	13	27	27	50	50	71	7
G	27	44	50	63	71	93	0
F	44	62	63	77	93	109	0
B	62	80	80	93	109	126	0
C	80	97	97	109	126	140	0
E	97	116	116	127	140	154	0
Total idle time for Machine Z							34

Total elapsed time T = 154 hours

Idle time of machine X

= T − {sum of processing time for all jobs on machine X}

= 154 − 116

= 38 hours.

Idle time for machine Y

= T − {sum of processing time for all jobs on machine Y}

= 154 − 90 = 64 hours.

Idle time for machine Z

= T − {sum of processing time for all jobs on machine Z}

= 154 − 120 = 34 hours.

(b) On machine Z if we put time 15 instead of 14 corresponding to jobs C and E.

$\therefore$ H = Y + Z for 7 jobs will be changed as follows :

30, 30, 27, 34, 26, 30, 35

Min. (Z) = 15

Now, Min. (Z) $\geqslant$ Max. (Y)

The problem can be converted into 2 machine.

Optimal sequence will remain unchanged.

❄ ❄ ❄

8 PROBABILITY DISTRIBUTIONS

CHAPTER OUTLINE

INTRODUCTION

The concept of probability is associated with the possible outcomes of a random experiment. If a real number is associated with each outcome of a random experiment, we can reach to real result of a random experiment. This result changes from outcome to outcome and is called a variable. Thus, the concept of random variable is connected with the outcomes of a random experiment. This will be clear from the following illustration :

If two balanced coins are tossed simultaneously, the sample space obtained S = {HH, HT, TH, TT}. Here, the results of the random experiment are 0 head, 1 head and 2 heads. If the number of heads is denoted by X, then X is a variable taking values 0, 1, 2. Such a variable is called random variable.

IMPORTANT FORMULAE

Symbols :

X = Random variable

x = Value of a random variable X

$P(X = x)$ = Probability of x a value of X

$P(X = x_i) = p_i$, $i = 1, 2, 3, ..., n$ is probability mass function (p.m.f.) of X

(x_i, p_i), $i = 1, 2, 3, ..., n$ is probability distribution of X

$f(x)$ = Probability density function of X

(i) $f(x) \geqslant 0$ or $a < x < b$

(ii) Area under the curve $y = f(x)$ is equal to 1

$F(x) = P[X \leqslant x]$ for $x \in R$ is cumulative distribution function (c.d.f.) of X

$E(X)$ = Expected value of X

Var(X) = Variance of X.

1. Discrete Random Variable :

(1) Probability Mass Function (p.m.f.)

$p_i = P[X = x_i]$, $i = 1, 2, 3, ..., n$ such that

(i) $0 \leqslant p_i \leqslant 1, i = 1, 2, ..., n$

(ii) $\sum_{i=1}^{n} p_i = 1$

(2) Cumulative Distribution Function (c.d.f.)

$F(x) = P[X \leqslant x], x \in R$

Domain $= (-\infty, \infty)$, codomain $= [0, 1]$

$$F(x_i) = \sum_{i=1}^{n} p_i$$

(3) Expected value :

$$E(X) = \sum_{i=1}^{n} x_i p_i = \mu$$

$E(X) = \Sigma x\, p(x)$, $p(x) =$ p.m.f. of X

(4) Variance :

$$Var(X) = E(X - \mu)^2$$

$$Var(X) = \sum_{i=1}^{n} (x_i - \mu)^2 p_i$$

$$Var(X) = \sum_{i=1}^{n} x_i^2 p_i - \mu^2$$

$Var(X) = E(X^2) - [E(X)]^2 \qquad [\because \mu = E(X)]$

$Var(X) = \Sigma x^2 p(x) - [\Sigma x p(x)]^2$, $p(x) =$ p.m.f. of X.

2. Continuous Random Variable :

(1) Probability Density Function (p.d.f.)

$f(x)$ a real valued function such that

(i) $f(x) \geqslant 0$ for $a < x < b$

(ii) Area under the curve $y = f(x)$, $x = a$, $x = b$

i.e. $\int_a^b f(x)\,dx = 1$

(2) Distribution Function :

$$F(x_i) = P[X = x_i] = \int_{-\infty}^{x_i} f(x)\,dx$$

3. Binomial Distribution :

$P(X = x) = p(x) = {}^nC_x p^x q^{n-x}$, $x = 0, 1, 2, ..., n$

$0 < p < 1, q = 1 - p$

$n = 1, 2, 3, \ldots\ldots$

$= 0,$ otherwise.

[Notes :

(1) $p(x) \geqslant 0$ for all x

(2) $\sum_{x=0}^{n} p(x) = 1$

(3) Mean $= np$, i.e. $E(x) = np$, Variance $= npq$, i.e. $Var(x) = npq$

(4) Mean > Variance, i.e. $np > npq$

(5) Parameters : n and p

(6) $p =$ Probability of success

(7) $x =$ Value of random variable X i.e. Number of successes.]

4. Poisson Distribution :

$$P[X = x] = p(x) = \frac{e^{-m} m^x}{x!}, x = 0, 1, 2, \ldots$$

$m = np > 0$

$= 0,$ otherwise.

[Notes :

(1) $p(x) \geqslant 0$ for all x

(2) $\sum_{x=0}^{\infty} p(x) = 1$

(3) Mean $= m$, Variance $= m$

(4) Mean = Variance

(5) Parameter : $m = np$

(6) $p =$ Probability of success

(7) e (constant) $= 2.7183$.]

8.1 : RANDOM VARIABLE

Definition : A random variable is a function associating a real number with each outcome of a sample space S of a random experiment and it is denoted by capital letter such as X, Y, ...

Symbolically, $X : S \to R$

e.g. two unbiased coins are tossed together

$\therefore$ S = {HH, HT, TH, TT}, where H = head, T = tail.

Let X = Number of heads, then X(HH) = 2, X(HT) = 1, X(TH) = 1, X(TT) = 0.

Thus, each outcome of S is associated with either 2 or 1 or 0. i.e. X is a function with domain S and range set {0, 1, 2}. Here, X is a random variable.

Remarks :

1. Small letter x is used to denote a particular value taken by X.
2. Each value of X corresponds to an event defined on S.
3. S may be a finite sample space or a countably infinite sample space. Hence, X takes a finite number of values or countably infinite values.
4. Several random variables can be defined on the same sample space.

8.2 : TYPES OF RANDOM VARIABLE

Random variable are of two types viz (i) Discrete random variable and (ii) Continuous random variable.

8.2.1 : Discrete random variable

A random variable X which can assume countable number of isolated values (integers) is called a discrete random variable.

Examples :

(i) Number of children in a family
(ii) Number of attempts to hit the target
(iii) Number of heads in tossing a coin thrice
(iv) Number of stars in the sky
(v) Profit or loss made by an investor in a day.

Remarks :

1. A discrete random variable X assumes value which are non-negative whole numbers. [See examples (i) to (iv)]
2. A discrete random variable X can assume positive, negative and fractional isolated values [See example (v)]
3. Values of a discrete random variable have to jump from one possible value to the next and it cannot have any intermediate value.

8.2.2 : Continuous random variable

A random variable X which can assume all real values within a given interval is called a continuous random variable.

Examples :

(i) Price of a commodity
(ii) Age of a person
(iii) Life of an electric component (in hours)
(iv) Speed of a vehicle in km/hour
(v) Temperature of a city in degree celsius.

Remarks :

1. A continuous random variable X assumes any value in the given interval of the values. i.e., it moves continuously from one possible value to another without jump.
2. Values of a continuous random variable X are obtained by measurement.
3. The study of a continuous random variable requires calculus method such as integration and differentiation.

[**Notes :**

(1) Values of a discrete random variable are obtained by counting while that of a continuous random variable are obtained by measurement.

(2) The study of a discrete random variable requires only simple mathematical tools such as summation and difference, while the study of a continuous variable requires calculus methods such as integration and differentiation.]

8.3 : PROBABILITY DISTRIBUTION OF A DISCRETE RANDOM VARIABLE

- **Probability Distribution :** If X is a discrete random variable assuming the values in the range $\{x_1, x_2, x_3, ..., x_n\}$ with corresponding probabilities p_i $(i = 1, 2, 3, ..., n)$, then the set of ordered pairs (x_i, p_i), $i = 1, 2, 3, ..., n$ is called a probability distribution of a random variable X. Generally it is represented in tabular form as follows :

$X = x_i$	x_1	x_2	x_3	....	....	x_n	Total
$P[X = x_i] = p(x_i)$	p_1	p_2	p_3	...	...	p_n	$\Sigma p_i = 1$

Example :

If a balanced coin is tossed thrice, then

S = {TTT, TTH, THT, HTT, THH, HTH, HHT, HHH}

Let X = Number of heads

The range of X is {0, 1, 2, 3}

$$\text{Now, } p_1 = P[X = 0] = \frac{1}{8}$$

$$p_2 = P[X = 1] = \frac{3}{8}$$

$$p_3 = P[X = 2] = \frac{3}{8}$$

$$p_4 = P[X = 3] = \frac{1}{8}$$

The probability distribution of X is represented in the following table :

$X = x_i$	0	1	2	3	Total
$p_i = P[X = x_i]$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	1

8.3.1 : Probability Mass Function (p.m.f.)

Let the possible values of a discrete random variable X be denoted by $x_1, x_2, ..., x_n$ with the corresponding probabilities $p_i = P[X = x_i]$, $i = 1, 2, ..., n$. If there is a function f such that $f(x_i) = p_i = P[X = x_i]$, for all possible values of X, then 'f' is called the probability mass function (p.m.f.) of x.

[**Notes :**

(1) $0 \leqslant p_i \leqslant 1$. $i = 1, 2, 3, ..., n$

(2) $\sum_{i=1}^{n} p_i = 1$

(3) Sometimes p.m.f. may be given in the form of a function i.e.

$P[X=x] = {}^nC_x p^x, \quad x=0, 1, 2, 3, 4$
$\qquad = 0, \qquad$ otherwise

(4) $P[X=x] = p^x, \quad x=1, 2, \ldots$
$\qquad = 0, \quad$ otherwise.]

8.3.2 : Cumulative Distribution Function (c.d.f.)

Let X be a discrete random variable and its probability distribution is as follows :

$X = x_i$	x_1	x_2	...	...	x_n	Total
$f(x) = p_i = P[X=x_i]$	p_1	p_2	...	...	p_n	1

The cumulative distribution function (c.d.f.) of X is defined as follows and is denoted by $F(x)$.

$F(x) = P[X \leqslant x], \; x \in R$

$$= \sum_{x_i \leqslant x} f(x_i)$$

[**Notes :**

(1) Domain of c.d.f. is $(-\infty, \infty)$
Codomain of c.d.f. is $[0, 1]$

(2) $F(x_i) = p_1 + p_2 + \ldots + p_i, \; i = 1, 2, 3, \ldots, n$
Thus,
$F(x_1) = p_1$
$F(x_2) = p_1 + p_2$
$F(x_3) = p_1 + p_2 + p_3$
$\vdots$
$F(x_n) = p_1 + p_2 + p_3 + \ldots\ldots + p_n = 1$

(3) c.d.f. of a discrete random variable X is also represented in tabular form as follows :

$X = x_i$	$F(x_i)$
x_1	p_1
x_2	$p_1 + p_2$
x_3	$p_1 + p_2 + p_3$
$\vdots$	$\vdots$
x_n	$p_1 + p_2 + \ldots + p_n$

]

(4) The c.d.f. of a discrete random variable can also be represented as follows :

$$\begin{aligned} F(x) &= 0, && x < x_1 \\ &= p_1, && x_1 \leqslant x < x_2 \\ &= p_1 + p_2, && x_2 \leqslant x < x_3 \\ &\;\vdots, && \;\vdots \\ &= p_1 + p_2 + \ldots + p_{n-1}, && x_{n-1} \leqslant x < x_n \\ &= 1, && x_n \leqslant x \end{aligned}$$

Remarks :

1. c.d.f. is often called as distribution function (d.f.)
2. In computing c.d.f. at x, we add probabilities sequentially up to $X = x$.
3. c.d.f. is defined for all $x \in R$. But since discrete r.v. takes isolated values, $F(x)$ is constant in between two successive values of X. It has jump at points x_i, $i = 1, 2, \ldots, n$ so $F(x)$ is a step function.

8.3.3 : Expected Value and Variance of a Random Variable

1. Expected Value : If X is a discrete random variable taking values $x_1, x_2, \ldots, x_n$ with the probabilities $p_1, p_2, \ldots, p_n$ respectively, then the expected value or mean of the random variable X is defined as follows and is denoted by $E(X)$ or μ.

$$\mu = E(X) = \sum_{i=1}^{n} x_i p_i = \sum_{i=1}^{n} x_i \, P[X = x_i]$$

[**Notes :**

(1) If p.m.f. of X is $p(x)$, then $E(X) = \Sigma x \cdot p(x)$

(2) $E(c) = 0$, c is constant.]

Remarks :

1. $E(X)$ is called as mathematical expectation of X.
2. The symbol used for $E(X)$ is μ i.e. $E(X) = \mu$.
3. $E(X)$ can be considered as the arithmetic mean or weighted average of the probability distribution of X.
4. $E(X)$ is considered to be the centre of gravity of the probability distribution of X.
5. The value of $E(X)$ can be positive, negative or zero.

2. Variance : If X is a discrete random variable taking values $x_1, x_2, \ldots, x_n$ with the probabilities $p_1, p_2, \ldots, p_n$ respectively and $\mu = E(X)$, then the variance of X is defined as follows and is denoted by Var(X) or

$\text{Var}(X) = \sigma^2$

$\text{Var}(X) = E(X - \mu)^2$

$$\therefore \; \text{Var}(X) = \sum_{i=1}^{n} (x_i - \mu)^2 \, p_i$$

[**Notes :**

(1) $$\begin{aligned} \text{Var}(X) &= \sum_{i=1}^{n} (x_i - \mu)^2 \, p_i \\ &= \Sigma x_i^2 p_i - 2\mu \Sigma x_i p_i + \mu^2 \Sigma p_i \\ &= \Sigma x_i^2 p_i - 2\mu^2 + \mu^2 \qquad (\because \Sigma x_i p_i = \mu, \; \Sigma p_i = 1) \end{aligned}$$

$\therefore \; \text{Var}(X) = \Sigma x_i^2 p_i - \mu^2$

$\therefore \; \text{Var}(X) = E(X^2) - [E(X)]^2$

(2) Var(X) is a measure of dispersion

$\therefore$ $\text{Var}(X) = \sigma^2$

(3) $\text{Var}(X) \geqslant 0$, always

(4) If p.m.f. of X is $p(x)$

$$\text{Var}(X) = \Sigma x^2 p(x) - \mu^2$$
$$= \Sigma x^2 p(x) - [E(X)]^2$$
$$\text{Var}(X) = \Sigma x^2 p(x) - [\Sigma x p(x)]^2$$

(5) $\text{Var}(C) = 0$ where c is constant

$\text{Var}(CX) = c^2 \text{Var}(X)$

(6) $\text{Var}(X \pm Y) = \text{Var}(X) + \text{Var}(Y)$]

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1. Let X represent the difference between number of heads and number of tails obtained when a coin is tossed 6 times. What are the possible values of X?

Solution : X = Difference between number of heads and number of tails.

A coin is tossed 6 times.

$\therefore$ number of heads and number of tails are shown in table as follows :

No. of heads	No. of tails	Difference
0	6	6
1	5	4
2	4	2
3	3	0
4	2	2
5	1	4
6	0	6

Hence, possible vlaues of X are {0, 2, 4, 6}

2. An urn contains 5 red and 2 black balls. Two balls are drawn at random. X denotes number of black balls drawn. What are the possible values of X?

Solution : 5 red and 2 black balls. Two balls are drawn at random.

X = number of black balls.

In 2 balls, drawn number of

Red balls	Black balls
2	0
1	1
0	2

Hence, the possible values of X are {0, 1, 2}.

3. Determine whether each of the following is a probability distribution. Give reasons for your answer.

(i)

x	0	1	2
$P(x)$	0.4	0.4	0.2

(ii)

x	0	1	2	3	4
$P(x)$	0.1	0.5	0.2	−0.1	0.3

(iii)

x	0	1	2
$P(x)$	0.1	0.6	0.3

(iv)

z	3	2	1	0	−1
$P(z)$	0.3	0.2	0.4	0.05	0.05

(v)

y	−1	0	1
$P(y)$	0.6	0.1	0.2

(vi)

x	0	1	2
$P(x)$	0.3	0.4	0.2

Solution :

(i)

x	0	1	2
$P(x)$	0.4	0.4	0.2

Here, $P(x) > 0$ for all values of x

$\Sigma P(x) = 0.4 + 0.4 + 0.2 = 1$

Hence, the given distribution is a probability distribution.

(ii)

x	0	1	2	3	4
$P(x)$	0.1	0.5	0.2	−0.1	0.3

Here, $P(x)$ for $x = 3$, $P(3) = -0.1 < 0$

Probability for an value of x cannot be negative.

Hence, the given distribution is not a probability distribution.

(iii)

x	0	1	2
$P(x)$	0.1	0.6	0.3

Here, $P(x) > 0$ for all values of x

$\Sigma P(x) = 0.1 + 0.6 + 0.3 = 1$

Hence, the given distribution is a probability distribution.

(iv)

z	3	2	1	0	-1
$P(z)$	0.3	0.2	0.4	0.05	0.05

Here, $P(z) > 0$ for all values of z

$\Sigma P(z) = 0.3 + 0.2 + 0.4 + 0.05 + 0.05 = 1$

Hence, the given distribution is a probability distribution.

(v)

y	-1	0	1
$P(y)$	0.6	0.1	0.2

Here, $P(y) > 0$ for all values of y

$\Sigma P(y) = 0.6 + 0.1 + 0.2$

$= 0.9 \neq 1$

Hence, the given distribution is not a probability distribution.

(vi)

x	0	1	2
$P(x)$	0.3	0.4	0.2

Here, $P(x) > 0$ for all values of x

$\Sigma P(x) = 0.3 + 0.4 + 0.2$

$= 0.9 \neq 1$

Hence, the given distribution is not a probability distribution.

4. Find the probability distribution of (i) number of heads in two tosses of a coin, (ii) number of tails in three tosses of a coin, (iii) number of heads in four tosses of a coin.

Solution :

(i) Two tosses of a coin :

Here, S = {HH, HT, TH, TT}

Let x = Number of heads

$\therefore$ HH means $x = 2$ $\quad \therefore P(x=2) = \frac{1}{4}$

HT, TH means $x = 1$ $\quad \therefore P(x=1) = \frac{2}{4}$

TT means $x = 0$ $\quad \therefore P(x=0) = \frac{1}{4}$

$\therefore$ the probability distribution of x is shown in the tabular form as follows :

$X = x$	0	1	2	Total
$P(x_i)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$	$\Sigma P(x_i) = 1$

(ii) Three tosses of a coin :

Here, S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

Let x = Number of heads

$\therefore$ probability distribution of x is obtained as follows :

$X = x$	$P_i = p_i$
0	$P(\text{TTT}) = \frac{1}{8}$
1	$P(\text{HTT, THT, TTH}) = \frac{3}{8}$
2	$P(\text{HHT, HTH, THH}) = \frac{3}{8}$
3	$P(\text{HHH}) = \frac{1}{8}$
Total	$\Sigma p_i = 1$

(iii) Four tosses of a coin :

Let x = Number of heads

4 tosses of a coin $\quad \therefore n = 4$, $n(\text{S}) = 2^4 = 16$

All possible values of x are 0, 1, 2, 3, 4.

The probability distribution is obtained as follows :

$X = x$	$P(x_i) = \frac{{}^nC_x}{n(\text{S})}$
0	$P(0) = \frac{{}^4C_0}{16} = \frac{1}{16}$
1	$P(1) = \frac{{}^4C_1}{16} = \frac{4}{16}$
2	$P(2) = \frac{{}^4C_2}{16} = \frac{6}{16}$
3	$P(3) = \frac{{}^4C_3}{16} = \frac{4}{16}$
4	$P(4) = \frac{{}^4C_4}{16} = \frac{1}{16}$
Total	$\Sigma P(x_i) = 1$

5. Find the probability distribution of the number of successes in two tosses of a die if success is defined as getting a number greater than 4.

Solution : Two tosses of a die.

$\therefore$ the sample space of the experiment is

S = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6),
(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6),
(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6),
(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6),
(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6),
(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}

Let x denotes the number greater than 4.

The possible number greater than 4 in a single throw of a pair of dice are given by (5, 5), (5, 6), (6, 5), (6, 6).

Since, a die is tossed two times 0, 1, 2 are the possible values of x.

$x = 0$, means no number greater than 4 n two tosses of a dice

$$\therefore\ P[x=0] = P\{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4) (4, 1), (4, 2), (4, 3), (4, 4)\}$$

$$= \frac{16}{36} = \frac{4}{9}$$

$x = 1$, means only one number is greater than 4 in two tosses of a die.

$$\therefore\ P(x=1) = P\{(1, 5), (1, 6), (2, 5), (2, 6), (3, 5), (3, 6), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (6, 1), (6, 2), (6, 3), (6, 4)\}$$

$$= \frac{16}{36} = \frac{4}{9}$$

$x = 2$, means two numbers are greater than 4 in two tosses of a die.

$$\therefore\ P(x=2) = P\{(5, 5), (5, 6), (6, 5), (6, 6)$$

$$= \frac{4}{36} = \frac{1}{9}$$

Therefore, the probability distribution $f(x)$ is obtained as follows :

$X = x$	0	1	2	Total
$P(x_i)$	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$	$\Sigma P(x_i) = 1$

6. A sample of 4 bulbs is drawn at random with replacement from a lot of 30 bulbs which includes 6 defective bulbs. Find the probability distribution of the number of defective bulbs.

Solution : Lot of 30 bulbs.

6 defective bulbs : 24 bulbs are non-defective.

4 bulbs are drawn at random with replacement.

Let x denote the number of defective bulbs.

$\therefore$ possible values of x are 0, 1, 2, 3, 4.

Now, p = probability that bulb is non-defective

$$= \frac{24}{30} = \frac{4}{5}$$

q = probability that bulb is defective

$$= \frac{6}{30} = \frac{1}{5} \text{ OR } 1 - p = 1 - \frac{4}{5} = \frac{1}{5}$$

$x = 0$, means all bulbs are non-defective

$$\therefore\ P(x=0) = \left(\frac{4}{5}\right)^4$$

$x = 1$, means 3 bulbs are non-defective and 1 bulb is defective.

$$\therefore\ P(x=1) = {}^4C_3 \times \left(\frac{4}{5}\right)^3\left(\frac{1}{5}\right)^1 = 4\left(\frac{4}{5}\right)^3\left(\frac{1}{5}\right)^1$$

$x = 2$, means 2 bulbs are non-defective and 2 bulbs are defective.

$$\therefore\ P(x=2) = {}^4C_2\left(\frac{4}{5}\right)^2\left(\frac{1}{5}\right)^2 = 6\left(\frac{4}{5}\right)^2\left(\frac{1}{5}\right)^2$$

$x = 3$, means 1 bulb is non-defective and 3 bulbs are defective.

$$\therefore\ P(x=3) = {}^4C_3\left(\frac{4}{5}\right)\left(\frac{1}{5}\right)^3 = 4\left(\frac{4}{5}\right)\left(\frac{1}{5}\right)^3$$

$x = 4$, means all 4 bulbs are defective.

$$\therefore\ P(x=4) = {}^4C_0\left(\frac{4}{5}\right)^0\left(\frac{1}{5}\right)^4 = \left(\frac{1}{5}\right)^4$$

$\therefore$ probability destribution of the number of defective bulbs x is obtained as follows :

x = No. of defective bulbs	$P_i = P(x_i)$
0	$\left(\frac{4}{5}\right)^4$
1	$4\left(\frac{4}{5}\right)^3\left(\frac{1}{5}\right)$
2	$6\left(\frac{4}{5}\right)^2\left(\frac{1}{5}\right)^2$
3	$4\left(\frac{4}{5}\right)\left(\frac{1}{5}\right)^3$
4	$\left(\frac{1}{5}\right)^4$

7. A coin is tossed so that the head is 3 times as likely to occur as tail. Find the probability distribution of number of tails in two tosses.

Solution : The head is 3 times as likely to occur as tail in two tosses.

For two tosses of coin S = {HH, TH, HT, TT}.

Let x denote number of tails in two tosses of a coin. Since, coin tossed twice the possible values of x are 0, 1, 2.

Probability of getting a tail in two tosses of a coin

$$q = \frac{1}{4} \text{ and } p = 1 - q = \frac{3}{4}$$

$$\therefore\ P(H) = 3 \times P(T) = \frac{3}{4}$$

Now, $x = 0$ means 0 tail and 2 heads

$$\therefore\ P(x=0) = {}^2C_0 p^2 \cdot q = 1 \times \left(\frac{3}{4}\right)^2 \left(\frac{1}{4}\right)^0 = \frac{9}{16}$$

$x = 1$, means 1 tail and 1 head

$$\therefore\ P(x=1) = {}^2C_1\ p^1 q^1 = 2 \times \frac{3}{4} \times \left(\frac{1}{4}\right) = \frac{6}{16}$$

$x = 2$, means 2 tails and 0 head

$$\therefore\ P(x=2) = {}^2C_2\ p^2 \cdot q^0 = 1 \times \left(\frac{1}{4}\right)^2 \times \left(\frac{3}{4}\right)^0 = \frac{1}{16}$$

Therefore, the probability distribution of tails is obtained as follows :

$X = x$	0	1	2
$P_i = P(x)$	$\frac{9}{16}$	$\frac{6}{16}$	$\frac{1}{16}$

8. A random variable X has the following probability distribution :

x	1	2	3	4	5	6	7
$P(x)$	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

Determine (i) k, (ii) $P(X < 3)$, (iii) $P(0 < X < 3)$ (iv) $P(X > 4)$.

Solution :

(i) Determine k :

x	1	2	3	4	5	6	7
$P(x)$	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

For the probability distribution

$\Sigma P(x) = 1$

$\therefore\ k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$

$\therefore\ 10k^2 + 9k = 1$

$\therefore\ 10k^2 + 9k - 1 = 0$

$\therefore\ 10k^2 + 10k - k - 1 = 0$

$\therefore\ 10k(k+1) - 1(k+1) = 0$

$\therefore\ (10k - 1)(k + 1) = 0$

$\therefore\ 10k - 1 = 0$ OR $k + 1 = 0$

$\therefore\ 10k = 1$ OR $k = -1$

$\therefore\ k = \frac{1}{10}$ OR $k = -1$

$k = -1$ is not possible because $P(x)$ cannot be negative.

$\therefore\ k = \frac{1}{10}.$

(ii) $P[X < 3]$:

$P[X < 3] = P[X = 1] + P[X = 2]$

$= k + 2k$

$P[X < 3] = 3k$

Putting $k = \frac{1}{10}$, we get

$P[X < 3] = 3 \times \frac{1}{10} = \frac{3}{10}.$

(iii) $P[0 < X < 3]$

$P[0 < X < 3] = P[X = 1] + P[X = 2]$

$= k + 2k$

$= 3k = 3 \times \frac{1}{10} = \frac{3}{10}$ $\left(\because k = \frac{1}{10}\right)$

(iv) $P[X > 4]$:

$P[X > 4] = P[X = 5] + P[X = 6] + P[X = 7]$

$= k^2 + 2k^2 + 7k^2 + k$

$= 10k^2 + k$

Putting $k = \frac{1}{10}$, we get

$P[X > 4] = 10 \times \left(\frac{1}{10}\right)^2 + \frac{1}{10}$

$= 10 \times \frac{1}{100} + \frac{1}{10}$

$= \frac{1}{10} + \frac{1}{10} = \frac{2}{10}$

$= \frac{1}{5} = 0.2$

9. Find the expected value and variance of X using the following p.m.f.

x	−2	−1	0	1	2
$P(x)$	0.2	0.3	0.1	0.15	0.25

Solution :

We construct the following table to calculate $E(X)$ and Var (X) :

x_i	p_i	x_ip_i	$x_i^2p_i = x_ip_i \times x_i$
-2	0.2	-0.4	0.8
-1	0.3	-0.3	0.3
0	0.1	0	0
1	0.15	0.15	0.15
2	0.25	0.50	1.00
Total	1	0.65 -0.70 $\Sigma x_ip_i = -0.05$	$\Sigma x_i^2p_i = 2.25$

$E(X) = \Sigma\, x_ip_i = -0.05$

$\text{Var}(X) = \Sigma\, x_i^2p_i - [E(X)]^2$

$= 2.25 - (-0.05)^2$

$= 2.25 - 0.0025 = 2.2475$

Hence, $E(X) = -0.05$, $\text{Var}(X) = 2.2475$

10. Find expected value and variance of X, the number on the uppermost face of a fair die.

Solution :

A fair die is thrown.

X = the number obtained on the uppermost face of the die.

$\therefore$ $X = \{1, 2, 3, 4, 5, 6\}$

$\therefore$ The p.m.f. is obtained as follows :

$X = x$	1	2	3	4	5	6
$P[X = x]$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

We construct the following table to calculate $E(X)$ and $\text{Var}(X)$:

$X = x_i$	$p_i = P[X = x_i]$	x_ip_i	$x_i^2p_i = x_ip_i \times x_i$
1	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
2	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{4}{6}$
3	$\frac{1}{6}$	$\frac{3}{6}$	$\frac{9}{6}$
4	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{16}{6}$
5	$\frac{1}{6}$	$\frac{5}{6}$	$\frac{25}{6}$
6	$\frac{1}{6}$	$\frac{6}{6}$	$\frac{36}{6}$
Total	1	$\Sigma x_ip_i = \frac{21}{6}$	$\Sigma x_i^2p_i = \frac{91}{6}$

$E(X) = \Sigma\, x_ip_i = \frac{21}{6} = \frac{7}{2}$

$\therefore$ $E(X) = 3.5$

$\text{Var}(X) = \Sigma\, x_i^2p_i - (\Sigma\, x_ip_i)^2$

$= \frac{91}{6} - \left(\frac{21}{6}\right)^2$

$= \frac{91}{6} - \frac{441}{36}$

$= \frac{546 - 441}{36}$

$= \frac{105}{36}$

$= \frac{35}{12}$

$\therefore$ $\text{Var}(X) = \frac{35}{12}$

Hence, $E(X) = 3.5$, $\text{Var}(X) = \frac{35}{12}$

11. Find the mean of number of heads in three tosses of a fair coin.

Solution : Three tosses of a fair coin are done.

$\therefore$ S = {HHH, HHT, HTH, THH, THT, HTT, TTT}

Let x be the number of heads.

The possible values of X are 1, 2, 3.

The probability distribution of X is obtained as follows :

$X = x$	p_i	x_ip_i
0	$\frac{1}{8}$	$0 \times \frac{1}{8} = 0$
1	$\frac{3}{8}$	$1 \times \frac{3}{8} = \frac{3}{8}$
2	$\frac{3}{8}$	$2 \times \frac{3}{8} = \frac{6}{8}$
3	$\frac{1}{8}$	$3 \times \frac{1}{8} = \frac{3}{8}$
Total	$\Sigma p_i = 1$	$\Sigma x_ip_i = \frac{12}{8}$

Mean number of heads :

$\mu = E(x)$

$= \Sigma\, x_ip_i$

$= \frac{12}{8} = 1.5$

12. Two dice are thrown simultaneously. If X denotes the number of sixes, find the expectation of X.

Solution : When a fair die is tossed twice, then the sample space S has $6 \times 6 = 36$ sample points.

$\therefore\ n(S) = 36$

Let X denotes the number of sixes.

$\therefore$ X can take the value 0, 1, 2.

When X = 0, i.e. no six, then

$X = \{(1, 1),(1, 2), (1, 3), (1, 4), (1, 5), (2, 1),(2, 2), (2, 3), (2, 4), (2, 5), (3, 1),(3, 2), (3, 3), (3, 4), (3, 5), (4, 1),(4, 2), (4, 3), (4, 4), (4, 5), (5, 1),(5, 2), (5, 3), (5, 4), (5, 5)\}$

$\therefore\ n(X) = 25$

$\therefore\ P[X = 0] = \dfrac{25}{36}$

When X = 1, i.e. one six, then

$X = \{(1, 6), (2, 6), (3, 6), (4, 6), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5)\}$

$\therefore\ n(X) = 10$

$\therefore\ P[X = 1] = \dfrac{10}{36}$

When X = 2, i.e. two sixes, then $X = \{(6, 6)\}$

$\therefore\ n(X) = 1$

$\therefore\ P[X = 2] = \dfrac{1}{36}$

$\therefore$ the required probability distribution is

$X = x$	0	1	2
$P[X = x]$	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$

Now, $E(X) = \Sigma x P(x)$

$$= 0 \times \frac{25}{36} + 1 \times \frac{10}{36} + 2 \times \frac{1}{36}$$

$$= 0 + \frac{10}{36} + \frac{2}{36}$$

$$= \frac{12}{36} = \frac{1}{3}$$

$\therefore\ E(X) = \dfrac{1}{3}$

---

13. Two numbers are selected at random (without replacement) from the first six positive integers. Let X denote the larger of the two numbers. Find $E(X)$.

Solution : First six positive numbers are : 1, 2, 3, 4, 5, 6.

Two numbers are selected at random without replacement.

$\therefore\ n(S) = 6^2 = 36.$

Among these which are not the larger of two numbers are 6 pairs. (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6).

$\therefore\ n(S) = 30.$

$X = x$ larger of two	X	p_i	$x_i p_i$
2	(1, 2), (2, 1)	$\frac{2}{30}$	$\frac{4}{30}$
3	(1, 3), (3, 1), (2, 3), (3, 2)	$\frac{4}{30}$	$\frac{12}{30}$
4	(1, 4), (4, 1), (2, 4), (4, 2), (3, 4), (4, 3)	$\frac{6}{30}$	$\frac{24}{30}$
5	(1, 5), (5, 1), (2, 5), (5, 2), (3, 5), (5, 3), (4, 5), (5, 4)	$\frac{8}{30}$	$\frac{40}{30}$
6	(1, 6), (6, 1), (2, 6), (6, 2) (3, 6), (6, 3), (4, 6), (6, 4) (5, 6), (6, 5)	$\frac{10}{30}$	$\frac{60}{30}$
Total		$\Sigma p_i = 1$	$\Sigma x_i p_i = \frac{140}{30}$

Now, $E(X) = \Sigma x_i p_i = \dfrac{140}{30} = \dfrac{14}{3} = 4.667.$

---

14. Let X denote the sum of the numbers obtained when two fair dice are rolled. Find the variance of X.

Solution :

Two fair dice are rolled.

$\therefore\ S = \{(1, 1), (1, 2), \ldots, (1, 6), (2, 1), (2, 2), \ldots, (2, 6), (3, 1), (3, 2), \ldots, (3, 6), (4, 1), (4, 2), \ldots, (4, 6), (5, 1), (5, 2), \ldots, (5, 6), (6, 1), (6, 2), \ldots, (6, 6)\}$

$\therefore\ n(S) = 36$

Let X = the sum of the two numbers

$\therefore$ X assumes values 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12

We construct the following table to calculate the variance of the sum i.e. Var (X) :

Sum of numbers $X = x_i$	**Favourable outcomes**	$P(X = x_i) = p_i$	$x_i p_i$	$x_i^2 p_i$
2	(1, 1)	$P(X=2)=\frac{1}{36}$	$\frac{2}{36}$	$\frac{4}{36}$
3	(1, 2), (2, 1)	$P(X=3)=\frac{2}{36}$	$\frac{6}{36}$	$\frac{18}{36}$
4	(1, 3), (3, 1), (2, 2)	$P(X=4)=\frac{3}{36}$	$\frac{12}{36}$	$\frac{48}{36}$
5	(1, 4), (4, 1), (2, 3), (3, 2)	$P(X=5)=\frac{4}{36}$	$\frac{20}{36}$	$\frac{100}{36}$
6	(1, 5), (5, 1), (2, 4), (4, 2), (3, 3)	$P(X=6)=\frac{5}{36}$	$\frac{30}{36}$	$\frac{180}{36}$
7	(1, 6), (6, 1) (2, 5), (5, 2), (3, 4), (4, 3)	$P(X=7)=\frac{6}{36}$	$\frac{42}{36}$	$\frac{294}{36}$
8	(2, 6), (6, 2), (4, 4), (3, 5), (5, 3)	$P(X=8)=\frac{5}{36}$	$\frac{40}{36}$	$\frac{320}{36}$
9	(3, 6), (6, 3), (4, 5), (5, 4)	$P(X=9)=\frac{4}{36}$	$\frac{36}{36}$	$\frac{324}{36}$
10	(4, 6), (6, 4), (5, 5)	$P(X=10)=\frac{3}{36}$	$\frac{30}{36}$	$\frac{300}{36}$
11	(5, 6), (6, 5)	$P(X=11)=\frac{2}{36}$	$\frac{22}{36}$	$\frac{242}{36}$
12	(6, 6)	$P(X=12)=\frac{1}{36}$	$\frac{12}{36}$	$\frac{144}{38}$
Total		1	$\Sigma x_i p_i$ $=\frac{252}{36}$	$\Sigma x_i^2 p_i$ $=\frac{1974}{36}$

The expected value of the sum of the two numbers : $E(X) = \Sigma\, x_i p_i = \frac{252}{36} = 7$

Variance of X :

$$\text{Var}(X) = \Sigma x_i^2 p_i - [E(X)]^2 = \frac{1974}{36} - (7)^2 = 54.8333 - 49 = 5.8333$$

15. A class has 15 students whose ages are 14, 17, 15, 14, 21, 17, 19, 20, 16, 18, 20, 17, 16, 19 and 20 years. If X denotes the age of a randomly selected student, find the probability distribution of X. Find the mean and variance of X.

Solution : Let X be the age of a randomly selected student.

There are 15 students.

The probability distribution of X is obtained as follows :

$X = x$	$P = p_i$	$x_i p_i$	$x_i^2 p_i = x_i p_i \times x_i$
14, 14	$P(14) = \frac{2}{15}$	$\frac{28}{15}$	$\frac{28}{15} \times 14 = \frac{392}{15}$
15	$P(15) = \frac{1}{15}$	$\frac{15}{15}$	$\frac{15}{15} \times 15 = \frac{225}{15}$
16, 16	$P(16) = \frac{2}{15}$	$\frac{32}{15}$	$\frac{32}{15} \times 16 = \frac{512}{15}$
17, 17, 17	$P(17) = \frac{3}{15}$	$\frac{51}{15}$	$\frac{51}{15} \times 17 = \frac{867}{15}$
18	$P(18) = \frac{1}{15}$	$\frac{18}{15}$	$\frac{18}{15} \times 18 = \frac{324}{15}$
19, 19	$P(19) = \frac{2}{15}$	$\frac{38}{15}$	$\frac{38}{15} \times 19 = \frac{722}{15}$
20, 20, 20	$P(20) = \frac{3}{15}$	$\frac{60}{15}$	$\frac{60}{15} \times 20 = \frac{1200}{15}$
21	$P(21) = \frac{1}{15}$	$\frac{21}{15}$	$\frac{21}{15} \times 21 = \frac{441}{15}$
Total	$\Sigma p_i = 1$	$\Sigma x_i p_i = \frac{263}{15}$	$\Sigma x_i^2 p_i = \frac{4683}{15}$

Mean of X :

$$\mu = E(X)$$
$$= \Sigma x_i p_i$$
$$= \frac{263}{15} = 17.5333$$

Variance of X :

$$\sigma^2 = \text{Var}(X)$$
$$= E(X^2) - [E(X)]^2$$
$$= \Sigma x_i^2 p_i - \mu^2$$
$$= \frac{4683}{15} - (17.5333)^2$$
$$= 312.2 - 307.4166 = 4.7834^*$$

[* **Note :** *Answer given in the textbook is incorrect.*]

16. 70% of the members favour and 30% oppose a proposal in a meeting. The random variable X takes the value 0 if a member opposes the proposal and the value 1 if a member is in favour. Find $E(X)$ and Var(X).

Solution : Probability that a member favours a proposal

$$= 70\% = \frac{70}{100} = 0.7$$

Probability that a member opposes a proposal

$$= 30\% = \frac{30}{100} = 0.3$$

Proposal	$X = x$	p_i	$x_i p_i$	$x_i^2 p_i = x_i p_i \times x_i$
Oppose	0	0.3	0	$0 \times 0 = 0$
Favour	1	0.7	0.7	$1 \times 0.7 = 0.7$
Total		$\Sigma p_i = 1$	$\Sigma x_i p_i = 0.7$	$\Sigma x_i^2 p_i = 0.7$

$E(X) = \Sigma x_i p_i = 0.7$

$$\begin{aligned} \text{Var}(X) &= \Sigma x_i^2 p_i - [E(X)]^2 \\ &= 0.7 - (0.7)^2 \\ &= 0.7 - 0.49 \\ &= 0.21 \end{aligned}$$

EXAMPLES FOR PRACTICE : 8.1

1. A page in a book can have at most 300 words. Write all possible values of a random variable which shows the number of misprints on a page.

2. There are ten cars in a showroom of which seven are petrol cars. A car is selected at random. Write all possible values of a random variable that selected car having diesel engine.

3. Obtain the probability distribution of number of sixes in two tosses of a fair die.

4. Verify whether the following can be regarded as p.m.f. probability distribution of X for the given values of X :

(i)

$X = x$	0	1	2	3	4
$P(X = x)$	0.4	0.3	0.05	0.15	0.10

(ii)

$X = x$	-2	-1	0	1
$P(X = x)$	0.08	-0.4	0.5	0.18

(iii)

$X = x$	2	4	6	8	10
$P(X = x)$	0.32	0.18	0.10	0.25	0.05

(iv) $p(x) = \frac{1}{31}\,{}^5C_{x-1}$, $x = 1, 2, 3, 4, 5$

$= 0$, otherwise

(v) $p(x) = \frac{2x+1}{15}$ $x = 0, 1, 2, 3.$

5. For the following probability distribution of X, find the value of k :

$X = x$	0	1	2	3
$P(X = x)$	k	$\frac{k}{2}$	$\frac{k}{4}$	$2k$

6. A random variable X has the following probability distribution :

$X = x$	-1	0	1	2
$P(X = x)$	0	$\frac{1}{2}$	k^2	$\frac{k}{2}$

Find (i) the value of k,
(ii) $P(X < 2)$, $P(1 \leqslant X \leqslant 2)$.

7. For the following probability distribution of X, find (i) value of k (ii) $P(8 < X < 17)$.

$X = x$	5	8	11	14	17	20
$P(X = x)$	0.12	$k + 0.03$	$2k$	0.35	0.16	$k - 0.06$

8. The probability distribution of a random variable X is defined as follows :

$$\begin{aligned} P(X = x) = p(x) &= 0.16, & x &= 0 \\ &= ax, & x &= 1 \\ &= a(5 - x), & x &= 2, 3, 4 \end{aligned}$$

Find (i) the value of a,
(ii) the probability that X assumes an odd number.

9. In a packet of 8 screws, the number of defective screws is 2. From the box 2 screws are drawn together at random. Obtain the probability distribution of the number of defective screws.

10. Find $E(X)$ and $\text{Var}(X)$ of a random variable X having p.m.f. given below :

$X = x$	-1	0	1	2
$P(X = x)$	0.1	0.3	0.4	0.2

11. Find mean and standard deviation of X having probability distribution as given below :

$X = x$	0	1	2	4
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$

12. A discrete random variable X takes values -1, 0 and 2 with probabilities $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$ respectively. Find Var (X).

13. Discrete random variable X has p.m.f.

$p(x) = k, \ x = 1, 3, 5, 7$

$\quad = 0$, otherwise

Find $E(X)$ and $\text{Var}(X)$.

14. A fair coin is tossed 3 times. If X denotes the number of heads obtained, find $E(X)$.

15. A random variable X has p.m.f.

$p(x) = \frac{x}{15}, \ x = 1, 2, 3, 4, 5$

$\quad = 0$, otherwise

Find $E(X)$ and $\text{Var}(X)$.

16. A fair dice is rolled. A person receives ₹ 5 for each odd number obtained on a die and ₹ 10 for each even number obtained on a die. Find the expected gain of the person.

17. The probability distribution of a random variable X is as follows :

x	2	4	6	8	10
$p(x)$	k	$6k$	$3k$	$2k$	$12k$

Find k, $E(X)$ and $\text{Var}(X)$.

18. Find mean, variance and standard deviation of a random variable having its probability distribution as follows :

$X = x$	-2	-1	0	1	2	3
$p(x)$	$1.2k$	0.2	$0.4k$	0.1	0.05	k

19. A box contains 8 tickets numbered 1 to 8. A ticket is drawn at random from the box. If X denotes the number on the ticket drawn, write the probability distribution of X.

Hence, find (i) $P(X > 6)$ (ii) $P(X \leqslant 2)$ (iii) $P(2 < X < 7)$ (iv) $P(X > 7)$.

20. It is known that a box of 8 batteries contains 3 defective pieces and a person randomly selects 2 batteries from this box. Find the probability distribution of the number of defective batteries.

Answers

1. $X = \{1, 2, 3, ..., 300\}$

2. $X = \{1, 2, 3\}$

3.

$X = x$	0	1	2
$P(x_i)$	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$

4. (i) Yes (ii) No (iii) No (iv) Yes (v) No. **5.** $k = \frac{4}{15}$

6. (i) $k = \frac{1}{2}$ (ii) $\frac{3}{4}, \frac{1}{2}$ **7.** (i) $k = 0.10$ (ii) 0.55

8. (i) $a = 0.12$ (ii) 0.36

9.

$X = x$	0	1	2
$P(x_i)$	$\frac{15}{28}$	$\frac{12}{28}$	$\frac{1}{28}$

10. $E(X) = 0.7$, $\text{Var}(X) = 0.81$ **11.** Mean $= \frac{5}{3}$, S.D. $= 1.247$

12. $\frac{19}{16}$ **13.** $k = \frac{1}{4}$, $E(X) = 4$, $\text{Var}(X) = 5$

14. 1.5 **15.** $E(X) = \frac{11}{3}$, $\text{Var}(X) = \frac{14}{9}$

16. ₹ 7.5

17. $k = \frac{1}{24}$, $E(X) = 7.5$, $\text{Var}(X) = 7.75$

18. $k = \frac{1}{4}$, mean $= 0.15$, variance $= 3.9275$, S.D. $= 1.9818$

19.

$X = x$	1	2	3	4	6	7	8
$P(x_i)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

(i) $\frac{1}{4}$ (ii) $\frac{1}{4}$ (iii) $\frac{1}{2}$ (iv) $\frac{1}{8}$.

20.

$X = x$	0	1	2
$P(X = x)$	$\frac{10}{28}$	$\frac{15}{28}$	$\frac{3}{28}$

8.4 : PROBABILITY DISTRIBUTION OF A CONTINUOUS RANDOM VARIABLE

The probability distribution of a continuous random variable is represented by a continuous function called the probability density function (p.d.f.).

8.4.1 : Probability Density Function (p.d.f.)

Let X be a continuous random variable taking the values in the interval (a, b). The probability density function (p.d.f.) of X is an integrable function f that satisfies the following conditions :

(1) $f(x) \geqslant 0$ for all $x \in (a, b)$

(2) $\int_a^b f(x)\,dx = 1$

(3) For any real numbers c and d such that $a \leqslant c < d \leqslant b$.

$$P[X \in (c, d)] = \int_c^d f(x)\,dx$$

[Notes :

(1) p.d.f. of a continuous random variable is different from the p.m.f. of a discrete random variable.

(2) Both p.m.f. and p.d.f. are positive at possible values of the random variable. However, the p.d.f. is positive over an entire interval.]

8.4.2 : Cumulative Distribution Function (c.d.f.)

The cumulative distribution function (c.d.f.) of a continuous random variable x is denoted by F and is defined by

$F(x) = 0$ for all $x \geqslant a$

$= \int_a^x f(x)\,dx$ for all $x \geqslant a$

[Notes :

(1) It is a non-decreasing continuous function.

(2) c.d.f. of a discrete random variable is a step function while the c.d.f. of a continuous random variable is a continuous function.]

EXERCISE 8.2	Textbook pages 144 and 145

1. Check whether each of the following is a p.d.f.

(i) $f(x) = \begin{cases} x & \text{for } 0 \leqslant x \leqslant 1, \\ 2 - x & \text{for } 1 < x \leqslant 2. \end{cases}$

(ii) $f(x) = 2$ **for** $0 < x < 1$.

Solution :

(i) $f(x) = \begin{cases} x & \text{for } 0 \leqslant x \leqslant 1, \\ 2 - x & \text{for } 1 < x \leqslant 2. \end{cases}$

$f(x)$ is p.d.f. of X if,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

Now, $\int_0^1 x\,dx + \int_1^2 (2 - x)\,dx$

$$= \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2$$

$$= \left(\frac{1}{2} - 0\right) + \left[(4-2) - \left(2 - \frac{1}{2}\right)\right]$$

$$= \frac{1}{2} + \left(2 - 2 + \frac{1}{2}\right)$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

Hence, $f(x) = x, \quad 0 \leqslant x \leqslant 1$

$= 2 - x, \quad 1 \leqslant x \leqslant 2$

is pdf of random variable X.

(ii) $f(x) = 2, 0 < x < 1$

For p.d.f. of X, we have

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

Now, $\int_0^1 2\,dx = [2x]_0^1$

$= 2 - 0$

$= 2 > 1$

Hence, $f(x) = 2$, $0 < x < 1$ is not p.d.f. of x.*

[*Note : *Answer given in the textbook is incorrect.*]

2. The following is the p.d.f. of a r.v. X.

$$f(x) = \begin{cases} \frac{x}{8} & \text{for } 0 < x < 4 \\ 0 & \text{otherwise} \end{cases}$$

Find (i) $P(x < 1.5)$, **(ii)** $P(1 < x < 2)$, **(iii)** $P(x > 2)$.

Solution :

Given p.d.f. of X is

$$f(x) = \begin{cases} \frac{x}{8} & \text{for } 0 < x < 4 \\ 0 & \text{otherwise.} \end{cases}$$

(i) $P(x < 1.5)$

$$P(x < 1.5) = \int_0^{1.5} \frac{x}{8}\, dx$$

$$= \left[\frac{x^2}{16}\right]_0^{1.5}$$

$$= \frac{(1.5)^2}{16} - 0$$

$$= \frac{2.25}{16}$$

Hence, $P[x < 1.5]$ is $\frac{2.25}{16}$

(ii) $P(1 < x < 2)$

$$P(1 < x < 2) = \int_1^2 \frac{x}{8}\, dx$$

$$= \left[\frac{x^2}{16}\right]_1^2$$

$$= \frac{4}{16} - \frac{1}{16}$$

$$= \frac{3}{16}$$

Hence, $P(1 < x < 2)$ is $\frac{3}{16}$.

(iii) $P(x > 2)$

$$P(x > 2) = \int_2^4 \frac{x}{8}\, dx = \left[\frac{x^2}{16}\right]_2^4$$

$$= \frac{16}{16} - \frac{4}{16}$$

$$= \frac{12}{16} = \frac{3}{4}$$

Hence, $P(x > 2)$ is $\frac{3}{4}$

3. It is felt that error in measurement of reaction temperature (in celesus) in an experiment is a continuous r.v. with p.d.f.

$$f(x) = \begin{cases} \frac{x^3}{64} & \text{for } 0 \leqslant x \leqslant 4, \\ 0 & \text{otherwise} \end{cases}$$

(i) Verify whether $f(x)$ is a p.d.f.

(ii) Find $P(0 < x \leqslant 1)$.

(iii) Find probability that X is between 1 and 3.

Solution :

$$f(x) = \begin{cases} \frac{x^3}{64} & \text{for } 0 \leqslant x \leqslant 4, \\ 0 & \text{otherwise} \end{cases}$$

(i) $f(x)$ is a p.d.f. if $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$\text{Now, } \int_0^4 \frac{x^3}{64}\, dx = \left[\frac{x^4}{256}\right]_0^4$$

$$= \frac{256}{256} - 0 = 1$$

$$\text{Hence, } f(x) = \begin{cases} \frac{x^3}{64} & \text{for } 0 \leqslant x \leqslant 4, \\ 0 & \text{otherwise} \end{cases} \text{ is a p.d.f.}$$

(ii) $P(0 < x \leqslant 1)$:

$$P(0 < x \leqslant 1) = \int_0^1 \frac{x^3}{64}\, dx = \left[\frac{x^4}{256}\right]_0^1$$

$$= \frac{1}{256} - 0 = \frac{1}{256}$$

Hence, $P(0 < x \leqslant 1)$ is $\frac{1}{256}$.

(iii) $P(1 < x < 3)$:

$$P(1 < x < 3) = \int_1^3 \frac{x^3}{64}\, dx = \left[\frac{x^4}{256}\right]_1^3$$

$$= \frac{81}{256} - \frac{1}{256}$$

$$= \frac{80}{256} = \frac{5}{16}$$

Hence, probability that X is between 1 and 3 is $\frac{5}{16}$.

4. Find k if the following function represents the p.d.f. of a r.v. X.

(i) $f(x) = \begin{cases} kx & \text{for } 0 < x < 2, \\ 0 & \text{otherwise} \end{cases}$

Also find $P\left[\frac{1}{4} < X < \frac{1}{2}\right]$

(ii) $f(x) = \begin{cases} kx(1-x) & \text{for } 0 < x < 1, \\ 0 & \text{otherwise} \end{cases}$

Also find (a) $P\left[\frac{1}{4} < X < \frac{1}{2}\right]$, (b) $P\left[X < \frac{1}{2}\right]$.

Solution :

(i) $f(x) = \begin{cases} kx & \text{for } 0 < x < 2, \\ 0 & \text{otherwise.} \end{cases}$

$f(x)$ is a p.d.f. of a r.v. X if

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\therefore \int_{0}^{2} kx\,dx = \left[\frac{kx^2}{2}\right]_0^2 = \frac{k\cdot 4}{2} - 0 = 2k = 1$$

$$\therefore\ 2k = 1 \qquad \therefore\ k = \frac{1}{2}$$

Hence, $k = \frac{1}{2}$

Now, $P\left[\frac{1}{4} < X < \frac{1}{2}\right] = \int_{1/4}^{1/2} \frac{1}{2}x\,dx = \left[\frac{x^2}{4}\right]_{1/4}^{1/2}$

$$= \frac{\frac{1}{4}}{4} - \frac{\frac{1}{16}}{4}$$

$$= \frac{1}{16} - \frac{1}{64} - \frac{4-1}{64} = \frac{3}{64}$$

Hence, $P\left[\frac{1}{4} < X < \frac{1}{2}\right]$ is $\frac{3^*}{64}$.

[* **Note :** *Answer given in the textbook is incorrect.*]

(ii) $f(x) = \begin{cases} kx(1-x) & \text{for } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$

$f(x)$ is p.d.f. of a r.v. X if

$$\int_0^1 kx(1-x)\,dx = 1$$

$$\therefore \int_0^1 kx\,dx - \int_0^1 kx^2dx = 1$$

$$\therefore \left[\frac{kx^2}{2}\right]_0^1 - \left[\frac{kx^3}{3}\right]_0^1 = 1$$

$$\therefore \frac{k}{2} - \frac{k}{3} = 1$$

$$\therefore\ 3k - 2k = 6 \qquad \therefore\ k = 6$$

(a) $P\left[\frac{1}{4} < X < \frac{1}{2}\right] = \int_{1/4}^{1/2} 6x(1-x)\,dx$

$$= \int_{1/4}^{1/2} 6x\,dx - \int_{1/4}^{1/2} 6x^2\,dx$$

$$= \left[\frac{6x^2}{2}\right]_{1/4}^{1/2} - \left[\frac{6x^3}{3}\right]_{1/4}^{1/2}$$

$$= \left[\frac{6}{8} - \frac{6}{32}\right] - \left[\frac{6}{24} - \frac{6}{192}\right]$$

$$= \frac{24-6}{32} - \frac{48-6}{192} = \frac{18}{32} - \frac{42}{192}$$

$$= \frac{108-42}{192} = \frac{66}{192} = \frac{11}{32}$$

Hence, $P\left[\frac{1}{4} < X < \frac{1}{2}\right] = \frac{11}{32}$

(b) $P\left[X < \frac{1}{2}\right] = \int_0^{1/2} 6x\,dx - \int_0^{1/2} 6x^2\,dx$

$$= \left[\frac{6x^2}{2}\right]_0^{1/2} - \left[\frac{6x^3}{3}\right]_0^{1/2}$$

$$= \frac{6}{8} - \frac{6}{24}$$

$$= \frac{18-6}{24} = \frac{12}{24} = \frac{1^*}{2}$$

[* **Note :** *Answer given in the textbook is incorrect.*]

5. Let X be the amount of time for which a book is taken out of library by a randomly selected student and suppose that X has p.d.f.

$$f(x) = \begin{cases} 0.5x & \text{for } 0 \leqslant x \leqslant 2, \\ 0 & \text{otherwise} \end{cases}$$

Calculate (i) $P(X \leqslant 1)$, (ii) $P(0.5 \leqslant X \leqslant 1.5)$, (iii) $P(X \geqslant 1.5)$.

Solution :

(i) $P(X \leqslant 1) = \int_{-\infty}^{0} f(x)\,dx$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_0^1 f(x)\,dx$$

$$= 0 + \int_0^1 f(x)\,dx \qquad \ldots [\because f(x) = 0, \text{ for } x < 0]$$

$$= \int_0^1 0.5x\,dx = 0.5\int_0^1 x\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 = \frac{1}{4}[x^2]_0^1$$

$$= \frac{1}{4}[1-0] = \frac{1}{4}.$$

(ii) $P(0.5 \leqslant X \leqslant 1.5) = \int_{0.5}^{1.5} f(x)\,dx$

$$= \int_{0.5}^{1.5} 0.5x\,dx = 0.5\int_{0.5}^{1.5} x\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_{0.5}^{1.5} = \frac{1}{4}[x^2]_{0.5}^{1.5}$$

$$= \frac{1}{4}[2.25 - 0.25]$$

$$= \frac{2}{4} = \frac{1}{2}.$$

(iii) $P(X \geqslant 1.5) = \int_{1.5}^{\infty} f(x)\,dx$

$$= \int_{1.5}^{2} f(x)\,dx + \int_2^{\infty} f(x)\,dx$$

$$= \int_{1.5}^{2} f(x)\,dx + 0 \quad \ldots [\because f(x) = 0 \text{ for } x > 2]$$

$$= \int_{1.5}^{2} 0.5x\,dx = 0.5\int_{1.5}^{2} x\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_{1.5}^{2} = \frac{1}{4}[x^2]_{1.5}^{2}$$

$$= \frac{1}{4}[4-2.25] = \frac{1}{4}[1.75]$$

$$= \frac{1}{4}\left[\frac{175}{100}\right] = \frac{1}{4}\left[\frac{7}{4}\right] = \frac{7}{16}$$

6. Suppose X is the waiting time (in minutes) for a bus and its p.d.f. is given by

$$f(x) = \begin{cases} \frac{1}{5} & \text{for } 0 \leqslant x \leqslant 5, \\ 0 & \text{otherwise} \end{cases}$$

Find the probability that (i) waiting time is between 1 and 3 minutes, (ii) waiting time is more than 4 minutes.

Solution :

$$f(x) = \frac{1}{5} \quad \text{for } 0 \leqslant x \leqslant 5$$
$$= 0 \quad \text{otherwise.}$$

(i) Probability that waiting time is between 1 and 3 minutes.

$$P[1 < x < 3] = \int_{1}^{3} \frac{1}{5}\,dx = \left[\frac{x}{5}\right]_{1}^{3}$$

$$= \frac{3}{5} - \frac{1}{5}$$

$$= \frac{2}{5}$$

(ii) Probability that waiting time is more than 4 minutes.

$$P[4 < x < 5] = \int_{4}^{5} \frac{1}{5}\,dx = \left[\frac{x}{5}\right]_{4}^{5}$$

$$= \frac{5}{5} - \frac{4}{5}$$

$$= \frac{1}{5}$$

7. Suppose error involved in making a certain measurement is a continuous r.v. X with p.d.f.

$$f(x) = \begin{cases} k(4-x^2) & \text{for } -2 \leqslant x \leqslant 2, \\ 0 & \text{otherwise} \end{cases}$$

Compute (i) $P(X > 0)$, (ii) $P(-1 < X < 1)$, (iii) $P(X < -0.5 \text{ or } X > 0.5)$.

Solution :

$$f(x) = k(4-x^2), \quad \text{for } -2 \leqslant x \leqslant 2;$$
$$= 0, \quad \text{otherwise.}$$

To determine k :

For the p.d.f. of a r.v. X, we have

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\therefore \int_{-2}^{2} k(4-x^2)\,dx = 1$$

$$\therefore k\left[4x - \frac{x^3}{3}\right]_{-2}^{2} = 1$$

$$\therefore k\left[\left(8-\frac{8}{3}\right)-\left(-8-\frac{-8}{3}\right)\right] = 1$$

$$\therefore k\left[8-\frac{8}{3}+8-\frac{8}{3}\right] = 1$$

$$\therefore k\left[16-\frac{16}{3}\right] = 1$$

$$\therefore k\left[\frac{48-16}{3}\right] = 1$$

$$\therefore k \times \frac{32}{3} = 1$$

$$\therefore k = \frac{3}{32}$$

(i) $P[X > 0]$:

$$P[X > 0] = \int_{0}^{2} \frac{3}{32}(4-x^2)\,dx$$

$$= \frac{3}{32}\left[4x - \frac{x^3}{3}\right]_{0}^{2}$$

$$= \frac{3}{32}\left[8-\frac{8}{3}\right]$$

$$= \frac{3}{32}\left[\frac{24-8}{3}\right]$$

$$= \frac{3}{32} \times \frac{16}{3} = \frac{1}{2}$$

(ii) $P[-1 < X < 1]$:

$$P[-1 < X < 1] = \int_{-1}^{1} \frac{3}{32}(4-x^2)\,dx$$

$$= \frac{3}{32}\left[4x - \frac{x^3}{3}\right]_{-1}^{1}$$

$$= \frac{3}{32}\left[\left(4-\frac{1}{3}\right)-\left(-4-\frac{-1}{3}\right)\right]$$

$$= \frac{3}{32}\left[4-\frac{1}{3}+4-\frac{1}{3}\right]$$

$$= \frac{3}{32}\left[8-\frac{2}{3}\right]$$

$$= \frac{3}{32}\left[\frac{24-2}{3}\right]$$

$$= \frac{3}{32} \times \frac{22}{3}$$

$$= \frac{22}{32} = \frac{11}{16}$$

(iii) $P[X < -0.5 \text{ or } X > 0.5]$:

$$P[X < -0.5 \text{ or } X > 0.5]$$

$$= P[X < -0.5] \cup P[X > 0.5]$$

$$= P[X < -0.5] + P[X > 0.5]$$

$$= \int_{-2}^{-1/2} f(x)\,dx + \int_{1/2}^{2} f(x)\,dx$$

$$= \int_{-2}^{-1/2} \frac{3}{32}(4 - x^2)\,dx + \int_{1/2}^{2} \frac{3}{32}(4 - x^2)\,dx$$

$$= \frac{3}{32}\left[4x - \frac{x^3}{3}\right]_{-2}^{-1/2} + \frac{3}{32}\left[4x - \frac{x^3}{3}\right]_{1/2}^{2}$$

$$= \frac{3}{32}\left[\left(-2 + \frac{1}{24}\right) - \left(-8 + \frac{8}{3}\right)\right.$$

$$\left. + \frac{3}{32}\left(8 - \frac{8}{3}\right) - \left(2 - \frac{1}{24}\right)\right]$$

$$= \frac{3}{32}\left[\frac{-47}{24} + \frac{16}{3}\right] + \frac{3}{32}\left[\frac{16}{3} - \frac{47}{24}\right]$$

$$= \frac{3}{32}\left[\frac{-47}{24} + \frac{16}{3} + \frac{16}{3} - \frac{47}{24}\right]$$

$$= \frac{3}{32}\left[-\frac{94}{24} + \frac{32}{3}\right]$$

$$= \frac{3}{32}\left[\frac{-94 + 256}{24}\right]$$

$$= \frac{3}{32} \times \frac{162}{24}$$

$$= \frac{81}{128}$$

Hence, $k = \frac{3}{32}$, $P[X > 0] = \frac{1}{2}$.

$P[-1 < X < 1] = \frac{11}{16}$ and

$P[-0.5 < X > 0.5] = \frac{81}{128}$

8. Following is the p.d.f. of a continuous r.v. X.

$$f(x) = \begin{cases} \frac{x}{8} & \text{for } 0 < x < 4, \\ 0 & \text{otherwise} \end{cases}$$

(i) Find expression for the c.d.f. of X.

(ii) Find $F(x)$ at $x = 0.5$, 1.7, and 4*.

[* **Note :** *Question has been modified.*]

Solution :

(i) Let $F(x)$ be the c.d.f. of X

$$\therefore\ F(x) = F[X \leqslant x] = \int_{-\infty}^{x} f_x(x)\,dx$$

$$= \int_{-\infty}^{0} f_x(x)\,dx + \int_{0}^{x} f_x(x)\,dx$$

$$= 0 + \int_{0}^{x} \frac{x}{8}\,dx$$

$$= \left[\frac{x^2}{16}\right]_0^x$$

$$= \frac{x^2}{16}$$

$$\therefore\ F(x) = \frac{x^2}{16} \text{ for } 0 < x < 4$$

(ii) $F(x)$ at $x = 0.5$

$$F(0.5) = \frac{(0.5)^2}{16} = \frac{0.25}{10} = \frac{25}{100 \times 16} = \frac{1}{64}$$

$F(x)$ at $x = 1.7$:

$$F(1.7) = \frac{(1.7)^2}{16} = \frac{2.89}{16}$$

$F(x)$ at $x = 4$

$$F(4) = \frac{(4)^2}{16} = \frac{16}{16} = 1$$

9. The p.d.f. of a continuous r.v. X is

$$f(x) = \begin{cases} \frac{3x^2}{8} & \text{for } 0 < x < 2, \\ 0 & \text{otherwise} \end{cases}$$

Determine the c.d.f. of X and hence find

(i) $P(X < 1)$, (ii) $P(X < -2)$, (iii) $P(X > 0)$,

(iv) $P(1 < X < 2)$.

Solution :

$$f(x) = \frac{3x^2}{8} \quad \text{for } 0 < x < 2$$

$$= 0 \quad \text{otherwise.}$$

Let $F(x)$ be the c.d.f. of X

$$\therefore\ F(x) = P[X \leqslant x] = \int_{-\infty}^{x} f_x(x)\,dx$$

$$= \int_{-\infty}^{0} f_x(x)\,dx + \int_{0}^{x} f_x(x)\,dx$$

$$= 0 + \int_{0}^{x} \frac{3x^2}{8}\,dx$$

$$= \left[\frac{3x^3}{24}\right]_0^x = \frac{x^3}{8}$$

$$\therefore\ F(x) = \frac{x^3}{8} \text{ for } 0 < x < 2$$

(i) $P[X<1]$:

$$P[X<1]=F(1)=\frac{(1)^3}{8}=\frac{1}{8}$$

(ii) $P[X<-2]$:

$$P[X<-2]=F(-2)=0 \qquad \ldots (\because\ 0<x<2)$$

(iii) $P[X>0]$:

$$P[X>0]=F(2)-F(0)$$

$$=\frac{(2)^3}{8}+\frac{0}{8}$$

$$=\frac{8}{8}+0=1$$

(iv) $P[1<X<2]$:

$$P[1<X<2]=F(2)-F(1)$$

$$=\frac{(2)^3}{8}-\frac{(1)^3}{8}$$

$$=\frac{8}{8}-\frac{1}{8}$$

$$=\frac{7}{8}$$

10. If a r.v. X has p.d.f.

$$f(x)=\begin{cases}\dfrac{c}{x} & \text{for } 1<x<3,\\ 0 & \text{otherwise}\end{cases},\ c>0.$$

Find c, $E(X)$, and Var(X). Also find $F(x)$.

Solution :

Given p.d.f. of r.v. X

$$f(x)=\frac{c}{x},\quad 1<x<3,\ c>0$$

For p.d.f. of X, we have

$$\int_{-\infty}^{\infty} f(x)\,dx=1$$

$$\therefore\ \int_1^3 \frac{c}{x}\,dx=1$$

$$\therefore\ c\left[\log x\right]_1^3=1$$

$$\therefore\ c[\log 3-\log 1]=1$$

$$\therefore\ c\log\left(\frac{3}{1}\right)=1$$

$$\therefore\ c\log 3=1$$

$$\therefore\ c=\frac{1}{\log 3}$$

E(X) :

$$E(X)=\int_1^3 x f(x)\,dx$$

$$=\int_1^3 x\cdot\frac{1}{x\log 3}\,dx$$

$$=\int_1^3 \frac{1}{\log 3}\,dx$$

$$=\frac{1}{\log 3}[x]_1^3$$

$$=\frac{1}{\log 3}(3-1)$$

$$=\frac{2}{\log 3}$$

Hence, $E(X)=\dfrac{2}{\log 3}$

Var(X) :

$$\text{Var}(X)=\int_{-\infty}^{\infty} x^2\cdot f(x)\,dx-\left[\int_{-\infty}^{\infty} x\cdot f(x)\,dx\right]^2$$

$$=\int_1^3 x^2\cdot\frac{1}{\log 3\cdot x}\,dx-\left[\int_1^3 x\cdot\frac{1}{\log 3\cdot x}\,dx\right]^2$$

$$=\int_1^3 \frac{x}{\log 3}\,dx-\left[\frac{2}{\log 3}\right]^2$$

$$=\frac{1}{\log 3}\left[\frac{x^2}{2}\right]_1^3-\left[\frac{2}{\log 3}\right]^2$$

$$=\frac{1}{\log 3}\left[\frac{9}{2}-\frac{1}{2}\right]-\left[\frac{2}{\log 3}\right]^2$$

$$=\frac{4}{\log 3}-\frac{4}{(\log 3)^2}=\frac{4[\log 3-1]}{[\log 3]^2}$$

Hence, $\text{Var}(X)=\dfrac{4[\log 3-1]}{[\log 3]^2}$

Since $F(x)$ is a p.d.f., the c.d.f. is given by

$$F(x)=\int_0^x \frac{1}{x\log 3}\,dx=\frac{1}{\log 3}\int_0^x \frac{1}{x}\,dx=\frac{1}{\log 3}\left[\log x\right]_0^x$$

$$\therefore\ F(x)=\frac{\log x}{\log 3}$$

EXAMPLES FOR PRACTICE 8.2

1. Verify whether the following functions are p.d.f. of continuous r.v. X.

(i) $f(x)=3x^2$ for $0<x<1$

$=0$ otherwise.

(ii) $f(x)=\dfrac{1}{3}$ for $0\leqslant x\leqslant 3$

$=0$ otherwise.

(iii) $f(x)=2x$ for $0<x\leqslant 1$

$=4-4x$ for $1<x<2$

$=0$ otherwise.

(iv) $f(x)=\dfrac{x+1}{8}$ for $2<x<4$

$=0$ otherwise.

2. Show that the function $f(x)$ defined by

$$f(x)=\frac{1}{7}, \quad 1<x<\infty$$
$$=0, \quad \text{otherwise,}$$

is a p.d.f. for a r.v. X. Hence, find $P(3<X<10)$.

3. The p.d.f. of a r.v. X is defined as follows :

$$f(x)=\frac{2k}{5}, \quad -2<x<3$$
$$=0, \quad \text{otherwise,}$$

Find k, $P[X<1]$ and $P[0<X<2]$.

4. Find the c.d.f. $F(x)$ associated with the following p.d.f. $f(x)$:

$$f(x)=\frac{1}{x^3}, \quad 2<x<\infty$$
$$=0, \quad \text{otherwise.}$$

5. If the p.d.f. of a continuous random variable X is given by

$$f(x)=\frac{1}{3}, \quad 1\leqslant x\leqslant 4$$
$$=0, \quad \text{otherwise,}$$

then find $E(X)$ and $\text{Var}(X)$.

6. Following is the p.d.f. of a continuous r.v. X :

$$f(x)=3x^2, \quad 0\leqslant x\leqslant 1$$
$$=0, \quad \text{otherwise,}$$

calculate, mean and variance of X.

7. If the p.d.f. of a continuous random variable X is defined as follows.

$$f(x)=\frac{1}{2}x, \quad 1\leqslant x\leqslant 3$$
$$=0, \quad \text{otherwise,}$$

Find $E(X)$, $\text{Var}(X)$ and also obtain c.d.f. of X.

8. If the p.d.f. of continuous random variable X is

$$f(x)=\frac{x^3}{3}, \quad 0\leqslant x\leqslant 2$$
$$=0, \quad \text{otherwise,}$$

Find (i) c.d.f. of X (ii) Mean and variance of X.

9. The probability distribution function of continuous random variable X is given by

$$f(x)=\frac{x}{4}, \ 0<x<2$$
$$=0, \ \text{otherwise,}$$

find $P(x\leqslant 1)$.

10. Find mean and variance of the continuous random variable x, whose p.d.f. is given by

$$f(x)=6x(1-x), \ 0<x<1$$
$$=0, \quad \text{otherwise.}$$

11. Let X be a random variable with p.d.f.

$$f(x)=k\sqrt{x}, \quad 0<x<1$$
$$=0, \quad \text{otherwise.}$$

(i) Evaluate k.

(ii) Find $P(0.3<X<0.6)$.

12. Find k, if the function f defined by

$$f(x)=kx, \quad 0<x<2$$
$$=0, \quad \text{otherwise}$$

is the p.d.f. of a r.v. X. Also, find $P\left(\frac{1}{4}<X<\frac{1}{3}\right)$.

13. Find the c.d.f. $F(x)$ associated with the following p.d.f. $f(x)$:

$$f(x)=12x^2(1-x), \quad 0<x<1$$
$$=0, \quad \text{otherwise.}$$

Also, find $P\left(\frac{1}{3}<X<\frac{1}{2}\right)$ by using p.d.f. and c.d.f.

14. Given the p.d.f. of a continuous r.v. X as

$$f(x)=\frac{x^2}{3}, -1<x<2$$
$$=0, \ \text{otherwise.}$$

Determine the c.d.f. of X and hence find $P(X<1)$, $P(X\leqslant -2)$, $P(X>0)$, $P(1<X<2)$.

15. The following is the p.d.f. (Probability Density Function) of a continuous random variable X :

$$f(x)=\frac{x}{32}, \quad 0<x<8$$
$$=0, \quad \text{otherwise.}$$

(a) Find the expression for c.d.f. (Cumulative Distribution Function) of X.

(b) Also, find its values at $x=0.5$ and 9.

16. Suppose the life in hours of certain kind of bulbs has the probability density function

$$f(x)=\frac{100}{x^2}, \quad x\geqslant 100$$
$$=0, \quad \text{otherwise.}$$

Find the probability that the bulbs will last up to 150 hours.

Answers

1. (i) Yes (ii) Yes (iii) No (iv) Yes **2.** $\frac{5}{7}$

3. $k=\frac{1}{2}$, $P[X<1]=\frac{3}{5}$, $P[0<X<2]=\frac{2}{5}$

4. $F(x)=\frac{1}{4}-\frac{1}{2x^2}$ **5.** $E(X)=\frac{5}{2}$, $\text{Var}(X)=\frac{3}{4}$

6. Mean $=\frac{3}{4}$, Variance $=\frac{3}{80}$

7. $E(X)=\frac{13}{3}$, $\text{Var}(X)=\frac{17}{3}$, $F(x)=\frac{x^2-1}{4}$

8. (i) $F(x)=\frac{x^4}{12}$ (ii) Mean $=\frac{32}{15}$, Variance $=\frac{64}{45}$ **9.** $\frac{1}{8}$

10. Mean $=0.5$, Variance $=0.05$

11. (i) $\frac{3}{2}$ (ii) $(0.6)^{\frac{3}{2}}-(0.3)^{\frac{3}{2}}$

12. $k=\frac{1}{2}$, $P\left(\frac{1}{4}<X<\frac{1}{3}\right)=\frac{7}{576}$

13. $F(x)=4x^3-3x^4$, $P\left(\frac{1}{3}<X<\frac{1}{2}\right)=\frac{29}{144}$

14. $F(x)=\frac{x^3}{9}+\frac{1}{9}$, $P(X<1)=\frac{2}{9}$,

$P(X\leqslant -2)=0$, $P(X>0)=\frac{8}{9}$,

$P(1<X<2)=\frac{7}{9}$.

15. (a) $F(x)=\frac{x^2}{64}$ (b) $F(0.5)=\frac{1}{256}$, $F(9)=1$.

16. $\frac{1}{3}$.

8.5 : BINOMIAL DISTRIBUTION

8.5.1 : Bernoulli Trial

An experiment is dichotomous if it has only two possible outcomes one is 'success' and the other is 'failure'. For example, A manufactured item is 'defective' or 'non-defective'. Here if the occurence of 'non-defective' is considered success then occurrence of 'defective' is failure.

A dichotomous experiment is called a **Bernoulli trial.**

- **Sequence of Bernoulli Trials :** A sequence of dichotomous experiment is called a sequence of Bernoulli trials if it satisfies the following conditions :
  (1) The trials are independent.
  (2) The probability of success remains same in all trials.

[Notes :

(1) The probability of success in a Bernoulli trial is denoted by p and the probability of failure is denoted by q such that $p+q=1$ that is $q=1-p$.

(2) When drawing is done with replacement, then draws are Bernoulli trials. But if it is done without replacement, then draws are not Bernoulli trials.]

8.5.2 : Binomial Distribution

Consider a Binomial experiment. Let the outcomes in each trial be a success (S) or failure (F) with probability p and $(1-p)=q$ respectively.

Let a random variable X = Number of successes

$\therefore\ 0\leqslant X\leqslant n$.

We want to find the probability of an event that the n trials result in exactly x successes and $(n-x)$ failures.

Let one particular sequence of x consecutive successes followed by $(n-x)$ failures is

$$\underbrace{\text{S S S} \ldots \text{S}}_{x \text{ successes}} \quad \underbrace{\text{F F F} \ldots \text{F}}_{(n-x) \text{ failures}}$$

Let $A=\{\text{S S S} \ldots \text{S } x \text{ times}\}$

$B=\{\text{F F F} \ldots \text{F } (n-x) \text{ times}\}$

Since trials are independent,

$$\begin{aligned}P(A\cap B)&=P(A).P(B)\\&=P(S).P(S)\ \ldots\ x \text{ times}\times P(F).P(F) \quad \ldots (n-x) \text{ times.}\\&=p.p.p\ \ldots\ x \text{ times}\times q.q.q \quad \ldots (n-x) \text{ times}\\&=p^x(q)^{n-x}\end{aligned}$$

Now, the number of possible orders of x successes and $(n-x)$ failures in n independent trials is ${}^nC_x$ and these are all mutually disjoint orders. Hence, probability that n trials result in exactly x successes and $(n-x)$ failures in any order is given by

$P[X=x]={}^nC_x p^x q^{n-x}$, $0\leqslant x\leqslant n$

This is known as Binomial distribution.

Definition : A discrete random varaible X is said to follow a Binomial distribution with parameters n and p if its probability mass function (p.m.f.) is given by

$$\begin{aligned}P[X=x]&=p(x)\\&={}^nC_x p^x q^{n-x}, \quad && x=0, 1, 2, \ldots n\\& && 0<p<1,\ q=1-p\\& && n=1, 2, 3 \ldots\\&=0, && \text{otherwise}\end{aligned}$$

$P[X=x]$ for every x is xth term in the expansion of $(q+p)^n$.

Binomial distribution can be represented in tabular form as follows :

$X = x$	0	1	2	…	x	…	n
$p(x)$	q^n	$^nC_1pq^{n-1}$	$^nC_2p^2q^{n-2}$	…	$^nC_xp^xq^{n-x}$	…	p^n

From the above table we can see that,

(i) $p(x) \geqslant 0$ for all x

(ii) $\sum_{x=0}^{n} p(x) = \sum_{x=0}^{n} {}^nC_xp^xq^{n-x}$

$= (q+p)^n$

$= 1$

[**Notes :**

(1) $X \sim B(n, p)$ denotes the notation that X follows Binomial distribution with parameters n and p.

(2) If $X \sim B(n, p)$, $Y = (n - X)$, then $Y \sim B(n, q)$

(3) Mean $= E(X) = np$, Variance $= \text{Var}(X) = npq$

(4) Mean > Variance i.e. $np > npq$.]

- **Conditions of Binomial Distribution :**

The binomial distribution is used whenever each of the following is satisfied :

(i) Trials of experiment are Bernoulli trials.

(ii) The number of trials (n) is finite.

(iii) All n trials are independent.

(iv) The probability p of success is constant from trial to trial i.e. the trials are repeated under identical conditions.

(v) The random variable X denotes the number of successes x in n independent Bernoulli trials.

8.5.3 : Mean and Variance of Binomial Distribution

Let $X \sim B(n, p)$

Mean :

$\mu = E(x) = np$

Variance :

$\sigma^2 = \text{Var}(x) = npq$

EXERCISE 8.3 | **Textbook page 150 and 151**

1. A die is thrown 4 times. If 'getting an odd number' is a success, find the probability of (i) 2 successes, (ii) at least 3 successes, (iii) at most 2 successes.

Solution : Here, $n = 4$,

p = probability of getting odd number on a die

$= \frac{3}{6} = \frac{1}{2}$

$\therefore\ q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

Let X denote the odd numbers in throwing a die.

$\therefore\ x$ = Number of successes.

Here, $X \sim B(n, p)$ with $n = 4$, $p = \frac{1}{2}$ and $q = \frac{1}{2}$

$$\therefore\ p(X = x) = \binom{n}{x} p^x q^{n-x}$$

$$= \binom{4}{x}\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{4-x}$$

$$= \binom{4}{x}\left(\frac{1}{2}\right)^4$$

(i) 2 successes : $X = 2$

$$P(X = 2) = \binom{4}{x}\left(\frac{1}{2}\right)^4$$

$$= \frac{4!}{2!\,2!} \times \frac{1}{16}$$

$$= \frac{4 \times 3}{2} \times \frac{1}{16}$$

$$= 6 \times \frac{1}{16} = \frac{3}{8} = 0.375$$

Hence, $P(X = 2)$ is 0.375

(ii) At least 3 successes : $P(X \geqslant 3)$

i.e. $X = 3$, $X = 4$

$$\therefore\ P(X \geqslant 3) = P(3) + P(4)$$

$$= \binom{4}{3}\left(\frac{1}{2}\right)^4 + \binom{4}{4}\left(\frac{1}{2}\right)^4$$

$$= \frac{4}{16} + \frac{1}{16}$$

$$= \frac{5}{16} = 0.3125$$

Hence, $P(X \geqslant 3)$ is 0.3125.

(iii) At most 2 successes : $P(X \leqslant 2)$

$\therefore\ X = 0, 1, 2$

$$\therefore\ P(X \leqslant 2) = P(0) + P(1) + P(2)$$

$$= \binom{4}{0}\left(\frac{1}{2}\right)^4 + \binom{4}{1}\left(\frac{1}{2}\right)^4 + \binom{4}{2}\left(\frac{1}{2}\right)^4$$

$$= \frac{1}{16} + \frac{4}{16} + \frac{6}{16}$$

$$= \frac{11}{16} = 0.6875$$

Hence, $P(X \leqslant 2)$ is 0.6875.

2. A pair of dice is thrown 3 times. If getting a doublet is considered a success, find the probability of two successes.

Solution : Here, $n=3$,

$X=\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$

$\therefore$ probability of getting a doublet in throwing a pair of dice $p=\frac{6}{36}=\frac{1}{6}$

and $q=1-p=1-\frac{1}{6}=\frac{5}{6}$

Here, $X \sim B(n, p)$ with $n=3$, $p=\frac{1}{6}$ and $q=\frac{5}{6}$

$$\therefore\ P(X=x)=\binom{n}{x}p^x q^{n-x}$$

$$=\binom{3}{x}\left(\frac{1}{6}\right)^x\left(\frac{5}{6}\right)^{3-x}$$

Probability of two successes : $x=2$

$$P(X=2)=\binom{3}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)$$

$$=3\times\frac{1}{36}\times\frac{5}{6}=\frac{5}{72}$$

Hence, $P[X=2]$ is $\frac{5}{72}$

[Note : *Answer given in the textbook is incorrect.*]

3. There are 10% defective items in a large bulk of items. What is the probability that a sample of 4 items will include not more than one defective item?

Solution : Here, $n=4$, X = Defective item

p = Probability of defective item

$=10\%=\frac{10}{100}=0.1$

$\therefore\ q=1-p=1-0.1=0.9$

Now, $X \sim B(n, p)$ with $n=4$, $p=0.1$ and $q=0.9$

$$\therefore\ P(X=x)=\binom{n}{x}p^x q^{n-x}$$

$$=\binom{4}{x}(0.1)^x(0.9)^{4-x}$$

P(Not more than one defective)

$=P(x \leqslant 1)$

$=P(X=0)+P(X=1)$

$=\binom{4}{0}(0.1)^0(0.9)^4+\binom{4}{1}(0.1)(0.9)^3$

$=1\times(0.9)^4+4\times0.1\times(0.9)^3$

$=(0.9)^3[0.9+0.4]$

$=(0.9)^3(1.3)=1.3(0.9)^3$

4. Five cards are drawn successively with replacement from a well-shuffled deck of 52 cards. Find the probability that

(i) all the five cards are spades.

(ii) only 3 cards are spades.

(iii) none is a spade.

Solution : Here, $n=5$, X = card of spade

p = Probability that card is spade

$=\frac{1}{13}$ ... ($\because$ spade cards = 13)

$\therefore\ q=1-p=1-\frac{1}{13}=\frac{12}{13}$

Now, $X \sim B\left(5, \frac{1}{13}\right)$

$$\therefore\ P(X=x)=\binom{n}{x}p^x q^{n-x}$$

$$=\binom{5}{x}\left(\frac{1}{13}\right)^x\left(\frac{12}{13}\right)^{5-x}$$

(i) P[All the five cards are spades]

$=P[X=5]$

$=\binom{5}{5}\left(\frac{1}{13}\right)^5\left(\frac{12}{13}\right)^0$

$=1\times\frac{1}{(13)^5}$

$\therefore\ P(X=5)=\frac{1}{(13)^5}$

(ii) P[Only 3 cards are spades]

$=P[X=3]$

$=\binom{5}{3}\left(\frac{1}{13}\right)^3\left(\frac{12}{13}\right)^2$

$=10\times\frac{1}{(13)^3}\frac{(12)^2}{(13)^2}$

$=\frac{10\times(12)^2}{(13)^5}$

(iii) P[None card is spade]

$=P[X=0]$

$=\binom{5}{0}\left(\frac{1}{13}\right)^0\left(\frac{12}{13}\right)^5=1\times\left(\frac{12}{13}\right)^5=\left(\frac{12}{13}\right)^5$

[Note : *Answers given in the textbook are incorrect.*]

5. The probability that a bulb produced by a factory will fuse after 200 days of use is 0.2. Let X denote the number of bulbs (out of 5) that fuse after 200 days of use. Find the probability of

(i) $X=0$, (ii) $X \leqslant 1$, (iii) $X > 1$, (iv) $X \geqslant 1$.

Solution : Here, $n=5$,

$X=$ Number of bulbs fuse after 200 days

$p=$ probability that bulb fuses $=0.2$

$\therefore\ q=1-p=1-0.2=0.8$

Now, $X \sim B(5, 0.2)$

$$\therefore\ P(X=x)=\binom{n}{x}p^x q^{n-x}$$

$$=\binom{5}{x}(0.2)^x(0.8)^{5-x}$$

(i) $P(X=0)=\binom{5}{0}(0.2)^0(0.8)^5$

$$=1\times 1\times (0.8)^5$$

$$=(0.8)^5$$

(ii) $P[X\leqslant 1]=P(X=0)+P(X=1)$

$$=(0.8)^5+\binom{5}{1}(0.2)^1(0.8)^4$$

$$=(0.8)^5+5\times 0.2\times (0.8)^4$$

$$=(0.8)^4[0.8+1]$$

$$=1.8(0.8)^4$$

(iii) $P[X>1]=1-P[X\leqslant 1]$

$$=1-1.8(0.8)^4$$

(iv) $P[X\geqslant 1]=1-P[X=0]$

$$=1-(0.8)^5$$

6. 10 balls are marked with digits 0 to 9. If 4 balls are selected with replacement, what is the probability that none is marked 0?

Solution : Here, $n=4$,

$X=\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$p=\frac{1}{10}\qquad \therefore\ q=1-p=1-\frac{1}{10}=\frac{9}{10}$

$\therefore\ X\sim B\left(4, \frac{1}{10}\right)$

$$\therefore\ P(X=x)=\binom{n}{x}p^x q^{n-x}$$

$$=\binom{4}{x}\left(\frac{1}{10}\right)^x\left(\frac{9}{10}\right)^{4-x}$$

$P[\text{None is marked } 0]=P[X=0]$

$$=\binom{4}{0}\left(\frac{1}{10}\right)^0\left(\frac{9}{10}\right)^4$$

$$=1\times 1\times\left(\frac{9}{10}\right)^4$$

$$=\left(\frac{9}{10}\right)^4$$

7. In a multiple choice test with three possible answers for each of the five questions, what is the probability of a candidate getting four or more correct answers by random choice?

Solution : Here, $n=5$, $X=$ Number of correct answers

$p=$ probability of correct answer

$$=\frac{1}{3}$$

$\therefore\ q=1-p=1-\frac{1}{3}=\frac{2}{3}$

Now, $X\sim B\left(5, \frac{1}{3}\right)$

$$\therefore\ P(X=x)=\binom{n}{x}p^x q^{n-x}$$

$$=\binom{5}{x}\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{5-x}$$

$P[\text{Four or more correct answers}]$

$=P[X\geqslant 4]$

$=P[X=4]+P[X=5]$

$$=\binom{5}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)+\binom{5}{5}\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^0$$

$$=5\times\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)+1\times\left(\frac{1}{3}\right)^5$$

$$=\frac{10}{243}+\frac{1}{243}$$

$$=\frac{11}{243}.$$

8. Find the probability of throwing at most 2 sixes in 6 throws of a single die.

Solution : Here, $n=6$, $X=$ Number of sixes

$p=\frac{1}{6}\quad \therefore\ q=1-p=1-\frac{1}{6}=\frac{5}{6}$

Now, $X\sim B\left(6, \frac{1}{6}\right)$

$$\therefore\ P(X=x)=\binom{n}{x}p^x q^{n-x}$$

$$=\binom{6}{x}\left(\frac{1}{6}\right)^x\left(\frac{5}{6}\right)^{n-x}$$

$P[\text{at most 2 sixes}]$

$=P[X\leqslant 2]$

$=P(X=0)+P(X=1)+P[X=2]$

$$=\binom{6}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^6+\binom{6}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^5+\binom{6}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^4$$

$$= 1 \times 1 \times \left(\frac{5}{6}\right)^6 + 6 \times \frac{1}{6} \times \left(\frac{5}{6}\right)^5 + 15 \times \frac{1}{36} \times \left(\frac{5}{6}\right)^4$$

$$= \left(\frac{5}{6}\right)^4 \left[\left(\frac{5}{6}\right)^2 + \frac{5}{6} + \frac{15}{36}\right]$$

$$= \left(\frac{5}{6}\right)^4 \left[\frac{25}{36} + \frac{5}{6} + \frac{15}{36}\right]$$

$$= \left(\frac{5}{6}\right)^4 \left[\frac{25+30+15}{36}\right]$$

$$= \left(\frac{5}{6}\right)^4 \left[\frac{70}{36}\right] = \left(\frac{70}{36}\right)\left(\frac{5}{6}\right)^4$$

[**Note :** *Answer given in the textbook is incorrect.*]

9. Given that $X \sim B(n, p)$,

(i) if $n = 10$ and $p = 0.4$, Find $E(X)$ and $Var(X)$.

(ii) if $p = 0.6$ and $E(X) = 6$, find n and $Var(X)$.

(iii) if $n = 25$, $E(X) = 10$, find p and $Var(X)$.

(iv) if $n = 10$, $E(X) = 8$, find $Var(X)$.

Solution :

(i) Given : $n = 10$, $p = 0.4$

$\therefore\ q = 1 - p = 1 - 0.4 = 0.6$

$E(X) = \mu = np$

$= 10 \times 0.4 = 4$

$Var(X) = npq$

$= 10 \times 0.4 \times 0.6 = 2.4$

(ii) Given : $p = 0.6$

$\therefore\ q = 1 - p = 1 - 0.6 = 0.4,\ E(X) = 6$

Now, $E(X) = \mu = np$

$\therefore\ 6 = n \times 0.6$

$\therefore\ n = \frac{6}{0.6} = 10^*.$

$Var(X) = npq$

$= 10 \times 0.6 \times 0.4 = 2.4$

(iii) Given : $n = 25$, $E(X) = 10$

$E(X) = 10$

$\therefore\ np = 10$

$\therefore\ 25p = 10$

$\therefore\ p = \frac{10}{25} = 0.4^*$

$\therefore\ q = 1 - p = 1 - 0.4 = 0.6$

$Var(X) = npq$

$= 25 \times 0.4 \times 0.6$

$= 10 \times 0.6 = 6^*$

(iv) Given : $n = 10$, $E(X) = 8$

$E(X) = np \qquad \therefore\ 8 = 10p$

$\therefore\ p = \frac{8}{10} \qquad \therefore\ q = 1 - \frac{8}{10} = \frac{2}{10}$

$Var(X) = npq$

$= 10 \times \frac{8}{10} \times \frac{2}{10}$

$= \frac{16}{10} = 1.6^*$

[* **Note :** *Answers given in the textbook are incorrect.*]

EXAMPLES FOR PRACTICE 8.3

1. If for a binomial distribution, number of trials is 9, the variance is 2, and the probability of success is greater than that of failure, find the probability of both (a) success (b) failure.
2. In a certain factory there are 25% unskilled workers. Find the probability that in a sample of 5 workers selected from this factory exactly 4 are unskilled workers.
3. X is a binomial variate with $E(X) = 4$ and standard deviation of $X = \sqrt{3}$. Find the values of n, p and q.
4. Six fair coins are tossed simultaneously. Find the probability of getting exactly three heads.
5. On an average A can solve 60% the problems. What is the probability of A solving exactly 5 problems out of 6?
6. A coin is tossed 4 times. Find the probability of getting exactly 3 heads.
7. In a binomial distribution, mean and variance are 12 and 4 respectively. Find the parameters of the distribution.
8. For a binomial distribution, probability of success is 1/4 and the mean is 12.5, find the remaining parameters of the distribution.
9. The probability that A wins a game of chess against B is 2/3. Find the probability that A wins at least one game out of 4 games he plays against B.
10. For a binomial distribution, the number of trials is 5 and $P(X = 4) = P(X = 3)$. Find the probability of success and also obtain $P(X > 2)$.
11. Find the probability of guessing correctly at least eight of the ten answers in a True or False objective test.

12. The probability that a bomb dropped from an aeroplane will strike a target is $\frac{1}{5}$. If four bombs are dropped, find the probability that :

(a) exactly two will strike the target.

(b) at least one will strike the target.

13. A card is drawn at random and replaced four times from a well shuffled pack of 52 cards. Find the probability that –

(a) two diamond cards are drawn

(b) at least one diamond card is drawn.

14. For a binomial distribution, mean is 6 and variance is 2. Find n and p.

15. If a fair die is rolled four times, what is the probability that two of the rolls will show '1'?

16. Find the probability of guessing correctly at most three of the seven answers in a True or False objective test.

17. If X has binomial distribution with $n=20$, $p=\frac{1}{10}$. Find $E(X)$ and $\text{Var}(X)$.

Answers

1. (a) $\frac{2}{3}$ (b) $\frac{1}{3}$ **2.** $\frac{15}{1024}$ **3.** $n=16, p=\frac{1}{4}, q=\frac{3}{4}$

4. $\frac{5}{16}$ **5.** $\frac{2916}{15625}$ **6.** $\frac{1}{4}$ **7.** $n=18, p=\frac{2}{3}$

8. $n=50, q=\frac{3}{4}=0.75$ **9.** $\frac{80}{81}=0.988$

10. $p=\frac{2}{3}, P(X>2)=0.790$ **11.** $\frac{7}{128}$ **12.** (a) $\frac{96}{625}$ (b) $\frac{369}{625}$

13. (a) 0.2109 (b) 0.6836 **14.** $n=9,\ p=\frac{2}{3}$ **15.** 0.12

16. $\frac{1}{2}$ **17.** $E(X)=2, \text{Var}(X)=1.8.$

8.6 : POISSON DISTRIBUTION

Poisson distribution is very important discrete probability distribution for countably infinite sample space.

If n is very large and p is very small such that $np=m$ is constant, then we can derive Poisson distribution from Binomial distribution by taking $m=np$.

Poisson distribution is named after its founder, the French mathematician Simeon D. Poisson (1781–1840). It was applied to know the probability of deaths in the Persian army resulting from the kick of a horse. Also, it was applied to know the probability of the number of suicides among women and children. Thus, this is a discrete probability distribution for rare happenings.

Definition :

A discrete random variable X is said to follow Poisson distribution with parameter m if its probability mass function $p(x)$ is given by

$$P[X=x]=p(x)=\frac{e^{-m}m^x}{x!},\ x=0,1,2,\ \dots$$

$$m=np>0$$

$$=0 \qquad \text{, otherwise}$$

where $e=$ constant $=2.7183$

[**Notes :**

(1) $X \sim P(m)$ is the notation denoting X follows Poisson distribution with parameter 'm'.

(2) $p(x) \geqslant 0$, for all x and

$$\sum_{x=0}^{\infty} p(x)=\sum_{x=0}^{\infty}=\frac{e^{-m}\ m^x}{x!}$$

$$=e^{-m}+e^{-m}.m+\frac{e^{-m}\ m^2}{2!}+\frac{e^{-m}\ m^3}{3!}+\dots$$

$$=e^{-m}\left[1+m+\frac{m^2}{2!}+\frac{m^3}{3!}+\dots\right]$$

$$=e^{-m}\times e^m=e^0$$

$$=1 \quad \boxed{\therefore\ e^m=1+m+\frac{m^2}{2!}+\frac{m^2}{3!}=\dots}$$

(3) Mean $=E(X)=m$ and Variance $=\text{Var}(X)=m$

$\therefore$ Mean $=$ Variance $=m$.]

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1. If X has Poisson distribution with m = 1, then find $P(X \leqslant 1)$ given $e^{-1}=0.3678$.

Solution : $X \sim p(m=1)$, $e^{-1}=0.3678$

$$\therefore\ P[X=x]=\frac{e^{-m}m^x}{x!}$$

$$=\frac{e^{-1}(1)^x}{x!}=0.3678\times\frac{1}{x!}$$

$$P[X \leqslant 1]=P[X=0]+P[X=1]$$

$$=0.3678\times\frac{1}{0!}+0.3678\times\frac{1}{1!}$$

$$=0.3678+0.3678$$

$$=0.7356$$

Hence, $P[X \leqslant 1]$ is 0.7356.

[**Note :** *Answer given in the textbook is incorrect.*]

2. If $X \sim P\left(\frac{1}{2}\right)$, then find $P(X=3)$ given $e^{-0.5}=0.6065$.

Solution :

Given : $X \sim P\left(\frac{1}{2}\right)$, $e^{-0.5}=0.6065$

$$\therefore\ P[X=x]=\frac{e^{-m}m^{x}}{x!}$$

$$=\frac{e^{-0.5}(0.5)^{x}}{x!}$$

$$=0.6065\times\frac{(0.5)^{x}}{x!}$$

Now, $P[x=3]=0.6065\times\frac{(0.5)^{3}}{3!}$

$$=0.6065\times\frac{0.125}{6}$$

$$=\frac{0.0758}{6}$$

$$=0.0126$$

Hence, $P(X=3)$ is 0.0126.

3. If X has Poisson distribution with parameter m and $P(X=2)=P(X=3)$, then find $P(X\geqslant 2)$. Use $e^{-3}=0.0497$.

Solution : $X \sim P(m)$

$$\therefore\ P[X=x]=\frac{e^{-m}m^{x}}{x!}$$

$\therefore\ P[X=2]=\frac{e^{-m}m^{2}}{2!}$ and

$$P[X=3]=\frac{e^{-m}m^{3}}{3!}$$

Now, $P(X=2)=P(X=3)$

$$\therefore\ \frac{e^{-m}m^{2}}{2!}=\frac{e^{-m}m^{3}}{3!}$$

$$\therefore\ \frac{m^{2}}{2}=\frac{m^{3}}{6}$$

$$\therefore\ \frac{1}{2}=\frac{m}{6}$$

$$\therefore\ \frac{6}{2}=m$$

$$\therefore\ m=3$$

Now, $P[X\geqslant 2]=1-P(X<2)$

$$=1-[P(X=0)+P(X=1)]$$

$$=1-\left[\frac{e^{-3}(3)^{0}}{0!}+\frac{e^{-3}(3)^{1}}{1!}\right]$$

$$=1-e^{-3}[1+3]$$

$$=1-0.0497(4)$$

$$=1-0.1988$$

$$=0.8012$$

Hence, $P[X\geqslant 2]$ is 0.8012

[**Note :** *Answer given in the textbook is incorrect.*]

4. The number of complaints which a bank manager receives per day follows a Poisson distribution with parameter m = 4. Find the probability that the manager receives (i) only two complaints on a given day, (ii) at most two complaints on a given day. Use $e^{-4}=0.0183$.

Solution : Here, X = Number of complaints a bank manager receives per day

$X \sim P(m=4)$, $e^{-4}=0.0183$

$$\therefore\ P[X=x]=\frac{e^{-m}m^{x}}{x!}$$

$$=\frac{e^{-4}(4)^{x}}{x!}$$

$$=0.0183\times\frac{(4)^{x}}{x!}$$

(i) P[Only two complaints on a given day]

$$=P[X=2]$$

$$=0.0183\times\frac{(4)^{2}}{2!}$$

$$=0.0183\times\frac{16}{2}$$

$$=0.0183\times 8$$

$$=0.1464$$

Hence, probability that only two complaints on a given day is 0.1464.

(ii) P[At most two complaints on a given day]

$$=P[X\leqslant 2]$$

$$=P[X=0]+P[X=1]+P[X=2]$$

$$=0.0183\times\frac{4^{0}}{0!}+0.0183\times\frac{(4)^{1}}{1!}+0.0183\frac{(4)^{2}}{2!}$$

$$=0.0183\,(1+4+8)$$

$$=0.0183\times 13=0.2379$$

Hence, probability that at most two complaints on a given day is 0.2379.

[**Note :** *Answers given in the textbook are incorrect.*]

5. A car firm has 2 cars, which are hired out day by day. The number of cars hired on a day follows Poisson distribution with mean 1.5. Find the probability that

(i) no car is used on a given day, (ii) some demand is refused on a given day, given $e^{-1.5} = 0.2231$.

Solution : X = Number of cars hired on a day

$X \sim P(m = 1.5)$, $e^{-1.5} = 0.2231$

$$\therefore P[X = x] = \frac{e^{-m}m^x}{x!}$$

$$= \frac{e^{-1.5}(1.5)^x}{x!}$$

$$= 0.2231 \times \frac{(1.5)^x}{x!}$$

(i) P[No car is used on a given day]

$= P[X = 0]$

$$= 0.2231 \times \frac{(1.5)^0}{0!} = 0.2231 \times 1 = 0.2231$$

Hence, the probability that no car is used on a given day is 0.2231.

(ii) P[Some demand is refused on a given day]

$= P[X > 2]$

$= 1 - P(X \leqslant 2)$

$= 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$

$$= 1 - \left[0.2231 \times \frac{(0.5)^0}{0!} + 0.2231 \times \frac{(1.5)^1}{1!} + 0.2231 \times \frac{(1.5)^2}{2!}\right]$$

$= 1 - [0.2231 \times 1 + 0.2231 \times 1.5 + 0.2231 \times 1.125]$

$= 1 - [0.2231(1 + 1.5 + 1.125)]$

$= 1 - [0.2231 \times 3.625] = 1 - 0.8087 = 0.1913$*

Hence, some demand is refused on a given day is 0.1913.

[* **Note :** *Answer given in the textbook is incorrect.*]

6. Defects on plywood sheet occur at random with the average of one defect per 50 sq ft. Find the probability that such a sheet has (i) no defect, (ii) at least one defect. Use $e^{-1} = 0.3678$.

Solution :

X = Number of defects on a plywood sheet

$m = 1$, $e^{-1} = 0.3678$

$\therefore X \sim p(m = 1)$

Hence, $p(x) = \frac{e^{-m}\, m^x}{x!}$ $\quad \therefore p(x) = \frac{e^{-1}1^x}{x!} = 0.3678\, \frac{1}{x!}$

(i) P[No defect] i.e. $P[X = 0]$:

$$\therefore P[X = 0] = p(0) = 0.3678 \times \frac{1}{0!} = 0.3678 \times 1$$

$\therefore P[X = 0] = 0.3678$

Hence, the probability that sheet will have no defect is 0.3678

(ii) P[At least one defect] i.e. $P[X \geqslant 1]$

$\therefore P[X \geqslant 1] = 1 - P(0)$

$= 1 - 0.3678$

$= 0.6322$

Hence, the probability that sheet will have at least one defect is 0.6322.

7. It is known that, in a certain area of a large city, the average number of rats per bungalow is five. Assuming that the number of rats follows Poisson distribution, find the probability that a randomly selected bungalow has (i) exactly 5 rats, (ii) more than 5 rats, (iii) between 5 and 7 rats, inclusive.

[Given $e^{-5} = 0067$.]

Solution :

X = Number of rats

$m = 5$, $e^{-5} = 0.0067$

$\therefore X \sim P(m = 5)$

Hence, $p(x) = \frac{e^{-m}m^x}{x!}$

$$\therefore p(x) = \frac{e^{-5}(5)^x}{x!} = 0.0067 \times \frac{5^x}{x!}$$

(i) P[Exactly 5 rats] i.e. $P[X = 5]$:

$$\therefore p(5) = 0.0067 \times \frac{5^5}{5!}$$

$$= 0.0067 \times \frac{3125}{120}$$

$$= \frac{20.9375}{120}$$

$= 0.1745$

(ii) P[More than 5 rats] i.e. $P[X > 5] = 1 - P[X \leqslant 5]$:

$P[X > 5) = 1 - [P(X = 0) + P(X = 1) + P(X = 2)$
$+ P(X = 3) + P(X = 4) + P(X = 5)$

$= 1 - [p(0) + p(1) + p(2) + p(3) + p(4) + p(5)]$

$$= 1 - \left[0.0067\left(\frac{5^0}{0!} + \frac{5^1}{1!} + \frac{5^2}{2!} + \frac{5^3}{3!} + \frac{5^4}{4!} + \frac{5^5}{5!}\right)\right]$$

$$= 1 - \left[0.0067\left(1 + 5 + \frac{25}{2} + \frac{125}{6} + \frac{625}{24} + \frac{3125}{120}\right)\right]$$

$$= 1 - \left[0.0067\left(6 + \frac{1500 + 2500 + 3125 + 3125}{120}\right)\right]$$

$$= 1 - \left[0.0067\left(6 + \frac{10250}{120}\right)\right]$$

$= 1 - [0.0067(6 + 85.417)]$

$= 1 - [0.0067(91.417)]$

$= 1 - 0.6125 = 0.3875$

$\therefore\ P[X > 5) = 0.3875$

(iii) P[Between 5 and 7 (inclusive) rats] i.e. $P[X = 5, 6, 7]$:

$P[X = 5 \text{ or } 6 \text{ or } 7] = P(X = 5) + P(X = 6) + P(X = 7)$

$= p(5) + p(6) + p(7)$

$$= 0.0067\left[\frac{5^5}{5!} + \frac{5^6}{6!} + \frac{5^7}{7!}\right]$$

$$= 0.0067\left[\frac{3125}{120} + \frac{15625}{720} + \frac{78125}{5040}\right]$$

$= 0.0067(26.042 + 21.701 + 15.501)$

$= 0.0067 \times 63.244$

$= 0.4237$

Hence, the probability that there are between 5 and 7 (inclusive) rats in randomly selected bungalow is 0.4237.

[**Note :** *Answers given in the textbook are incorrect.*]

EXAMPLES FOR PRACTICE 8.4

1. The average number of customers who appear at a counter of a certain bank per minute is 2. Find the probability that during a given minute exactly 3 customers appear at the counter. [Given : $e^{-2} = 0.135$]

2. The random variable X defined as X = number of defective articles, follows Poisson distribution. If for a sample of 400 articles $P[X = 3] = 5P[X = 5]$, find p, the probability of a defective article.

3. If X is a Poisson variate and $P[X = 1] = P[X = 2]$, find its mean and variance.

4. Find the probability that at the most 2 defective bolts will be found in a box of 200 bolts, if it is known that 2% of such bolts are expected to be defective. Assume Poisson distribution given that $e^{-4} = 0.0183$.

5. In a company on an average 3 workers are absent per day. Assuming Poisson distribution, find the probability that on a particular day exactly 2 workers are absent given that $e^{-3} = 0.0498$.

6. The probability that a person will react to a drug is 0.002. Out of 2000 individuals checked find the probability that at least one individual reacts to the drug. [Given : $e^{-4} = 0.018$]

7. If for a Poisson distribution $3P(2) = P(3)$, find the mean and variance of the distribution.

8. If X is a Poisson random variable such that $P(X = 3) = P(X = 4)$, find the mean and variance of the distribution.

9. If X is a Poisson variate with mean 3, find $P[X \geqslant 2]$. [Given : $e^{-3} = 0.0498$]

10. If X is a Poisson variate and $P(X = 3) = P(X = 4)$, then prove that $P(X = 2) = 8e^{-4}$.

11. In a factory, 1% defective items is found in a lot. If a random sample of 200 items is taken, find the probability that (i) at most 3 items are defective and (ii) at least 4 items are defective. [Given : $e^{-2} = 0.1353$]

12. If x has Poisson distribution with parameter $m = 1$, find $P[x \leqslant 1]$. [Use : $e^{-1} = 0.367879$]

13. If a random variable X follows Poisson distribution such that

$P(X = 1) = P[X = 2]$, then find $P[X \geqslant 1]$

[Use : $e^{-2} = 0.1353$]

Answers

1. $p(3) = 0.18$ **2.** $m = 2,\ p = \frac{m}{n} = \frac{2}{400} = 0.005$

3. Mean = 2, Variance = 9 **4.** 0.2379

5. $P[X = 2] = 0.22410$ **6.** $P[X \geqslant 1] = 0.982$

7. Mean = 9, Variance = 9 **8.** Mean = 4, Variance = 2

9. $P(X \geqslant 2) = 0.8008$

11. (i) 0.8569, (ii) 0.1431

12. 0.7358 **13.** 0.8647.

MISCELLANEOUS EXERCISE – 8

(Textbook pages 153 to 157)

I. Choose the correct alternative :

1. $F(x)$ is c.d.f. of discrete r.v. X whose p.m.f. is given by $P(x) = k\binom{4}{x}$, for $x = 0, 1, 2, 3, 4$ and $P(x) = 0$ otherwise then $F(5) = \ldots\ldots\ldots$

(a) $\frac{1}{16}$ (b) $\frac{1}{8}$ (c) $\frac{1}{4}$ (d) 1

2. $F(x)$ is c.d.f. of discrete r.v. X whose distribution is

X_i	-2	-1	0	1	2
P_i	0.2	0.3	0.15	0.25	0.1

then $F(-3) = \dots\dots\dots$

(a) 0 (b) 1 (c) 0.2 (d) 0.15

3. X : is number obtained on upper most face when a fair die is thrown then $E(x) = \dots\dots\dots$

(a) 3.0 (b) 3.5 (c) 4.0 (d) 4.5

4. If p.m.f. of r.v. X is given below.

x	0	1	2
$P(x)$	q^2	$2pq$	p^2

then Var $(x) = \dots\dots\dots$

(a) p^2 (b) q^2 (c) pq (d) $2pq$

5. The expected value of the sum of two numbers obtained when two fair dice are rolled is ....

(a) 5 (b) 6 (c) 7 (d) 8

6. Given p.d.f. of a continuous r.v. X as

$f(x) = \frac{x^2}{3}$ for $-1 < x < 2$

$= 0$ otherwise then

$F(1) = \dots\dots\dots$

(a) $\frac{1}{9}$ (b) $\frac{2}{9}$ (c) $\frac{3}{9}$ (d) $\frac{4}{9}$

7. X is r.v. with p.d.f. $f(x) = \frac{k}{\sqrt{x}}$, $0 < x < 4$

$= 0$ otherwise

then $E(x) = \dots\dots\dots$

(a) $\frac{1}{3}$ (b) $\frac{4}{3}$ (c) $\frac{2}{3}$ (d) 1

8. If $X \sim B\left(20, \frac{1}{10}\right)$ then $E(x) = \dots\dots\dots$

(a) 2 (b) 5 (c) 4 (d) 3

9. If $E(x) = m$ and $\text{Var}(x) = m$ then X follows ....

(a) Binomial distribution (b) Poisson distribution
(c) Normal distribution (d) none of the above

10. If $E(x) > \text{Var}(x)$ then X follows ....

(a) Binomial distribution (b) Poisson distribution
(c) Normal distribution (d) none of the above

Answers

1. (d) 1 **2.** (a) 0 **3.** (b) 3.5 **4.** (d) $2pq$ **5.** (c) 7 **6.** (b) $\frac{2}{9}$ **7.** (b) $\frac{4}{3}$ **8.** (a) 2 **9.** (b) Poisson distribution **10.** (a) Binomial distribution.

II. Fill in the blanks :

1. The values of discrete r.v. are generally obtained by ....

2. The values of continuous r.v. are generally obtained by ....

3. If X is discrete random variable takes the values $x_1, x_2, x_3, \dots x_n$, then $\sum_{i=1}^{n} P(x_i) = \dots\dots\dots$

4. If $F(x)$ is distribution function of discrete r.v. x with p.m.f. $P(x) = \frac{x-1}{3}$ for $x = 1, 2, 3$ and $P(x) = 0$ otherwise then $F(4) = \dots\dots\dots$

5. If $F(x)$ is distribution function of discrete r.v. x with p.m.f. $P(x) = k\binom{4}{x}$ for $x = 0, 1, 2, 3, 4$ and $P(x) = 0$ otherwise then $F(-1) = \dots\dots\dots$

6. $E(x)$ is considered to be .... of the probability distribution of x.

7. If x is continuous r.v. and $F(x_i) = P(X \leqslant x_i) = \int_{-\infty}^{x_i} f(x)dx$ then $F(x)$ is called ....

8. In Binomial distribution, probability of success .... from trial to trial.

9. In Binomial distribution, if n is very large and probability success p is very small such that $np = m$ (constant) then .... distribution is applied.

Answers

1. counting **2.** measurement **3.** 1 **4.** 1 **5.** 0
6. centre of gravity **7.** distribution function
8. remains constant / independent **9.** Poisson.

(III) State whether each of the following is *True* or *False* :

1. If $P(X = x) = k\binom{4}{x}$ for $x = 0, 1, 2, 3, 4$, then $F(5) = \frac{1}{4}$ when $F(x)$ is c.d.f.

2.

X	-2	-1	0	1	2
$P(X = x)$	0.2	0.3	0.15	0.25	0.1

If $F(x)$ is c.d.f. of discrete r.v. X, then $F(-3) = 0$.

3. X is the number obtained on upper most face when a die is thrown then $E(x) = 3.5$.

4. If p.m.f. of discrete r.v. X is

x	0	1	2
$P(X = x)$	q^2	$2pq$	p^2

then $E(x) = 2p$.

5. The p.m.f. of a r.v. X is

$$P(x) = \frac{2x}{n(n+1)}, \ x = 1, 2, \ldots\ldots\ldots n$$
$$= 0 \qquad \text{otherwise,}$$

then $E(x) = \frac{2n+1}{3}$.

6. If $f(x) = kx(1-x)$ for $0 < x < 1$
$$= 0 \qquad \text{otherwise,}$$
then $k = 12$.

7. If $X \sim B(n, p)$ and $n = 6$ and $P(x = 4)$ $= P(x = 2)$, then $p = \frac{1}{2}$.

8. If r.v. X assumes values 1, 2, 3, ... n with equal probabilities, then $E(x) = \frac{(n+1)}{2}$.

9. If r.v. X assumes the values 1, 2, 3, ..., 9 with equal probabilities, $E(x) = 5$.

Answers

1. False	**2.** True	**3.** True	**4.** True	**5.** True
6. False	**7.** True	**8.** True	**9.** True.	

IV. Solve the following problems :

PART–I

1. Identify the random variable as discrete or continuous in each of the following. Identify its range if it is discrete.

(i) An economist is interested in knowing the number of unemployed graduates in the town with a population of 1 lakh.

(ii) Amount of syrum prescribed by a physician.

(iii) A person on high protein diet is interested in the weight gained in a week.

(iv) Twelve of 20 white rats available for an experiment are male. A scientist randomly selects 5 rats and counts the number of female rats among them.

(v) A highway safety group is interested in the speed (km/hrs) of a car at a check point.

Solution :

(i) Here, X = the number of unemployed graduates in a town.

$\therefore$ X is discrete random variable. Population of a town is 1 lakh.

$\therefore$ Range of $X = \{0, 1, 2, \ldots, 99999, 100000\}$

(ii) Here, X = the amount of syrup prescribed by a physician.

$\therefore$ X is continuous random variable.

(iii) Here, X = the gain in weight in a week

$\therefore$ X is a continuous random variable.

(iv) Here, X = number of female rats selected on a specified day.

8 rats are female. A scientist selects 5 rats at random.

$\therefore$ X is discrete random variable.

Range of $X = \{0, 1, 2, 3, 4, 5\}$

(v) Here, X = speed (in km/hr) of a car at a check point.

$\therefore$ X is continuous random variable.

2. The probability distribution of a discrete r.v. X is as follows :

x	1	2	3	4	5	6
$P(X = x)$	k	$2k$	$3k$	$4k$	$5k$	$6k$

(i) Determine the value of k.

(ii) Find $P(X \leqslant 4)$, $P(2 < X < 4)$, $P(X \geqslant 3)$.

Solution :

(i) For the probability distribution of a discrete r.v. X, we have

$\Sigma P(X = x) = 1$

$\therefore k + 2k + 3k + 4k + 5k + 6k = 1$

$\therefore 21k = 1$

$\therefore k = \frac{1}{21}$

(ii) Now, the probability distribution is written as follows :

x	1	2	3	4	5	6
$P(X=x)$	$\frac{1}{21}$	$\frac{2}{21}$	$\frac{3}{21}$	$\frac{4}{21}$	$\frac{5}{21}$	$\frac{6}{21}$

$$P[X \leqslant 4] = p(1)+p(2)+p(3)+p(4)$$
$$= \frac{1}{21}+\frac{2}{21}+\frac{3}{21}+\frac{4}{21}=\frac{10}{21}$$

$$P[2 < X < 4] = p(3)$$
$$= \frac{3}{21}=\frac{1}{7}$$

$$P[X \geqslant 3] = p(3)+p(4)+p(5)+p(6)$$
$$= \frac{3}{21}+\frac{4}{21}+\frac{5}{21}+\frac{6}{21}=\frac{18}{21}=\frac{6}{7}$$

Hence, $k=\frac{1}{21}$, $P[X \leqslant 4]=\frac{10}{21}$, $P[2<X<4]=\frac{1}{7}$

$P(X \geqslant 3)=\frac{6}{7}$.

3. Following is the probability distribution of a r.v. X.

x	-3	-2	-1	0	1	2	3
$P(X=x)$	0.05	0.10	0.15	0.20	0.25	0.15	0.1

Find the probability that

(i) X is positive,

(ii) X is non-negative,

(iii) X is odd,

(iv) X is even.

Solution :

(i) $P[\text{X is positive}]$
$$= p(1)+p(2)+p(3)$$
$$= 0.25+0.15+0.10$$
$$= 0.50$$

(ii) $P[\text{X is non-negative}]$
$$= p(0)+p(1)+p(2)+p(3)$$
$$= 0.20+0.25+0.15+0.10$$
$$= 0.70$$

(iii) $P[\text{X is odd}]$
$$= p(-3)+p(-1)+p(1)+p(3)$$
$$= 0.05+0.15+0.25+0.10$$
$$= 0.55$$

(iv) $P[\text{X is even}]$
$$= p(-2)+p(0)+p(2)$$
$$= 0.10+0.20+0.15$$
$$= 0.45$$

4. The p.m.f. of a r.v. X is given by $P(X=x)=\begin{cases}\binom{5}{x}\frac{1}{2^5}, & x=0,\ 1,\ 2,\ 3,\ 4,\ 5.\\ 0 & \text{otherwise}\end{cases}$

Show that $P(X \leqslant 2) = P(X \geqslant 3)$.

Solution :

$$P(X=x)=\frac{{}^5C_x}{2^5}, \quad x=0,\ 1,\ 2,\ \ldots,\ 5;$$
$$=0, \quad \text{otherwise.}$$

The probability distribution of X is written as follows :

$X=x$	0	1	2	3	4	5
$P(X=x)=\frac{{}^5C_x}{2^5}$	$\frac{{}^5C_0}{2^5}=\frac{1}{32}$	$\frac{{}^5C_1}{2^5}=\frac{5}{32}$	$\frac{{}^5C_2}{2^5}=\frac{10}{32}$	$\frac{{}^5C_3}{2^5}=\frac{10}{32}$	$\frac{{}^5C_4}{2^5}=\frac{5}{32}$	$\frac{{}^5C_5}{2^5}=\frac{1}{32}$

Now, $P[X \leqslant 2] = p(0)+p(1)+p(2)$
$$= \frac{1}{32}+\frac{5}{32}+\frac{10}{32}$$
$$= \frac{16}{32}$$
$$= \frac{1}{2} \qquad \ldots (1)$$

$P[X \geqslant 3] = p(3)+p(4)+p(5)$
$$= \frac{10}{32}+\frac{5}{32}+\frac{1}{32}$$
$$= \frac{16}{32}$$
$$= \frac{1}{2} \qquad \ldots (2)$$

From (1) and (2), we can show that, $P[X \leqslant 2] = P[X \geqslant 3]$

5. In the following probability distribution of a r.v. X.

x	1	2	3	4	5
$P(x)$	$\frac{1}{20}$	$\frac{3}{20}$	a	$2a$	$\frac{1}{20}$

Find a and obtain the c.d.f. of X.

Solution :

For p.d.f. we have,

$\Sigma P(X=x)=1$

Let $b=2a$

$\therefore \frac{1}{20}+\frac{3}{20}+a+b+\frac{1}{20}=1$

$\therefore \frac{5}{20}+a+b=1$

$\therefore \frac{5}{20}+a+2a=1$ ... (Since $b=2a$)

$\therefore 3a=1-\frac{5}{20}$

$\therefore 3a=\frac{20-5}{20}$

$\therefore 3a=\frac{15}{20}$

$\therefore a=\frac{15}{20}\times\frac{1}{3}=\frac{1}{4}$

Now, $b=2a$

$\therefore b=2\times\frac{1}{4}$

$=\frac{1}{2}$

The c.d.f. of X is obtained as follows :

x_i	1	2	3	4	5
$F(x_i)$	$\frac{1}{20}$	$\frac{1}{20}+\frac{3}{20}=\frac{4}{20}$	$\frac{4}{20}+\frac{1}{4}=\frac{9}{20}$	$\frac{9}{20}+\frac{1}{2}=\frac{19^*}{20}$	$\frac{19}{20}+\frac{1}{20}=1$

[* **Note** : *Answer given in the textbook is incorrect.*]

6. A fair coin is tossed 4 times. Let X denote the number of heads obtained. Identify the probability distribution of X and state the formula for p.m.f. of X.

Solution :

A fair coin is tossed 4 times

$\therefore n=2^4=16$

X = number of heads.

The probability distribution is obtained as follows :

$X=x$	0	1	2	3	4
Favourable outcomes $m={}^4C_x$	${}^4C_0=1$	${}^4C_1=4$	${}^4C_2=6$	${}^4C_3=4$	${}^4C_4=1$
$P(X=x)=\frac{m}{n}$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

Formula for p.m.f. of X :

A fair coin is tossed 4 times.

$\therefore$ Number of trials $n=4$

Number of successes $x=0, 1, 2, 3, 4$

$p=$ probability of getting head $=\frac{1}{2}$

$\therefore q=1-p=\frac{1}{2}$

X is a bernoulli trial

$\therefore P(X=x)={}^nC_x p^x q^{n-x}$

$={}^4C_x\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{4-x}={}^4C_x\left(\frac{1}{2}\right)^4$

$\therefore P(X=x)=\frac{{}^4C_x}{16}, x=0, 1, 2, 3, 4$

7. Find the probability of the number of successes in two tosses of a die, where success is defined as (i) number greater than 4, (ii) 6 appears in at least one toss.

Solution :

A fair die tosses two times.

$\therefore$ S $= \{(1, 1), (1, 2), ..., (1, 6), (2, 1), (2, 2), ..., (2, 6), ..., (6, 1), (6, 2), ..., (6, 6)\}$

$\therefore n(\text{S}) = 36$

(i) X = number greater than 4

$\therefore \text{X} = \{0, 1, 2\}$ ... ($\because$ A die tosses two times)

Now, $P(\text{X}=0) = P\{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4)\}$

$$= \frac{16}{36} = \frac{4}{9}$$

$P(\text{X}=1) = P\{(1, 5), (1, 6), (2, 5), (2, 6), (3, 5), (3, 6), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (6, 1), (6, 2), (6, 3), (6, 4)\}$

$$= \frac{16}{36} = \frac{4}{9}$$

$P(\text{X}=2) = P\{(5, 5), (5, 6), (6, 5), (6, 6)\}$

$$= \frac{4}{36} = \frac{1}{9}$$

$\therefore$ the probability distribution is obtained as follows :

X = x	0	1	2	Total
$P(x_i)$	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$	$\Sigma P(x_i) = 1$

(ii) X = Number of sixes

$\therefore \text{X} = \{0, 1, 2\}$ ... ($\because$ A die tosses two times)

Now, $P(\text{X}=0) = \{(1, 1), (1, 2), ..., (1, 5), (2, 1), (2, 2), (2, 5), ..., (5, 1), (5, 2), ..., (5, 5)\}$

$$= \frac{25}{36}$$

$P(\text{X}=1) = P\{(1, 6), (2, 6), (3, 6), (4, 6), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5)\}$

$$= \frac{10}{36}$$

$P(\text{X}=2) = P\{(6, 6)\}$

$$= \frac{1}{36}$$

Therefore, the probability distribution of X is obtained as shown in the following table :

X = x	0	1	2	Total
$P(x_i)$	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$	$\Sigma P(x_i) = 1$

8. A random variable X has the following probability distribution.

x	1	2	3	4	5	6	7
$P(x)$	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

Determine (i) k, (ii) $P(\text{X}<3)$, (iii) $P(\text{X}>6)$, (iv) $P(0<\text{X}<3)$.

Solution :

(i) To determine k :

x	1	2	3	4	5	6	7
P(x)	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

For the probability distribution

$\Sigma P(x) = 1$

$\therefore k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$

$\therefore 10k^2 + 9k - 1 = 0$

$\therefore 10k^2 + 10k - k - 1 = 0$

$\therefore 10k(k+1) - 1(k+1) = 0$

$\therefore (10k-1)(k+1) = 0$

$\therefore 10k - 1 = 0$ OR $k + 1 = 0$

$\therefore 10k = 1$ OR $k = -1$

$\therefore k = \frac{1}{10}$ OR $k = -1$

$k = -1$ is not possible, therefore $k = \frac{1}{10}$.

(ii) $P(\text{X}<3) = P(\text{X}=1) + P(\text{X}=2)$

$= k + 2k$

$= 3k$

$= 3 \times \frac{1}{10} = \frac{3}{10}$ ... $\left(\because k = \frac{1}{10}\right)$

(iii) $P(\text{X}>6) = P(\text{X}=7)$

$= 7k^2 + k$

$= 7\left(\frac{1}{10}\right)^2 + \frac{1}{10}$ ... $\left(\because k = \frac{1}{10}\right)$

$= \frac{7}{100} + \frac{1}{10}$

$$= \frac{7+10}{100}$$

$$= \frac{17}{100}$$

(iv) $P(0 < x < 3) = P(X=1) + P(X=2)$

$$= k + 2k$$

$$= 3k$$

$$= \frac{3}{10} \qquad \ldots\left(\because k = \frac{1}{10}\right)$$

9. The following is the c.d.f. of a r.v. X.

x	-3	-2	-1	0	1	2	3	4
$F(x)$	0.1	0.3	0.5	0.65	0.75	0.85	0.9	1

Find the probability distribution of X and $P(-1 \leqslant X \leqslant 2)$.

Solution : We are given c.d.f. and a r.v. X. We have to find the probability distribution of X.

We know that

$F(x) = P(X \leqslant x)$

i.e. $P(X \leqslant x) = F(x)$

$\therefore\ P(X \leqslant -3) = F(-3)$

$\therefore\ P(X = -3) = 0.1$

Now, $P(X = -2) = F(-2) - F(-3) = 0.3 - 0.1 = 0.2$

$P(X = -1) = F(-1) - F(-2) = 0.5 - 0.3 = 0.2$

$P(X = 0) = F(0) - F(-1) = 0.65 - 0.5 = 0.15$

$P(X = 1) = F(1) - F(0) = 0.75 - 0.65 = 0.10$

$P(X = 2) = F(2) - F(1) = 0.85 - 0.75 = 0.10$

$P(X = 3) = F(3) - F(2) = 0.9 - 0.85 = 0.05$

$P(X = 4) = F(4) - F(3) = 1 - 0.9 = 0.1$

$\therefore$ Probability distribution of X is obtained as shown in the following table :

$X = x$	-3	-2	-1	0	1	2	3	4
$P(x_i)$	0.1	0.2	0.2	0.15	0.10	0.10	0.05	0.1

From the above table,

$P(-1 \leqslant X \leqslant 2) = P(X = -1) + P(X = 0) + P(X = 1) + P(X = 2)$

$$= 0.2 + 0.15 + 0.10 + 0.10$$

$$= 0.55$$

10. Find the expected value and variance of the r.v. X if its probability distribution is as follows :

(i)

x	1	2	3
$P(X=x)$	1/5	2/5	2/5

(ii)

x	-1	0	1
$P(X=x)$	1/5	2/5	2/5

(iii)

x	1	2	3	...	n
$P(X=x)$	$1/n$	$1/n$	$1/n$	...	$1/n$

(iv)

x	0	1	2	3	4	5
$P(X=x)$	1/32	5/32	10/32	10/32	5/32	1/32

Solution :

We prepare the following table to find $E(X)$ and Var (X) :

(i)

$X = x$ x_i	$P(X=x)$ p_i	$x_i p_i$	$x_i^2 p_i = x_i p_i \times x_i$
1	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$
2	$\frac{2}{5}$	$\frac{4}{5}$	$\frac{8}{5}$
3	$\frac{2}{5}$	$\frac{6}{5}$	$\frac{18}{5}$
Total	1	$\Sigma x_i p_i = \frac{11}{5}$	$\Sigma x_i^2 p_i = \frac{27}{5}$

Expected value :

$E(X) = \Sigma x_i p_i$

$$= \frac{11}{5}$$

Variance :

$\text{Var}(X) = E(X^2) - [E(X)]^2$

$$= \Sigma x_i^2 p_i - \left(\frac{11}{5}\right)^2$$

$$= \frac{27}{5} - \frac{121}{25}$$

$$= \frac{135-121}{25}$$

$$= \frac{14}{25}$$

Hence, $E(X) = \frac{11}{5}$ Var $(X) = \frac{14}{25}$

(ii)

x	-1	0	1
$P(X=x)$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{2}{5}$

We construct the following table to find $E(X)$ and $\mathrm{Var}(X)$:

$X=x$ x_i	$P(X=x)$ p_i	$x_i p_i$	$x_i^2 p_i = x_i p_i \times x_i$
-1	$\frac{1}{5}$	$\frac{-1}{5}$	$\frac{1}{5}$
0	$\frac{2}{5}$	0	0
1	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$
Total	1	$\Sigma x_i p_i = \frac{1}{5}$	$\Sigma x_i^2 p_i = \frac{3}{5}$

Expected value :

$E(X) = \Sigma x_i p_i = 1/5$

Variance :

$$\mathrm{Var}(X) = E(X^2) - [E(X)]^2$$

$$= \Sigma x_i^2 p_i - \left(\frac{1}{5}\right)^2$$

$$= \frac{3}{5} - \frac{1}{25} = \frac{15-1}{25} = \frac{14}{25}$$

Hence, $E(X) = \frac{1}{5}$, $\mathrm{Var}(X) = \frac{14}{25}$

(iii)

x	1	2	3	$\ldots$	n
$P(X=x)$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\ldots$	$\frac{1}{n}$

We construct the following table to find $E(X)$ and $\mathrm{Var}(X)$:

$X=x$ x_i	$P(X=x)$ p_i	$x_i p_i$	$x_i^2 p_i = x_i p_i \times x_i$
1	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1^2}{n}$
2	$\frac{1}{n}$	$\frac{2}{n}$	$\frac{2^2}{n}$
3	$\frac{1}{n}$	$\frac{3}{n}$	$\frac{3^2}{n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$\frac{1}{n}$	$\frac{n}{n}$	$\frac{n^2}{n}$
Total	1	$\Sigma x_i p_i = \frac{n+1}{2}$	$\Sigma x_i^2 p_i = \frac{(n+1)(2n+1)}{6}$

$$\Sigma x_i p_i = \frac{1}{n} + \frac{2}{n} + \frac{3}{n} + \ldots + \frac{n}{n}$$

$$= \frac{1}{n}(1+2+3+\ldots+n)$$

$$= \frac{1}{n}\left[\frac{n(n+1)}{2}\right] \quad \left[\because \sum_{1}^{n} i = \frac{n(n+1)}{2}\right]$$

$$= \frac{n+1}{2}$$

$$\Sigma x_i^2 p_i = \frac{1^2}{n} + \frac{2^2}{n} + \frac{3^2}{n} + \ldots + \frac{n^2}{n}$$

$$= \frac{1}{n}[1^2 + 2^2 + 3^2 + \ldots + n^2]$$

$$= \frac{1}{n}\left[\frac{n(n+1)(2n+1)}{6}\right] \quad \left[\because \sum_{1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}\right]$$

$$= \frac{(n+1)(2n+1)}{6}$$

Expected value :

$$E(X) = \Sigma x_i p_i$$

$$= \frac{n+1}{2}$$

Variance :

$$\mathrm{Var}(X) = E(X^2) - [E(X)]^2$$

$$= \Sigma x_i^2 p_i - \left(\frac{n+1}{2}\right)^2$$

$$= \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}$$

$$= \frac{n+1}{2}\left[\frac{2n+1}{3} - \frac{n+1}{2}\right]$$

$$= \frac{n+1}{2}\left[\frac{4n+2-3\text{n}-3}{6}\right]$$

$$= \frac{n+1}{2}\left[\frac{n-1}{6}\right]$$

$$= \frac{n^2-1}{12}$$

Hence, $E(X) = \frac{n+1}{2}$, $\mathrm{Var}(X) = \frac{n^2-1}{12}$

(iv)

x	0	1	2	3	4	5
$P(X=x)$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{10}{32}$	$\frac{5}{32}$	$\frac{1}{32}$

We construct the following table to compute $E(X)$ and $\mathrm{Var}(X)$:

$X=x$ x_i	$P(X=x)$ p_i	$x_i p_i$	$x_i^2 p_i = x_i p_i \times x_i$
0	$\frac{1}{32}$	0	0
1	$\frac{5}{32}$	$\frac{5}{32}$	$\frac{5}{32}$
2	$\frac{10}{32}$	$\frac{20}{32}$	$\frac{40}{32}$
3	$\frac{10}{32}$	$\frac{30}{32}$	$\frac{90}{32}$
4	$\frac{5}{32}$	$\frac{20}{32}$	$\frac{80}{32}$
5	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{25}{32}$
Total	1	$\Sigma x_i p_i = \frac{80}{32}$	$\Sigma x_i^2 p_i = \frac{240}{32}$

Expected value :

$E(X) = \Sigma x_i p_i = \frac{80}{32} = 2.5$

Variance :

$\text{Var}(X) = E(X^2) - [E(X)]^2$

$= \Sigma x_i^2 p_i - (2.5)^2$

$= \frac{240}{32} - 6.25$

$= 7.5 - 6.25$

$= 1.25$

Hence, $E(X) = 2.5$, $\text{Var}(X) = 1.25$

11. A player tosses two coins. He wins ₹ 10 if 2 heads appear, ₹ 5 if 1 head appears and ₹ 2 if no head appears. Find the expected value and variance of winning amount.

Solution :

2 fair coins are tossed.

$\therefore$ $S = \{HH, HT, TH, TT\}$

Let X = number of heads

$\therefore$ $X = \{0, 1, 2\}$

Now, $P(X=0) = \frac{1}{4}$, $P(X=1) = \frac{2}{4}$, $P(X=2) = \frac{1}{4}$

Let x_i = the amount received corresponds to the values of X.

We construct the following table to compute the expected winning amount and the variance of winning amount :

X	x_i ₹	$P(X=x)$ p_i	$x_i p_i$	$x_i^2 p_i = x_i p_i \times x_i$
0	2	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{4}{4}$
1	5	$\frac{2}{4}$	$\frac{10}{4}$	$\frac{50}{4}$
2	10	$\frac{1}{4}$	$\frac{10}{4}$	$\frac{100}{4}$
Total	–	1	$\Sigma x_i p_i = \frac{22}{4}$	$\Sigma x_i^2 p_i = \frac{154}{4}$

Expected winning amount :

$E(X) = \Sigma x_i p_i$

$= \frac{22}{4} = ₹\ 5.5$

Variance of winning amount :

$\text{Var}(X) = \Sigma x_i^2 p_i - (\Sigma x_i p_i)^2$

$= \frac{154}{4} - (5.5)^2$

$= 38.5 - 30.25$

$= 8.25 = ₹\ 8.25$

12. Let the p.m.f. of the r.v. X be

$$P(x) = \begin{cases} \frac{3-x}{10} & \text{for } x = -1, 0, 1, 2. \\ 0 & \text{otherwise} \end{cases}$$

Calculate $E(X)$ and $\text{Var}(X)$.

Solution :

$X=x$	$P(x_i) = \frac{3-x}{10}$	$x_i p(x_i)$	$x_i^2 p(x_i) = x_i p(x_i) x_i$
-1	$\frac{4}{10}$	$\frac{-4}{10}$	$\frac{4}{10}$
0	$\frac{3}{10}$	$\frac{0}{10}$	$\frac{0}{10}$
1	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{2}{10}$
2	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{4}{10}$
Total	$\Sigma p(x_i) = 1$	$\Sigma x_i p(x_i)$ $= 0$	$\Sigma x_i^2 p(x_i)$ $= \frac{10}{10} = 1$

$E(X) = \Sigma x_i p(x_i) = 0$

$\text{Var}(X) = \Sigma x_i^2 p(x_i) - [E(X)]^2$

$= 1 - 0 = 1$

13. Suppose error involved in making a certain measurement is a continuous r.v. X with p.d.f.

$$f(x)=\begin{cases} k(4-x^2) & \text{for } -2\leqslant x\leqslant 2,\\ 0 & \text{otherwise}\end{cases}$$

Compute (i) $P(X>0)$, (ii) $P(-1<X<1)$, (iii) $P(X<0.5 \text{ or } X>0.5)$.

Solution : *For solution, refer to the solution of example 7 of exrcise 8.2.*

14. The p.d.f. of the r.v. X is given by

$$f(x)=\begin{cases} \dfrac{1}{2a} & \text{for } 0<x<2a.\\ 0 & \text{otherwise}\end{cases}$$

Show that $P\left(X<\dfrac{a}{2}\right)=P\left(X>\dfrac{3a}{2}\right)$.

Solution :

$$f(x)=\frac{1}{2a},\ 0<x<2a\ (a>0)$$

$$=0, \quad \text{otherwise}$$

$$P\left[X<\frac{a}{2}\right]=\int_0^{a/2}\frac{1}{2a}dx$$

$$=\frac{1}{2a}[x]_0^{a/2}$$

$$=\frac{1}{2a}\left[\frac{a}{2}-0\right]$$

$$=\frac{1}{2a}\times\frac{a}{2}=\frac{1}{4}$$

$$\therefore\ P\left[X<\frac{a}{2}\right]=\frac{1}{4} \qquad \dots (1)$$

$$P\left[X>\frac{3a}{2}\right]=\int_{3a/2}^{2a}\frac{1}{2a}dx$$

$$=\frac{1}{2a}[x]_{3a/2}^{2a}$$

$$=\frac{1}{2a}\left[2a-\frac{3a}{2}\right]$$

$$=\frac{1}{2a}\left[\frac{4a-3a}{2}\right]$$

$$=\frac{1}{2a}\times\frac{a}{2}=\frac{1}{4}$$

$$\therefore\ P\left[X>\frac{3a}{2}\right]=\frac{1}{4} \qquad \dots (2)$$

From (1) and (2) it is shown that $P\left[X<\dfrac{a}{2}\right]=P\left[X>\dfrac{3a}{2}\right]$

15. Determine k if the p.d.f. of the r.v. is

$$f(x)=\begin{cases} ke^{-\theta x}, & \text{for } 0\leqslant x<\infty,\ \theta>0;\\ 0, & \text{otherwise}\end{cases}$$

Find $P\left(X>\dfrac{1}{\theta}\right)$ and determine M if

$$P(0<X<M)=\frac{1}{2}.$$

Solution :

$$f(x)=ke^{-\theta x},\quad 0\leqslant x<\infty,\ \theta>0$$

$$=0, \quad \text{otherwise.}$$

To determine k :

For the p.d.f. of a r.v. X, we have

$$\int_{-\infty}^{\infty}f(x)\,dx=1$$

$$\therefore\ \int_0^{\infty}ke^{-\theta x}dx=1$$

$$\therefore\ k\int_0^{\infty}e^{-\theta x}dx=1$$

$$\therefore\ k\left(\frac{1}{\theta}\right)[-e^{-x}]_0^{\infty}=1$$

$$\therefore\ k\frac{1}{\theta}[-e^{-\infty}-(-e^{-0})]=1$$

$$\therefore\ \frac{k}{\theta}\left[-\frac{1}{e^{\infty}}+\frac{1}{e^0}\right]=1$$

$$\therefore\ \frac{k}{\theta}[-0+1]=1$$

$$\therefore\ \frac{k}{\theta}=1 \qquad \therefore\ k=\theta.$$

$$P\left[x>\frac{1}{\theta}\right]=\int_{1/\theta}^{\infty}\theta\cdot e^{-\theta x}dx$$

$$=\left[\theta\times\frac{e^{-\theta x}}{-\theta}\right]_{1/\theta}^{\infty}$$

$$=[-e^{-\theta x}]_{1/\theta}^{\infty}$$

$$=[-e^{-\theta\infty}-(-e^{-1})]$$

$$=\left[-\frac{1}{e^{\theta\infty}}+\frac{1}{e^1}\right]$$

$$=\left[0+\frac{1}{e^1}\right]$$

$$=e^{-1}=\frac{1}{e}$$

$$P[0<X<M]=\frac{1}{2}$$

$$\therefore\ \int_0^{M}\theta\cdot e^{-\theta x}dx=\frac{1}{2}$$

$$\therefore \left[\theta \times \frac{e^{-\theta x}}{-\theta}\right]_0^M = \frac{1}{2}$$

$$\therefore \left[\theta \times \frac{e^{-\theta x}}{-\theta}\right]_0^M = \frac{1}{2}$$

$$\therefore -[e^{-\theta x}]_0^M = \frac{1}{2}$$

$$\therefore -\left[\frac{1}{e^{\theta x}}\right]_0^M = \frac{1}{2}$$

$$\therefore -\left[\frac{1}{e^{\theta M}} - \frac{1}{e^{0}}\right] = \frac{1}{2}$$

$$\therefore \frac{1}{e^{\theta M}} - \frac{1}{e^{0}} = -\frac{1}{2}$$

$$\therefore \frac{1}{e^{\theta M}} - 1 = -\frac{1}{2}$$

---

16. The p.d.f. of the r.v. X is given by

$$f(x) = \begin{cases} \dfrac{k}{\sqrt{x}} & \text{for } 0 < x < 4. \\ 0 & \text{otherwise} \end{cases}$$

Determine k, the c.d.f. of X and hence find $P(X \leqslant 2)$ and $P(x \geqslant 1)$.

Solution :

$$f(x) = \frac{k}{\sqrt{x}},\ 0 < x < 4$$

$$= 0, \quad \text{otherwise}$$

To determine k :

For the p.d.f. of a r.v. X, we have

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\therefore \int_0^4 \frac{k}{\sqrt{x}}\,dx = 1$$

$$\therefore k[2\sqrt{x}]_0^4 = 1$$

$$\therefore k[2\sqrt{4} - 0] = 1$$

$$\therefore 4k = 1$$

$$\therefore k = \frac{1}{4}$$

c.d.f. of X :

$$F(x_i) = \int_{-\infty}^{x_i} f(x)\,dx$$

$$\therefore F(x_i) = \int_0^x \frac{1}{4\sqrt{x}}\,dx$$

$$= \frac{1}{4}[2\sqrt{x}]_0^x$$

$$F(x_i) = \frac{\sqrt{x}}{2}$$

$$P[X \leqslant 2] = F(2)$$

$$= \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$P[X \geqslant 1] = 1 - P[X < 1]$$

$$= 1 - F(1)$$

$$= 1 - \frac{\sqrt{1}}{2}$$

$$= 1 - \frac{1}{2}$$

$$= \frac{1}{2}$$

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17. Let X denote the reaction temperature in celcius of a certain chemical process. Let X have the p.d.f.

$$f(x) = \begin{cases} \dfrac{1}{10} & \text{for } -5 \leqslant x \leqslant 5. \\ 0 & \text{otherwise} \end{cases}$$

Compute $P(X < 0)$.

Solution :

$$f(x) = \frac{1}{10} \text{ for } -5 \leqslant x \leqslant 5$$

$$= 0 \quad \text{otherwise}$$

$$P[X < 0] = \int_{-5}^{0} \frac{1}{10}\,dx$$

$$= \left[\frac{x}{10}\right]_{-5}^{0}$$

$$= 0 + \frac{5}{10}$$

$$= \frac{1}{2}$$

PART–II

1. Let $X \sim B(10, 0.2)$. Find (i) $P(X = 1)$, (ii) $P(X \geqslant 1)$, (iii) $P(X \leqslant 8)$.

Solution :

Given : $X \sim B(10, 0.2)$

$\therefore n = 10, p = 0.2$

$\therefore q = 1 - p = 1 - 0.2 = 8$

$$P(X = x) = \binom{n}{x} p^x q^{n-x}$$

$$= \binom{10}{x} (0.2)^x (0.8)^{10-x}$$

(i) $P(X = 1) = \binom{10}{1}(0.8)^1 (0.2)^9$

$$= 10 \times 0.8 \times (0.2)^9$$

$$= 8 \times (0.2)^9$$

(ii) $P(X \geqslant 1) = 1 - P(X=0)$

$$= 1 - \binom{10}{0}(0.8)^0(0.2)^{10}$$

$$= 1 - 1 \times 1 \times (0.2)^{10}$$

$$= 1 - (0.2)^{10}$$

(iii) $P(X \leqslant 8) = 1 - [P(X=9) + P(X=10)]$

$$= 1 - \left[\binom{10}{9}(0.8)^9(0.2)^1 + \binom{10}{10}(0.8)^{10}(0.2)^0\right]$$

$$= 1 - [10 \times (0.8)^9(0.2) + 1 \times (0.8)^{10} \times 1]$$

$$= 1 - [(0.8)^9(2 + 0.8)]$$

$$= 1 - 2.8(0.8)^9$$

[**Note :** *Answers given in the textbook are incorrect.*]

2. Let $X \sim B(n, p)$ (i) If $n = 10$ and $E(X) = 5$, find p and Var(X), (ii) If $E(X) = 5$ and Var(X) = 2.5, find n and p.

Solution :

(i) Given : $X \sim B(n, p)$

$n = 10$, $E(X) = 5$

Now, $E(X) = 5$

$\therefore\ np = 5$

$\therefore\ 10p = 5$

$\therefore\ p = \frac{5}{10} = \frac{1}{2}$

$\therefore\ q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

$\text{Var}(X) = npq$

$= np \times q$

$= 5 \times \frac{1}{2} = 2.5$

Hence, $p = \frac{1}{2}$, Var(X) = 2.5

(ii) $E(X) = 5$, Var(X) = 2.5

$\text{Var}(x) = 2.5$

$\therefore\ npq = 2.5$

Now, $E(X) = np = 5$

$\therefore\ 5q = 2.5$

$\therefore\ q = \frac{2.5}{5} = \frac{1}{2}$

$\therefore\ p = 1 - q = 1 - \frac{1}{2} = \frac{1}{2}.$

Now, put $p = \frac{1}{2}$ in $np = 5$, we get

$n \times \frac{1}{2} = 5$

$\therefore\ n = 10$

Hence, $n = 10$, $p = \frac{1}{2}$

3. If a fair coin is tossed 4 times, find the probability that it shows (i) 3 heads, (ii) head in the first 2 tosses and tail in last 2 tosses.

Solution :

Here, $n = 4$

X = Number of heads

$p = \frac{1}{2}\quad \therefore\ q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

$X \sim B\left(4, \frac{1}{2}\right)$

$$\therefore\ P(X = x) = \binom{4}{x}\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{4-x} = \binom{4}{x}\left(\frac{1}{2}\right)^4$$

(i) P[Getting 3 heads]

$= P(X=3)$

$= \binom{4}{3}\left(\frac{1}{2}\right)^4 = 4 \times \frac{1}{16}$

$= \frac{1}{4}$

(ii) P[Getting 2 heads and 2 tails]

$= P(X=2) \times P(X=2)$

$= \binom{4}{2}\left(\frac{1}{2}\right)^4 \times \binom{4}{2}\left(\frac{1}{2}\right)^4$

$= \frac{6}{16} \times \frac{6}{16}$

$= \frac{36}{256}$

$= \frac{9}{64}$

[**Note :** *Answers given in the textbook are incorrect.*]

4. The probability that a bomb will hit the target is 0.8. Find the probability that, out of 5 bombs, exactly 2 will miss the target.

Solution : Here, $n = 5$, X = Number of bombs that hit the target, p = probability that bomb will hit the target

$= 0.8$

$\therefore\ q = 1 - p = 1 - 0.8 = 0.2$

Here, $X \sim B(5, 0.8)$, $q = 0.2$

$\therefore\ P(X=x)=\binom{n}{x}p^x q^{n-x}$

$=\binom{5}{x}(0.8)^x(0.2)^{5-x}$

P[Exactly 2 bombs will miss the target]

$=P$[Exactly 3 bombs will hit the target]

$=P(X=3)$

$=\binom{5}{3}(0.8)^3(0.2)^2$

$=10\times 0.512\times 0.04$

$=0.2084$

[**Note :** *Answer given in the textbook is incorrect.*]

5. The probability that a lamp in the classroom will burn is 0.3. 3 lamps are fitted in the classroom. The classroom is unusable if the number of lamps burning in it is less than 2. Find the probability that the classroom cannot be used on a random occasion.

Solution : Here, $X=B(3,\ 0.3)$

$\therefore\ n=3,\ p=0.3$

$\therefore\ q=1-p=1-0.3=7$

X = Number of lamps burnng in the class room.

Here, $P(X=x)=\binom{n}{x}p^x q^{n-x}$

$=\binom{3}{x}(0.3)^x(0.7)^{3-x}$

P[Less than 2 bulbs burning]

$=P[X<2]$

$=P[X=0]+P[X=1]$

$=\binom{3}{0}(0.3)^0(0.7)^3+\binom{3}{1}(0.3)^1(0.7)^2$

$=1\times 1\times(0.7)^3+3\times 0.3\times(0.7)^2$

$=(0.7)^2[0.7+0.9]$

$=1.6(0.7)^2$

$=1.6\times 0.49$

$=0.784$

6. A large chain retailer purchases an electric device from the manufacturer. The manufacturer indicates that the defective rate of the device is 10%. The inspector of the retailer randomly selects 4 items from a shipment. Find the probability that the inspector finds at most one defective item in the 4 selected items.

Solution : Here, $n=4$

$p=$ probability of defective device

$=10\%=\frac{10}{100}=0.1$

$\therefore\ q=1-p=1-0.1=0.9$

$X\sim B(4,\ 0.1),\ q=0.9$

$\therefore\ P(X=x)=\binom{n}{x}p^x q^{n-x}$

$=\binom{4}{x}(0.1)^x(0.9)^{4-x}$

P[At most one defective device]

$=P[X\leqslant 1]$

$=P[X=0]+P[X=1]$

$=\binom{4}{0}(0.1)^0(0.9)^4+\binom{4}{1}(0.1)^1(0.9)^3$

$=1\times 1\times(0.9)^4+4\times 0.1\times(0.9)^3$

$=(0.9)^3[0.9+0.4]$

$=0.729\times 1.3$

$=0.9477$

7. The probability that a component will survive a check test is 0.6. Find the probability that exactly 2 of the next 4 components tested survive.

Solution : Here, $n=4$

$p=$ A component will survive a check test

$=0.6$

$\therefore\ q=1-p=1-0.6=0.4$

X = Number of components tested

Now, $X\sim B(4,\ 0.6)$ and $q=0.4$

$\therefore\ P[X=x]=\binom{n}{x}p^x q^{n-x}$

$=\binom{4}{x}(0.6)^x(0.4)^{4-x}$

P[Exactly two components tested survive]

$=P(X=2)$

$=\binom{4}{2}(0.6)^2(0.4)^2$

$=6\times 0.36\times 0.16$

$=0.3456$

8. An examination consists of 5 multiple choice questions, in each of which the candidate has to decide which one of 4 suggested answers is correct. A completely unprepared student guesses each answer completely randomly. Find the probability that this student gets 4 or more correct answers.

Solution : Here, $n = 5$

$p =$ probability of correct answer $= \dfrac{1}{4}$

$\therefore\ q = 1 - p = 1 - \dfrac{1}{4} = \dfrac{3}{4}$

X = Number of correct answer of question

Now, $X \sim B\left(5, \dfrac{1}{4}\right)$ and $q = \dfrac{3}{4}$

$$\therefore\ P(X = x) = \binom{n}{x} p^x q^{n-x}$$

$$= \binom{5}{x}\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{5-x}$$

P[Student gets 4 or more correct answers]

$= P(X = 4) + P(X = 5)$

$= \binom{5}{4}\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^1 + \binom{5}{5}\left(\frac{1}{4}\right)^5\left(\frac{3}{4}\right)^0$

$= \dfrac{5 \times 3}{4^5} + \dfrac{1}{4^5}$

$= \dfrac{15}{1024} + \dfrac{1}{1024} = \dfrac{16}{1024}$

$= \dfrac{1}{64}$

[**Note :** *Answer given in the textbook is incorrect.*]

9. The probability that a machine will produce all bolts in a production run within the specification is 0.9. A sample of 3 machines is taken at random. Calculate the probability that all machines will produce all bolts in a production run within the specification.

Solution : Here, $n = 3$

$p =$ Probability that a production run within the specification $= 0.9$

$\therefore\ q = 1 - p = 1 - 0.9 = 0.1$

X = Number of machines produce a production run within the specification

$X \sim B(3, 0.9)$ $\quad \therefore\ n = 3,\ p = 0.9,\ q = 0.1$

$$\therefore\ P(X = x) = \binom{n}{x} p^x q^{n-x}$$

$$= \binom{3}{x}(0.9)^x(0.1)^{3-x}$$

P[All machines will produce the bolts in a production run within the specification]

$= P[X = 3]$

$= \binom{3}{3}(0.9)^3(0.1)^0$

$= 1 \times 0.729 \times 1$

$= 0.729$

[**Note :** *Answer given in the textbook is incorrect.*]

10. A computer installation has 3 terminals. The probability that anyone terminal requires attention during a week is 0.1, independent of other terminals. Find the probabilities that (i) 0, (ii) 1 terminal requires attention during a week.

Solution : Here, $n = 3$

$p =$ Probability that terminal requires during a week $= 0.1$

$\therefore\ q = 1 - p = 1 - 0.1 = 0.9$

X = Number of terminals that require attention

$X \sim B(3, 0.1),\ p = 0.1,\ q = 0.9,\ n = 3$

$$\therefore\ P[X = x] = \binom{n}{x} p^x q^{n-x}$$

$$= \binom{3}{x}(0.1)^x(0.9)^{3-x}$$

(i) P[0 terminal requires attention during a week]

$= P[X = 0]$

$= \binom{3}{0}(0.1)^0(0.9)^3$

$= (0.9)^3$

$= 0.729$

(ii) P[1 terminal requires attention during a week]

$= P[X = 1]$

$= \binom{3}{1}\ (0.1)^1(0.9)^2$

$= 3 \times 0.1 \times 0.81$

$= 0.243$

[**Note :** *Answers given in the textbook are incorrect.*]

11. In a large school, 80% of the students like mathematics. A visitor asks each of 4 students selected at random, whether they like mathematics. (i) Calculate the probabilities of obtaining an answer yes from all of the selected students, (ii) Find the probability that the visitor obtains the answer yes from at least 3 students.

Solution : Here, $n = 4$

$p =$ Probability that a student likes mathematics

$= 80\%$

$$= \frac{80}{100} = 0.8$$

$\therefore\ q = 1 - p = 1 - 0.8 = 0.2$

X = Number of students selected

$X \sim B(4, 0.8),\ n = 4,\ p = 0.8,\ q = 0.2$

$$\therefore\ P(X = x) = \binom{n}{x} p^x q^{n-1}$$

$$= \binom{4}{x} (0.8)^x (0.2)^{4-x}$$

(i) P[All the students selected like mathematics]

$$= P[X = 4]$$

$$= \binom{4}{4} (0.8)^4 (0.2)^0$$

$$= 1 \times (0.8)^4 \times 1$$

$$= 0.4096$$

(ii) P[At least 3 students selected like mathematics]

$$= P[X \geqslant 3] = P[X = 3] + P[X = 4]$$

$$= \binom{4}{3} (0.8)^3 (0.2)^1 + \binom{4}{4} (0.8)^4 (0.2)^0$$

$$= 4 \times (0.8)^3 \times 0.2 + 0.4096$$

$$= 0.8 \times (0.8^3) + 0.4096$$

$$= 0.4096 + 0.4096 = 0.8192$$

[**Note :** *Answers given in the textbook are incorrect.*]

12. It is observed that it rains on 10 days out of 30 days. Find the probability that (i) it rains on exactly 3 days of a week, (ii) it rains on at most 2 days of a week.

Solution :

Let X = Number of days it rains in a week.

$\therefore$ possible values of X = 0, 1, 2, 3, 4, 5, 6, 7

p = Probability that it rains $= \frac{10}{30} = \frac{1}{3}$

$\therefore\ q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$

Given : $n = 7$ (Number of days in a week)

$X \sim B\left(7, \frac{1}{3}\right)$

Hence, $P[X = x] = {}^nC_x p^x q^{n-x}$

$$= {}^7C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{7-x}$$

(i) P [It rains on exactly 3 days of the week]

$$P[X = 3] = {}^7C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^4$$

$$= \frac{7 \times 6 \times 5}{3 \times 2 \times 1} \times \frac{1}{27} \times \frac{16}{81}$$

$$= \frac{35 \times 16}{27 \times 81} = \frac{560}{2187}$$

(ii) P[It will rain on at most 2 days of a week]

$$= P[X \leqslant 2]$$

$$= [P(X = 0) + P(X = 1) + P(X - 2)]$$

$$= {}^7C_0 \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^7 + {}^7C_1 \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^6 + {}^7C_2 \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^5$$

$$= 1 \times 1 \times \left(\frac{2}{3}\right)^7 + 7 \times \frac{1}{3} \times \left(\frac{2}{3}\right)^6 + \frac{7 \times 6}{2 \times 1} \times \frac{1}{9} \times \left(\frac{2}{3}\right)^5$$

$$= \left(\frac{2}{3}\right)^5 \left[\left(\frac{2}{3}\right)^2 + \frac{7}{3}\left(\frac{2}{3}\right)^1 + \frac{21}{9}\right]$$

$$= \left(\frac{2}{3}\right)^5 \left[\frac{4}{9} + \frac{14}{9} + \frac{21}{9}\right]$$

$$= \left(\frac{2}{3}\right)^5 \left[\frac{39}{9}\right]$$

$$= \left(\frac{39}{9}\right)\left(\frac{2}{3}\right)^5 = \frac{13}{3} \times \frac{32}{243} = \frac{416}{729}$$

[**Note :** *Answers given in the textbook are incorrect.*]

13. If X follows Poisson distribution such that $P(X = 1) = 0.4$ and $P(X = 2) = 0.2$, find variance of X.

Solution : $X \sim P(m)$

Given : $P(X = 1) = 0.4$ and $P(X = 2) = 0.2$,

$$p(x) = \frac{e^{-m} m^x}{x!}$$

$$\therefore\ P(X = 1) = p(1) = \frac{e^{-m} m^1}{1!}$$

$$\therefore\ 0.4 = e^{-m} \times m \quad \dots (1)$$

$$\therefore\ P(X = 2) = p(2) = \frac{e^{-m} m^2}{2!}$$

$$\therefore\ 0.2 = \frac{e^{-m} m^2}{2} \quad \dots (2)$$

From (1), $e^{-m} = \frac{0.4}{m}$. Put this result in (2), we get

$$0.2 = \frac{\frac{0.4}{m} \times m^2}{2}$$

$\therefore\ 0.4 = 0.4m$

$\therefore\ m = 1$

Hence, variance of $X = m = 1$.

[**Note :** *Answer given in the textbook is incorrect.*]

14. If X follow Poisson distribution with parameter m such that $\frac{P(X=x+1)}{P(X=x)}=\frac{m}{x+1}$. If $p(0)=0.1353^*$, find mean and variance of X. [Given : $e^{-2}=0.1353^*$]

[* **Note :** *Question has been modified.*]

Solution : $X \sim P(m)$

$$\therefore\ P[X=x]=\frac{e^{-m}\ m^x}{x!}$$

Now, $\frac{P[X=x+1]}{P[X=x]}=\frac{m}{x+1}$ $\quad\therefore\ P(x+1)=P(x)\frac{m}{x+1}$

Put $x=0$, we get

$$P[x=1]=P[x=0]\times\frac{m}{1}$$

$$\therefore\ \frac{e^{-m}m}{1!}=0.1353\times m \qquad \ldots[\because\ P(X=0)=0.1353)]$$

$$\therefore\ e^{-m}=0.1353$$

It is given that $e^{-2}=0.1353$

$$\therefore\ e^{-m}=e^{-2}$$

$$\therefore\ m=2 \qquad \ldots(\because\ \text{Base is same})$$

Hence, Mean = Variance = 2

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