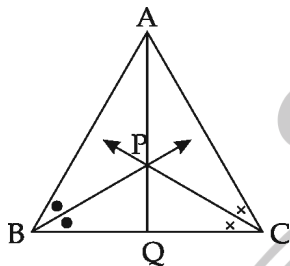


Challenging Questions

1. Similarity

- (1) Bisectors of $\angle B$ and $\angle C$ in $\triangle ABC$ meet each other at P. Line AP cuts the side BC at Q. Then prove that: $\frac{AP}{PQ} = \frac{AB+AC}{BC}$ (4 marks)



Proof:

In $\triangle ABQ$, ray BP bisects $\angle ABQ$. [Given]

$$\therefore \frac{AP}{PQ} = \frac{AB}{BQ} \quad \dots (i) \text{ [By property of an angle bisector of a triangle]}$$

In $\triangle ACQ$, ray CP bisects $\angle ACQ$. [Given]

$$\therefore \frac{AP}{PQ} = \frac{AC}{CQ} \quad \dots (ii) \text{ [By property of an angle bisector of a triangle]}$$

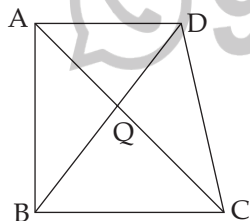
$$\therefore \frac{AP}{PQ} = \frac{AB}{BQ} = \frac{AC}{CQ} \quad \dots \text{ [From (i) and (ii)]}$$

$$\therefore \frac{AP}{PQ} = \frac{AB+AC}{BQ+CQ} \quad \text{[By theorem on equal ratios]}$$

$$\therefore \frac{AP}{PQ} = \frac{AB+AC}{BC}$$

$$\therefore \frac{AP}{PQ} = \frac{AB+AC}{BC}$$

- (2) In $\square ABCD$, side BC $\parallel$ side AD. Diagonal AC and diagonal BD intersect in point Q. If $AQ = \frac{1}{3} AC$, then show that $DQ = \frac{1}{2} BQ$. (4 marks)



Proof:

Side AD $\parallel$ side BC on transversal BD.

$$\angle ADB \cong \angle CBD \quad \dots (i) \text{ [Alternate angles]}$$

In $\triangle AQD$ and $\triangle CQB$,

$$\angle ADQ \cong \angle CBQ \quad \dots \text{ [From (i), B-Q-D]}$$

$$\angle AQD \cong \angle CQB \quad \dots \text{ [Vertically opposite angles]}$$

$$\therefore \triangle AQD \sim \triangle CQB$$

... [By AA test of similarity]

$$\therefore \frac{AQ}{CQ} = \frac{DQ}{BQ} \quad \dots (ii) \text{ [c.s.s.t.]}$$

$$\text{Now, } AQ = \frac{1}{3} AC \quad \dots \text{ [Given]}$$

$$\therefore 3AQ = AC$$

$$\therefore 3AQ = AQ + CQ$$

$$\therefore 3AQ - AQ = CQ$$

$$\therefore 2AQ = CQ$$

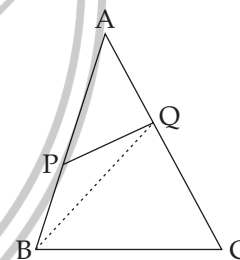
$$\therefore \frac{AQ}{CQ} = \frac{1}{2} \quad \dots (iii)$$

$$\therefore \frac{1}{2} = \frac{DQ}{BQ} \quad \dots \text{ [From (ii) and (iii)]}$$

$$\therefore DQ = \frac{1}{2} BQ$$

- (3) A line cuts two sides AB and side AC of $\triangle ABC$ in points P and Q respectively.

$$\text{Show that } \frac{A(\triangle ABC)}{A(\triangle APQ)} = \frac{AP \times AQ}{AB \times AC} \quad (4 \text{ marks})$$



Construction: Join seg BQ

Proof:

Considering $\triangle APQ$ and $\triangle ABQ$,

$$\frac{A(\triangle APQ)}{A(\triangle ABQ)} = \frac{AP}{AB} \quad \dots (i)$$

[Ratio of areas of two triangles having equal height is equal to the ratio of their corresponding bases]

Considering $\triangle ABQ$ and $\triangle ABC$

$$\frac{A(\triangle ABQ)}{A(\triangle ABC)} = \frac{AQ}{AC} \quad \dots (ii)$$

[Ratio of areas of two triangles having equal height is equal to the ratio of their corresponding bases]

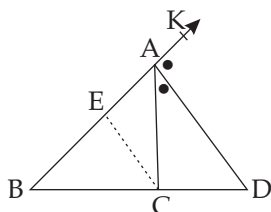
$$\frac{A(\triangle APQ)}{A(\triangle ABQ)} \times \frac{A(\triangle ABQ)}{A(\triangle ABC)} = \frac{AP}{AB} \times \frac{AQ}{AC}$$

... [Multiplying (i) and (ii)]

$$\therefore \frac{A(\triangle ABC)}{A(\triangle APQ)} = \frac{AP \times AQ}{AB \times AC}$$

- (4) In the adjoining figure, AD is the bisector of the exterior $\angle A$ of $\triangle ABC$. Seg AD intersects the side BC produced in D.

Prove that : $\frac{BD}{CD} = \frac{AB}{AC}$ (4 marks)



Construction : Draw seg CE $\parallel$ seg DA meeting BA at E

Proof :

In $\triangle ABD$,

seg CE $\parallel$ seg DA ... (Given)

$$\therefore \frac{BC}{CD} = \frac{BE}{EA} \quad \dots [\text{By B.P.T.}]$$

$$\therefore \frac{BC + CD}{CD} = \frac{BE + EA}{EA} \quad \dots [\text{By componendo}]$$

$$\therefore \frac{BD}{CD} = \frac{AB}{EA} \quad \dots (i) [\text{By B-E-A, B-C-D}]$$

seg CE $\parallel$ seg DA, on transversal BK

$$\angle KAD \cong \angle AEC \quad \dots (ii) [\text{Corresponding angles theorem}]$$

seg CE $\parallel$ seg DA, on transversal AC

$$\angle CAD \cong \angle ACE \quad \dots (iii) [\text{Alternate angles theorem}]$$

$$\text{Also, } \angle KAD \cong \angle CAD \quad \dots (iv) [\text{Ray AD bisects } \angle KAC]$$

$$\therefore \angle AEC \cong \angle ACE \quad \dots (v) [\text{From (ii), (iii) and (iv)}]$$

In $\triangle AEC$,

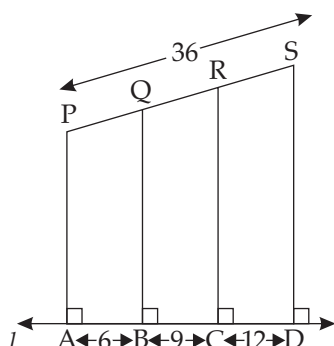
$$\angle AEC \cong \angle ACE \quad \dots [\text{From (v)}]$$

$$\therefore EA = AC \quad \dots (vi) [\text{By converse of isosceles triangle theorem}]$$

$$\therefore \frac{BD}{CD} = \frac{AB}{AC} \quad \dots [\text{From (i) and (vi)}]$$

- (5) In the adjoining figure, each of the segments PA, QB, RC and SD is perpendicular to line l . If $AB = 6$, $BC = 9$, $CD = 12$, $PS = 36$, then determine PQ, QR and RS. (4 marks)

Proof :



$$\left. \begin{array}{l} \text{Seg PA} \perp \text{line } l \\ \text{Seg QB} \perp \text{line } l \\ \text{Seg RC} \perp \text{line } l \\ \text{Seg SD} \perp \text{line } l \end{array} \right\} \dots [\text{Given}]$$

$$\therefore \text{seg PA} \parallel \text{seg QB} \parallel \text{seg RC} \parallel \text{seg SD} \quad \dots (i) [\because \text{lines perpendicular to same lines are parallel}]$$

Now, seg PA $\parallel$ seg QB $\parallel$ seg RC

$$\therefore \frac{PQ}{QR} = \frac{AB}{BC} \quad \dots [\text{Property of three parallel lines and their transversals}]$$

$$\therefore \frac{PQ}{QR} = \frac{6}{9} \quad \dots [\text{Given AB} = 6, \text{BC} = 9]$$

$$\therefore \frac{PQ}{QR} = \frac{2}{3} \quad \dots (ii)$$

Also, seg QB $\parallel$ seg RC $\parallel$ seg SD ... [From (i)]

$$\therefore \frac{QR}{RS} = \frac{BC}{CD} \quad \dots [\text{Property of three parallel lines and their transversals}]$$

$$\therefore \frac{QR}{RS} = \frac{9}{12} \quad \dots [\text{Given, BC} = 9, \text{CD} = 12]$$

$$\therefore \frac{QR}{RS} = \frac{3}{4} \quad \dots (iii)$$

$$\therefore PQ : QR : RS = 2 : 3 : 4 \quad \dots [\text{From (ii) and (iii)}]$$

Let the common multiple be x .

$$\therefore PQ = 2x, QR = 3x, RS = 4x$$

$$\text{Now, } PQ + QR + RS = PS \quad \dots [P-Q-R, Q-R-S]$$

$$\therefore 2x + 3x + 4x = 36$$

$$\therefore 9x = 36 \quad \therefore x = \frac{36}{9} = 4$$

$$\therefore PQ = 2x = 2 \times 4 = 8 \text{ units}$$

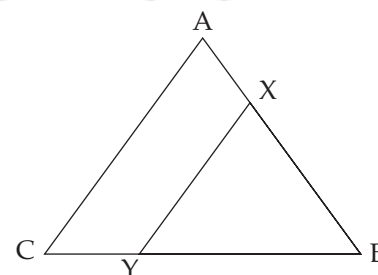
$$\therefore QR = 3x = 3 \times 4 = 12 \text{ units}$$

$$\therefore RS = 4x = 4 \times 4 = 16 \text{ units}$$

$$\therefore \boxed{PQ = 8 \text{ units, } QR = 12 \text{ units, } RS = 16 \text{ units}}$$

- (6) In the adjoining figure, $XY \parallel AC$ and XY divides the triangular region ABC into two equal areas. Determine $AX : AB$. (4 marks)

Proof :



seg XY $\parallel$ side AC on transversal BC.

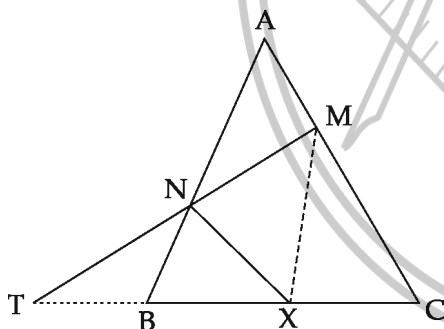
$$\angle XYB \cong \angle ACB \quad \dots (i) [\text{Corresponding angles}]$$

In $\triangle XYB$ and $\triangle ACB$,

$$\angle XYB \cong \angle ACB \quad \dots [\text{From (i)}]$$

$\angle ABC \cong \angle XBY$... [Common angle]
 $\therefore \triangle XYB \sim \triangle ACB$... [BY AA test of similarity]
 $\frac{A(\triangle XYB)}{A(\triangle ACB)} = \frac{XB^2}{AB^2}$... (ii) [By theorem on areas of similar triangles]
 Now, $A(\triangle XYB) = \frac{1}{2} A(\triangle ACB)$... [$\because$ seg XY divides the triangular region ABC into two equal areas]
 $\therefore \frac{A(\triangle XYB)}{A(\triangle ACB)} = \frac{1}{2}$... (iii)
 $\therefore \frac{XB^2}{AB^2} = \frac{1}{2}$... [From (ii) and (iii)]
 $\therefore \frac{XB}{AB} = \frac{1}{\sqrt{2}}$... [Taking square root on both sides]
 $\therefore 1 - \frac{XB}{AB} = 1 - \frac{1}{\sqrt{2}}$... [Subtracting both sides from 1]
 $\therefore \frac{AB - XB}{AB} = \frac{\sqrt{2} - 1}{\sqrt{2}} \therefore \frac{AX}{AB} = \frac{\sqrt{2} - 1}{\sqrt{2}}$... [A-X-B]
 $\therefore AX : AB = (\sqrt{2} - 1) : \sqrt{2}$
 $\therefore \boxed{AX : AB = (\sqrt{2} - 1) : \sqrt{2}}$

- (7) Let X be any point on side BC of $\triangle ABC$, XM and XN are drawn parallel to BA and CA. MN meets in T. Prove that $TX^2 = TB \cdot TC$. (4 marks)

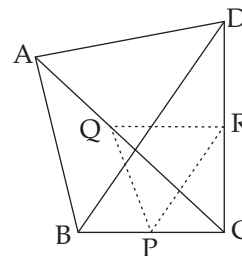


Proof :

In $\triangle TXM$,
 seg BN $\parallel$ seg XM ... [Given]
 $\therefore \frac{TN}{NM} = \frac{TB}{BX}$... (i) [By B.P.T.]
 In $\triangle TMC$,
 seg XN $\parallel$ seg CM ... [Given]
 $\therefore \frac{TN}{NM} = \frac{TX}{CX}$... (ii) [By B.P.T.]
 $\therefore \frac{TB}{BX} = \frac{TX}{CX}$... [From (i) and (ii)]
 $\therefore \frac{BX}{TB} = \frac{CX}{TX}$... [By invertendo]
 $\therefore \frac{BX + TB}{TB} = \frac{CX + TX}{TX}$... [By componendo]

$\therefore \frac{TX}{TB} = \frac{TC}{TX}$... [T-B-X, T-X-C]
 $\therefore \boxed{TX^2 = TB \cdot TC}$

- (8) Two triangles, $\triangle ABC$ and $\triangle DBC$, lie on the same side of the base BC. From a point P on BC, $PQ \parallel AB$ and $PR \parallel BD$ are drawn. They intersect AC at Q and DC at R. Prove that $QR \parallel AD$. (4 marks)

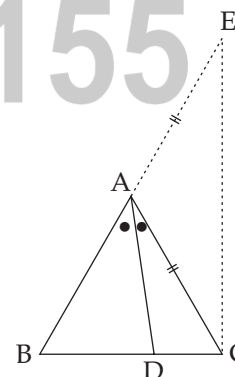


Proof :

In $\triangle CAB$,
 seg PQ $\parallel$ seg AB ... [Given]
 $\therefore \frac{CP}{PB} = \frac{CQ}{AQ}$... (i) [By B.P.T.]
 In $\triangle CBD$,
 seg PR $\parallel$ seg BD ... [Given]
 $\therefore \frac{CP}{PB} = \frac{CR}{RD}$... (ii) [By B.P.T.]
 In $\triangle ACD$,
 $\therefore \frac{CQ}{AQ} = \frac{CR}{RD}$... [From (i) and (ii)]
 $\therefore \boxed{\text{seg QR} \parallel \text{seg AD}}$... [By converse of B.P.T.]

- (9) In $\triangle ABC$, D is a point on BC such that $\frac{BD}{DC} = \frac{AB}{AC}$. Prove that AD is the bisector of $\angle A$. (Hint : Produce BA to E such that $AE = AC$. Join EC) (4 marks)

Proof :

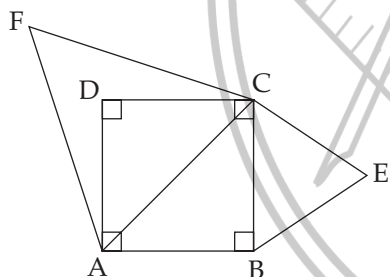


Proof : Seg BA is produced to point E such that $AE = AC$ and seg EC is drawn.

$\therefore \frac{BD}{DC} = \frac{AB}{AC}$... (i) [Given]

$AC = AE$... (ii) [By construction]
 $\therefore \frac{BD}{DC} = \frac{AB}{AE}$... (iii) [Substituting (ii) in (i)]
 $\therefore \text{seg AD} \parallel \text{seg EC}$... [By converse of B.P.T.]
 On transversal BE,
 $\angle BAD \cong \angle BEC$... [Corresponding angles theorem]
 $\therefore \angle BAD \cong \angle AEC$... (iv) [$\because$ B-A-E]
 On transversal AC,
 $\angle CAD \cong \angle ACE$... (v) [Alternate angles theorem]
 In $\triangle ACE$,
 $\text{seg AC} \cong \text{seg AE}$... [By construction]
 $\angle AEC \cong \angle ACE$... (vi) [By isosceles triangle theorem]
 $\therefore \angle BAD \cong \angle CAD$... [From (iv), (v) and (vi)]
 $\therefore$ **Ray AD is the bisector of $\angle BAC$.**

- (10) In the adjoining figure, $\square ABCD$ is a square. $\triangle BCE$ on side BC and $\triangle ACF$ on the diagonal AC are similar to each other. Then, show that $A(\triangle BCE) = \frac{1}{2} A(\triangle ACF)$ (4 marks)**



Proof :

$\square ABCD$ is a square. ... [Given]
 $\therefore AC = \sqrt{2} BC$... (i) [$\because$ Diagonal of a square = $\sqrt{2} \times$ side of square]
 $\triangle BCE \sim \triangle ACF$... [Given]
 $\therefore \frac{A(\triangle BCE)}{A(\triangle ACF)} = \frac{(BC)^2}{(AC)^2}$... (ii) [By theorem on areas of similar triangles]
 $\therefore \frac{A(\triangle BCE)}{A(\triangle ACF)} = \frac{(BC)^2}{(\sqrt{2} \cdot BC)^2}$... [From (i) and (ii)]
 $\therefore \frac{A(\triangle BCE)}{A(\triangle ACF)} = \frac{BC^2}{2BC^2}$
 $\therefore \frac{A(\triangle BCE)}{A(\triangle ACF)} = \frac{1}{2}$
 $\therefore$ **$A(\triangle BCE) = \frac{1}{2} A(\triangle ACF)$**

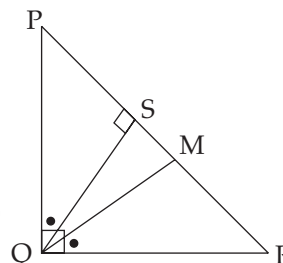


2. Theorem of Pythagoras

- (1) In $\triangle PQR$, $\angle PQR = 90^\circ$, as shown in figure, seg QS $\perp$ side PR, seg QM is angle bisector of $\angle PQR$.**

Prove that : $\frac{PM^2}{MR^2} = \frac{PS}{SR}$ (4 marks)

Proof :

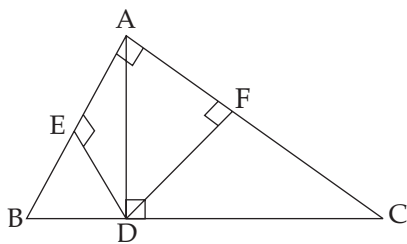


In $\triangle PQR$,
 Seg QM bisects $\angle PQR$... [Given]
 $\therefore \frac{PM}{MR} = \frac{PQ}{QR}$ [Property of an angle bisector of a triangle]
 $\therefore \frac{PM^2}{MR^2} = \frac{PQ^2}{QR^2}$... (i) [Squaring both sides]
 In $\triangle PQR$,
 $m\angle PQR = 90^\circ$... [Given]
 Seg QS $\perp$ hypotenuse PR ... [Given]
 $\therefore \triangle PQR \sim \triangle PSQ \sim \triangle QSR$... (ii) [Theorem on similarity of right angled triangles]
 $\triangle PSQ \sim \triangle PQR$... [From (ii)]
 $\therefore \frac{PQ}{PR} = \frac{PS}{PQ}$... [c.s.s.t.]
 $\therefore PQ^2 = PR \times PS$... (iii)
 Also, $\triangle QSR \sim \triangle PQR$... [From (ii)]
 $\therefore \frac{QR}{PR} = \frac{SR}{QR}$... [c.s.s.t.]
 $\therefore QR^2 = PR \times SR$... (iv)
 $\therefore \frac{PM^2}{MR^2} = \frac{PR \times PS}{PR \times SR}$... [From (i), (iii) and (iv)]
 $\therefore$ **$\frac{PM^2}{MR^2} = \frac{PS}{SR}$**

- (2) In $\triangle ABC$, $m\angle BAC = 90^\circ$, seg DE $\perp$ side AB, seg DF $\perp$ side AC, seg AD $\perp$ side BC.**

Prove : $A(\square AEDF) = \sqrt{AE \times EB \times AF \times FC}$ (4 marks)

Proof :



In $\triangle ADB$,
 $m\angle ADB = 90^\circ$... [Given]
 $\text{seg } DE \perp \text{side } AB$... [Given]
 $\therefore DE^2 = AE \times EB$... (i) [By property of geometric mean]

In $\triangle ADC$,
 $m\angle ADC = 90^\circ$... [Given]
 $\text{Seg } DF \perp \text{side } AC$... [Given]
 $\therefore DF^2 = AF \times FC$... (ii) [By property of geometric mean]

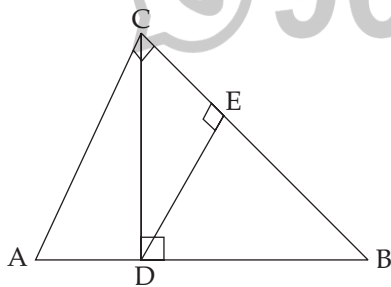
Multiplying (i) and (ii), we get
 $\therefore DE^2 \times DF^2 = AE \times EB \times AF \times FC$
 $\therefore DE \times DF = \sqrt{AE \times EB \times AF \times FC}$... (iii)

In $\square AEDF$,
 $\therefore m\angle EAF = m\angle AED = m\angle AFD = 90^\circ$... [Given]
 $\therefore m\angle EDF = 90^\circ$... [Remaining angle]
 $\therefore \square AEDF$ is a rectangle ... [By definition]
 $\therefore A(\square AEDF) = DE \times DF$... (iv) [Area of rectangle = $l \times b$]

From (iii) and (iv), we get
 $\therefore A(\square AEDF) = \sqrt{AE \times EB \times AF \times FC}$

- (3) In $\triangle ABC$, $\angle ACB = 90^\circ$,
 $\text{seg } CD \perp \text{side } AB$,
 $\text{seg } DE \perp \text{seg } CB$.
Show that : $CD^2 \times AC = AD \times AB \times DE$ (4 marks)

Proof :



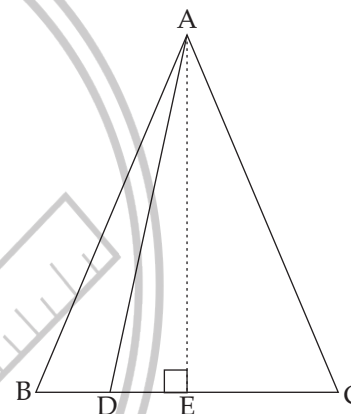
In $\triangle ACB$,
 $\angle ACB = 90^\circ$... [Given]
 $\text{seg } CD \perp \text{hypotenuse } AB$,
 $\therefore CD^2 = AD \times DB$... (i) [By property of geometric mean]

In $\triangle DEB$ and $\triangle ACB$,
 $\angle DEB \cong \angle ACB$... [Each is 90°]
 $\angle DBE \sim \angle ABC$... [Common angle]
 $\therefore \triangle DEB \cong \triangle ACB$... [By AA test of similarity]
 $\therefore \frac{DE}{AC} = \frac{DB}{AB}$... [c.s.s.t.]
 $\therefore AC = \frac{DE \times AB}{DB}$... (ii)
 $\therefore CD^2 \times AC = AD \times DB \times \frac{DE \times AB}{DB}$... [Multiplying (i) and (ii)]

$$\therefore CD^2 \times AC = AD \times AB \times DE$$

- (4) In an equilateral $\triangle ABC$, the side BC is trisected at D . Prove that $9AD^2 = 7AB^2$. (Hint : $AE \perp BC$) (4 marks)

Proof :



Construction : $AE \perp BC$ is drawn

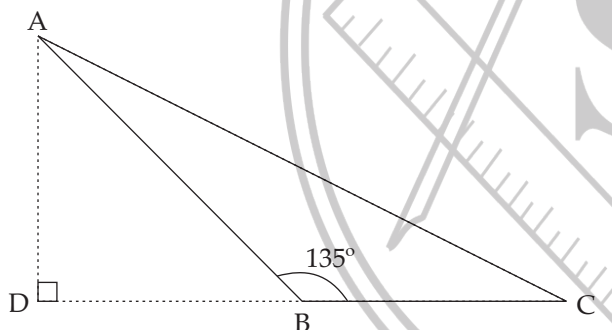
In $\triangle AED$,
 $m\angle AED = 90^\circ$... [By construction]
 $\therefore AD^2 = AE^2 + DE^2$... (i) [By Pythagoras theorem]
 $\therefore \angle ABC = 60^\circ$... (ii) [Angle of an equilateral triangle]

In $\triangle AEB$,
 $m\angle AEB = 90^\circ$... [By construction]
 $\therefore \angle ABE = 60^\circ$... [B-E-C]
 $\therefore \angle BAE = 30^\circ$... [Remaining angle of a triangle]
 $\therefore \triangle AEB$ is a $30^\circ - 60^\circ - 90^\circ$ triangle
 $\therefore AE = \frac{\sqrt{3}}{2} (AB)$... (iii) [Side opposite to 60°]
 $\therefore BE = \frac{1}{2} (AB)$... (iv) [Side opposite to 30°]
 $\therefore BD = \frac{1}{3} (BC)$... [Given]
 $\therefore BD = \frac{1}{3} (AB)$... (v) [Since, $BC = AB$, sides of an equilateral triangle]
 $\therefore DE + BD = BE$... [B-D-E]

$$\begin{aligned} \therefore DE &= BE - BD \\ \therefore DE &= \frac{1}{2} AB - \frac{1}{3} AB \quad \dots [\text{From (iv) and (v)}] \\ \therefore DE &= \frac{3AB - 2AB}{6} \\ \therefore DE &= \frac{1}{6} (AB) \quad \dots (\text{vi}) \\ \therefore AD^2 &= \left[\frac{\sqrt{3}}{2} AB \right]^2 + \left[\frac{1}{6} AB \right]^2 \quad \dots [\text{Substituting (iii), (vi) in (i)}] \\ \therefore AD^2 &= \frac{3}{4} AB^2 + \frac{1}{36} AB^2 \\ \therefore AD^2 &= \frac{27AB^2 + AB^2}{36} \\ \therefore AD^2 &= \frac{28AB^2}{36} \\ \therefore AD^2 &= \frac{7}{9} AB^2 \\ \therefore \boxed{9AD^2 = 7AB^2} \end{aligned}$$

- (5) In $\triangle ABC$, $\angle ABC = 135^\circ$.
Prove that : $AC^2 = AB^2 + BC^2 + 4A(\triangle ABC)$.
Construction : Draw seg $AD \perp$ side BC , such that D - B - C .
(4 marks)

Proof :



$$\begin{aligned} m \angle ABC + m \angle ABD &= 180^\circ \\ \dots (\text{Angles forming linear pair}) \end{aligned}$$

$$\begin{aligned} \therefore 135^\circ + m \angle ABD &= 180^\circ \\ \therefore m \angle ABD &= 180^\circ - 135^\circ \\ \therefore m \angle ABD &= 45^\circ \quad \dots (\text{i}) \end{aligned}$$

$$\begin{aligned} \text{In } \triangle ADB, \\ m \angle ADB &= 90^\circ \quad \dots (\text{Given}) \\ m \angle ABD &= 45^\circ \quad \dots [\text{From (i)}] \\ \therefore m \angle BAD &= 45^\circ \quad \dots (\text{ii}) (\text{Remaining angle}) \end{aligned}$$

$$\begin{aligned} \text{In } \triangle ABD, \\ \angle ABD &\cong \angle BAD \quad \dots [\text{From (i) and (ii)}] \\ \therefore \text{seg } AD &\cong \text{seg } DB \quad \dots (\text{iii}) (\text{Converse of isosceles triangle theorem}) \end{aligned}$$

$$\begin{aligned} \text{In } \triangle ADB, \\ m \angle ADB &= 90^\circ \quad (\text{Construction}) \\ \therefore AB^2 &= AD^2 + DB^2 \quad \dots (\text{iv}) \\ &\quad (\text{By Pythagoras theorem}) \end{aligned}$$

$$\begin{aligned} \text{In } \triangle ADC, \\ m \angle ADC &= 90^\circ \quad \dots (\text{Construction}) \\ \therefore AC^2 &= AD^2 + DC^2 \quad \dots (\text{By Pythagoras theorem}) \\ \therefore AC^2 &= AD^2 + (DB + BC)^2 \quad \dots (\because D - B - C) \\ \therefore AC^2 &= AD^2 + DB^2 + 2 \times DB \times BC + BC^2 \\ \therefore AC^2 &= AB^2 + BC^2 + 2 \times DB \times BC \quad \dots [\text{From (iv)}] \\ \therefore AC^2 &= AB^2 + BC^2 + 2 \times AD \times BC \quad \dots (\text{v}) [\text{From (iii)}] \end{aligned}$$

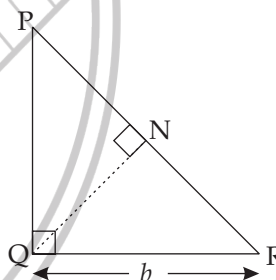
$$\text{Area of triangle} = \frac{1}{2} \times \text{base} \times \text{height}$$

$$\begin{aligned} \therefore A(\triangle ABC) &= \frac{1}{2} \times BC \times AD \\ \therefore 4A(\triangle ABC) &= 4 \times \frac{1}{2} \times BC \times AD \\ &\quad \dots (\text{Multiplying throughout by 4}) \\ \therefore 4A(\triangle ABC) &= 2 \times AD \times BC \quad \dots (\text{vi}) \end{aligned}$$

$$\therefore \boxed{AC^2 = AB^2 + BC^2 + 4A(\triangle ABC)}$$

- (6) In $\triangle PQR$ is a right angled triangle, right angled at Q such that $QR = b$ and $A(\triangle PQR) = a$.
If $QN \perp PR$, then show that $QN = \frac{2a \cdot b}{\sqrt{b^2 + 4a^2}}$.
(4 marks)

Proof :



$$\text{Area of triangle} = \frac{1}{2} \times \text{base} \times \text{height}$$

$$\begin{aligned} \therefore A(\triangle PQR) &= \frac{1}{2} \times QR \times PQ \\ \therefore a &= \frac{1}{2} \times b \times PQ \quad \dots (\text{Given}) \\ \therefore \frac{2a}{b} &= PQ \quad \dots (\text{i}) \end{aligned}$$

$$\begin{aligned} \text{Also,} \\ A(\triangle PQR) &= \frac{1}{2} \times PR \times QN \\ \therefore a &= \frac{1}{2} \times PR \times QN \quad \dots (\text{Given}) \\ \therefore QN &= \frac{2a}{PR} \quad \dots (\text{ii}) \end{aligned}$$

$$\begin{aligned} \text{In } \triangle PQR, \\ m \angle PQR &= 90^\circ \quad \dots (\text{Given}) \\ \therefore PR^2 &= PQ^2 + QR^2 \quad \dots (\text{By Pythagoras theorem}) \\ \therefore PR &= \sqrt{PQ^2 + QR^2} \quad \dots (\text{Taking square roots}) \end{aligned}$$

$$\therefore PR = \sqrt{\left(\frac{2a}{b}\right)^2 + b^2} \quad \dots[\text{From (i) and given}]$$

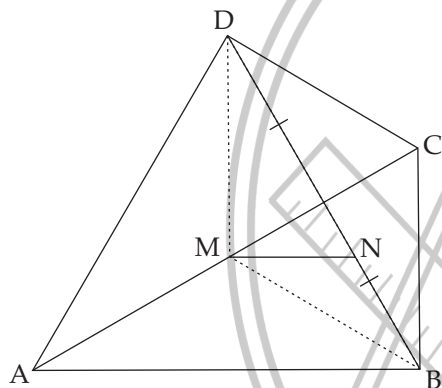
$$\therefore PR = \sqrt{\left(\frac{4a^2}{b^2}\right) + b^2}$$

$$\therefore PR = \frac{\sqrt{b^4 + 4a^2}}{b} \quad \dots(\text{iii})$$

$$\therefore QN = \frac{2a}{\sqrt{b^4 + 4a^2}} \quad \dots[\text{From (ii) and (iii)}]$$

$$\therefore QN = \frac{2ab}{\sqrt{b^4 + 4a^2}}$$

- (7) In $\square ABCD$ is a quadrilateral. M is the midpoint of diagonal AC and N is the midpoint of diagonal BD . Prove that : $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4MN^2$. (4 marks)



Given : $\square ABCD$ is a quadrilateral.

M and N are the midpoints of diagonal AC and BD respectively.

To prove : $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4MN^2$

Construction : Join seg DM and seg BM .

Proof :

In $\triangle ADC$,

seg DM is the median.

$$\therefore AD^2 + CD^2 = 2DM^2 + 2CM^2 \quad \dots [\text{Apollonius theorem}]$$

$$\therefore AD^2 + CD^2 = 2DM^2 + 2\left[\frac{1}{2}AC\right]^2 \quad \dots [M \text{ is the midpoint of } AC]$$

$$\therefore AD^2 + CD^2 = 2DM^2 + 2 \times \frac{1}{4}AC^2$$

$$\therefore AD^2 + CD^2 = 2DM^2 + \frac{1}{2}AC^2 \quad \dots (\text{i})$$

Similarly, in $\triangle ABC$ seg BM is the median.

$$AB^2 + BC^2 = 2BM^2 + \frac{1}{2}AC^2 \quad \dots (\text{ii})$$

$$\therefore AD^2 + CD^2 + AB^2 + BC^2 = 2DM^2 + \frac{1}{2}AC^2 + 2BM^2 + \frac{1}{2}AC^2 \quad \dots [\text{Adding (i) and (ii)}]$$

$$\therefore AB^2 + BC^2 + CD^2 + DA^2 = 2BM^2 + 2DM^2 + AC^2 \quad \dots (\text{iii})$$

In $\triangle DMB$,

seg MN is the median

$$\therefore BM^2 + DM^2 = 2MN^2 + 2BN^2 \quad \dots [\text{Apollonius theorem}]$$

$$\therefore BM^2 + DM^2 = 2MN^2 + 2\left[\frac{1}{2}BD\right]^2 \quad \dots [N \text{ is the midpoint of } BD]$$

$$\therefore BM^2 + DM^2 = 2MN^2 + 2 \times \frac{1}{4}BD^2$$

$$\therefore BM^2 + DM^2 = 2MN^2 + \frac{1}{2}BD^2$$

$$\therefore 2BM^2 + 2DM^2 = 4MN^2 + 2 \times \frac{1}{2}BD^2 \quad \dots [\text{Multiplying by } 2]$$

$$\therefore 2BM^2 + 2DM^2 = 4MN^2 + BD^2 \quad \dots (\text{iv})$$

$$\therefore AB^2 + BC^2 + CD^2 + DA^2 = (4MN^2 + BD^2) + AC^2 \quad \dots [\text{Substituting (iv) in (iii)}]$$

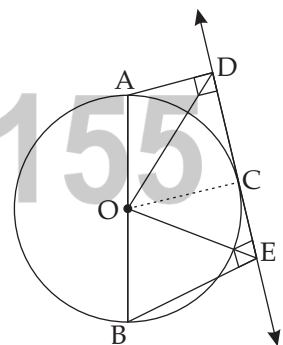
$$\therefore AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4MN^2$$



3. Circle

- (1) From the end points of a diameter of circle perpendiculars are drawn to a tangent of the same circle. Show that their feet on the tangent are equidistant from the centre of the circle.

(4 marks)



Given : (i) A circle with centre O .

(ii) Seg AB is the diameter of the circle.

(iii) Line l is tangent to the circle at point C .

(iv) Seg $AD \perp$ line l

(v) Seg $BE \perp$ line l

To Prove : $OD = OE$

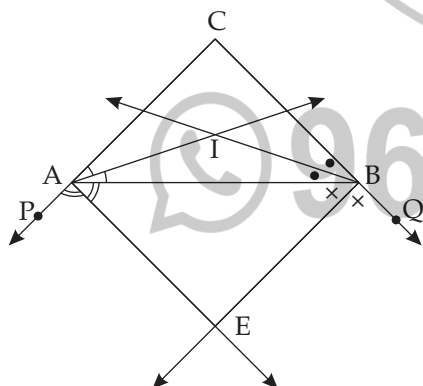
Construction : Draw seg OC

Proof :

Seg AD $\perp$ line l ... [Given]
 Seg OC $\perp$ line l ... [Radius is perpendicular to the tangent]
 Seg BE $\perp$ line l ... [Given]
 $\therefore$ Seg AD $\parallel$ seg OC $\parallel$ seg BE ... [Perpendiculars drawn to the same line are parallel to each other]
 $\therefore$ On transversal AB and DE,
 $\frac{AO}{OB} = \frac{DC}{CE}$... (i) [Property of three parallel lines and their transversals]
 But, AO = OB ... [Radii of the same circle]
 $\therefore \frac{AO}{OB} = 1$... (ii)
 $\therefore \frac{DC}{CE} = 1$... [From (i) and (ii)]
 $\therefore DC = CE$... (iii)
 In $\triangle OCD$ and $\triangle OCE$,
 seg OC $\cong$ seg OC ... [Common side]
 $\angle OCD \cong \angle OCE$... [Each is a right angle]
 seg DC $\cong$ seg CE ... [From (iii)]
 $\therefore \triangle OCD \cong \triangle OCE$... [By SAS test of congruence]
 $\therefore$ seg OD $\cong$ seg OE ... (c.s.c.t)
 $\therefore$ **OD = OE**

- (2) The bisectors of the angles A, B of $\triangle ABC$ intersect in I, the bisectors of the corresponding exterior angles intersect in E. Prove that $\square AIBE$ is cyclic. (4 marks)

Solution :

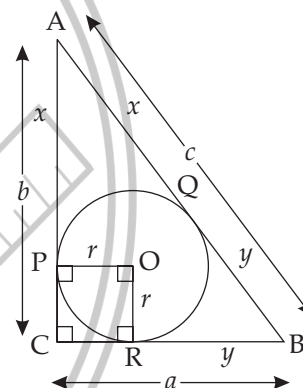


Proof : Take point P and Q as show in fig.

$m\angle CAB + m\angle BAP = 180^\circ$... [Linear pair axiom]
 $\therefore \frac{1}{2} m\angle CAB + \frac{1}{2} m\angle BAP = \frac{1}{2} \times 180^\circ$
 ... [Multiplying throughout by $\frac{1}{2}$]

$\therefore m\angle IAB + m\angle BAE = 90^\circ$... [$\because$ Ray AI and ray AE bisects $\angle CAB$ and $\angle BAP$ respectively]
 $\therefore m\angle IAE = 90^\circ$... (i) [Angle addition property]
 Similarly,
 $\therefore m\angle IBE = 90^\circ$... (ii)
 $\therefore m\angle IAE + m\angle IBE = 90^\circ + 90^\circ$... [Adding (i) and (ii)]
 $\therefore m\angle IAE + m\angle IBE = 180^\circ$
 $\therefore$ **$\square AIBE$ is cyclic** ... [If opposite angles of a quadrilateral are supplementary then quadrilateral is cyclic]

- (3) In a right angled $\triangle ABC$, $\angle ACB = 90^\circ$. A circle is inscribed in the triangle with radius r . a, b, c are the lengths of the sides BC, AC and AB respectively. Prove that $2r = a + b - c$. (4 marks)



Proof :

Let the centre of the inscribed circle be 'O'
 Let AP = AQ = x ... (i)
 CP = CR = y ... (ii)
 BR = BQ = z ... (iii)

(The lengths of the two tangent segments to a circle drawn from an external point are equal)
 $a + b - c = BC + AC - AB$
 $\therefore a + b - c = CR + RB + AP + PC - (AQ + QB)$
 $(B - R - C, A - P - C, A - Q - B)$
 $\therefore a + b - c = y + z + x + y - (x + z)$
 [From (i), (ii) and (iii)]

$\therefore a + b - c = y + z + x + y - x - z$
 $\therefore a + b - c = 2y$
 $\therefore a + b - c = 2CP$... (iv) [From (ii)]

In $\square PCRO$
 $m\angle OPC = m\angle ORC = 90^\circ$
 ... (Radius is perpendicular to tangent)
 $m\angle PCR = 90^\circ$... (Given)
 $\therefore m\angle POR = 90^\circ$... (Remaining angle)

$\therefore \square PCRO$ is a rectangle ... (By definition)
 $\therefore CP = OR$... (v)
 (Opposite sides of a rectangle)
 $\therefore a + b - c = 2 OR$... [From (iv) and (v)]
 $\therefore a + b - c = 2r$

- (4) If two consecutive angles of cyclic quadrilateral are congruent, then prove that one pair of opposite sides is congruent and other is parallel.**

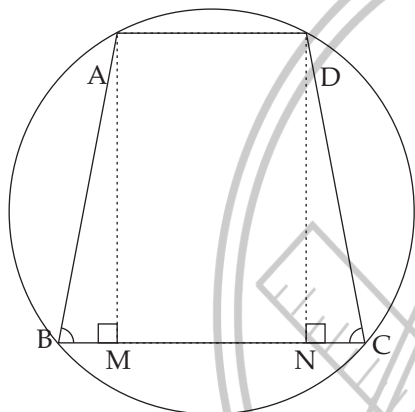
(4 marks)

Given : $\square ABCD$ is a cyclic quadrilateral

$$\angle ABC \cong \angle BCD$$

To Prove : side $DC \cong$ side AB , $AD \parallel BC$

Construction : Draw seg AM and seg DN both perpendicular to side BC .



Proof :

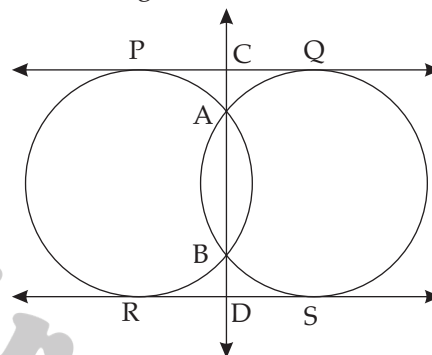
I $\angle ABC \cong \angle BCD$... (i) [Given]
 $\angle ABC + \angle ADC = 180^\circ$... (ii) [Opposite angles of a cyclic quadrilateral are supplementary]
 $\angle BCD + \angle ADC = 180^\circ$... [From (i) and (ii)]
 $\therefore$ Side $AD \parallel$ side BC ... [Interior angles test]
 In $\triangle DNC$ and $\triangle AMB$,
 seg $DN \cong$ seg AM ... [Perpendicular distance between two parallel lines]
 $\angle DNC \cong \angle AMB$... [Each is 90°]
 $\angle DCN \cong \angle ABM$... [Given B-M-N-C]
 $\therefore \triangle DNC \cong \triangle AMB$... [By SAA test of congruence]
 $\therefore$ **side $DC \cong$ side AB** ... [c.s.c.t.]

- (5) As shown in the adjoining figure, two circles intersect each other in points A and B. Two tangents touch these circles in points P, Q and R, S as shown. Line AB intersects seg PQ in C and seg RS in D. Show that C and D are midpoints of seg PQ and seg RS respectively.** (3 marks)

Given : Two circles intersect each other in points A and B.

Line PQ and RS are the common tangents and line CD is a common secant.

To Prove : C and D are midpoints of seg PQ and seg RS.



Proof :

Line CP is a tangent and line CD is a secant.

$$\therefore CP^2 = CA \times CB \quad \dots (i) \text{ [Tangent secant segment theorem]}$$

$$\text{Similarly, } CQ^2 = CA \times CB \quad \dots (ii)$$

$$\therefore CP^2 = CQ^2 \quad \dots \text{ [From (i) and (ii)]}$$

$$\therefore CP = CQ \quad \dots \text{ [Taking square root on both sides]}$$

$$\therefore \text{C is the midpoint of seg PQ} \quad \dots \text{ [P - C - Q]}$$

Line RD is a tangent and line CD is a secant

$$\therefore RD^2 = DB \times DA \quad \dots (iii) \text{ [Tangent secant segment theorem]}$$

$$\text{Similarly, } SD^2 = DB \times DA \quad \dots (iv)$$

$$\therefore RD^2 = SD^2 \quad \dots \text{ [From (iii) and (iv)]}$$

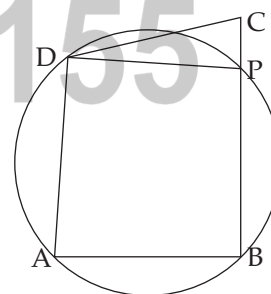
$$\therefore RD = SD \quad \dots \text{ [Taking square root on both sides]}$$

$$\therefore \text{D is the midpoint of seg RS} \quad \dots \text{ [R - D - S]}$$

- (6) $\square ABCD$ is a parallelogram. A circle passing through D, A, B cuts BC in P. Prove that $DC = DP$.** (3 marks)

Given : $\square ABCD$ is a parallelogram.

Prove : $DC = DP$



Proof :

$\square ABPD$ is a cyclic quadrilateral.

[By definition]

$$\therefore \angle BAD = \angle DPC \quad \dots (i) \text{ [Exterior angle of a cyclic quadrilateral equals to the interior opposite angle]}$$

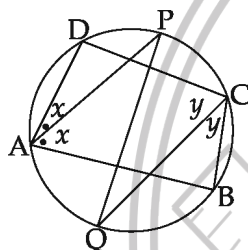
$\square ABCD$ is a parallelogram ... [Given]
 $\therefore \angle BAD = \angle DCB$... [Opposite angles of a parallelogram are equal]
 $\therefore \angle BAD = \angle DCP$... (ii) [C - P - B]
 In $\triangle DCP$,
 $\angle DPC = \angle DCP$... [From (i) and (ii)]
 $\therefore DC = DP$... [Converse of isosceles triangle theorem]

- (7) In a cyclic quadrilateral ABCD, the bisectors of opposite angles A and C meet the circle at P and Q respectively. Prove that PQ is a diameter of the circle. (4 marks)**

Given : $\square ABCD$ is a cyclic quadrilateral.

Ray AP and ray CQ bisect $\angle BAD$ and $\angle BCD$

To Prove : seg PQ is a diameter of the circle.



Proof :

$\angle DAP \cong \angle BAP$... ($\because$ ray AP bisects $\angle DAB$)
 Let $m \angle DAP = m \angle BAP = x^\circ$... (i)
 $\angle DCQ \cong \angle BCQ$... ($\because$ ray CQ bisects $\angle DCB$)
 Let, $m \angle DCQ = m \angle BCQ = y^\circ$... (ii)
 $\square ABCD$ is cyclic ... (Given)
 $\therefore m \angle DAB + m \angle DCB = 180^\circ$
 (Opposite angles of a cyclic quadrilateral are supplementary)
 $\therefore m \angle DAP + m \angle BAP + m \angle DCQ + m \angle BCQ = 180^\circ$
 ... (Angle addition property)
 $\therefore x + x + y + y = 180^\circ$... [From (i) and (ii)]
 $\therefore 2x + 2y = 180^\circ$... (iii)
 $\therefore m \angle DAP = \frac{1}{2} m (\text{arc DP})$
 ... (Inscribed angle theorem)
 $\therefore x = \frac{1}{2} m (\text{arc DP})$... [From (i)]
 $\therefore m (\text{arc DP}) = 2x^\circ$... (iv)
 $m \angle DCQ = \frac{1}{2} m (\text{arc DQ})$
 ... (Inscribed angle theorem)
 $\therefore y = \frac{1}{2} m (\text{arc DQ})$... [From (ii)]
 $\therefore m (\text{arc DQ}) = 2y^\circ$... (v)

$m (\text{arc DP}) + m (\text{arc DQ}) = 2x + 2y$
 ... [Adding (iv) and (v)]

$\therefore m (\text{arc PDQ}) = 180^\circ$
 ... [Arc addition property and from (iii)]

$\therefore$ Arc PDQ is a semicircle

$\therefore$ **seg PQ is a diameter of the circle.**

- (8) In $\triangle ABC$, $\angle A$ is an obtuse angle, P is the circumcentre of $\triangle ABC$. (4 marks)**

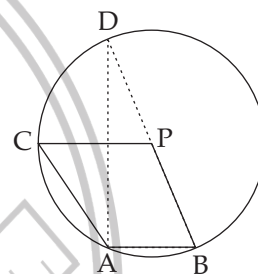
Prove that $\angle PBC = \angle A - 90^\circ$

(4 marks)

Given : P is centre of the circle.

To Prove : $\angle PBC = \angle A - 90^\circ$

Construction : Extend seg BP to intersect the circle at point D, B - P - D. Join seg AD.

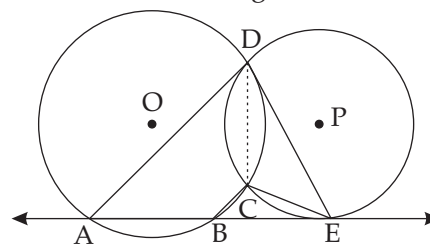


Proof :

Seg BD is the diameter
 $\therefore \angle BAD = 90^\circ$... (i) [Angles inscribed in a semicircle]
 $\therefore \angle DBC \cong \angle DAC$... [Angles inscribed in a same arc are congruent]
 $\therefore \angle PBC = \angle DAC$... (ii) [B - P - D]
 Now, $\angle BAC = \angle BAD + \angle DAC$... [Angle addition property]
 $\therefore \angle A = 90^\circ + \angle PBC$... [From (i) and (ii)]
 $\therefore \angle A - 90^\circ = \angle PBC$
 $\therefore$ **$\angle PBC = \angle A - 90^\circ$**

- (10) Two circles with centre O and P intersect each other in point C and D. Chord AB of the circle with centre O touches the circle with centre P in point E. Prove that $\angle ADE + \angle BCE = 180^\circ$ (4 marks)**

Construction : Draw seg CD.



Proof :

In $\triangle BCE$,

$$\therefore \angle CBE + \angle CEB + \angle BCE = 180^\circ \dots (i) \text{ [Sum of measures of angles of a triangle is } 180^\circ]$$

$\square ABCD$ is a cyclic quadrilateral, $\angle CBE$ is its exterior angle.

$$\therefore \angle CBE = \angle ADC \dots (ii) \text{ [Exterior angle of a cyclic quadrilateral equals to interior opposite angle]}$$

$$\angle CED = \angle EDC \dots (iii) \text{ [Angles in alternate segments]}$$

$$\therefore \angle ADC + \angle EDC + \angle BCE = 180^\circ \dots \text{[Substituting (ii) and (iii) in (i)]}$$

$$\therefore \boxed{\angle ADE + \angle BCE = 180^\circ} \dots \text{[Angle addition property]}$$



4. Geometric Constructions

- (1) Draw a $\triangle ABC$ with side $BC = 6$ cm, $\angle B = 45^\circ$ and $\angle A = 100^\circ$, then construct a triangle whose sides are $\frac{4}{7}$ times the corresponding sides of $\triangle ABC$. (4 marks)

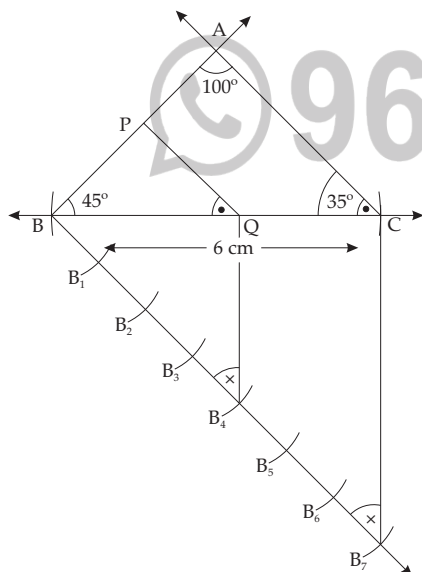
Solution :

Analysis :

$$\begin{aligned} \text{In } \triangle ABC, \\ m\angle A = 100^\circ \\ m\angle B = 45^\circ \end{aligned}$$

$$\therefore m\angle C = 35^\circ \text{ (Remaining angles of a triangle)}$$

(Analytical Figures)

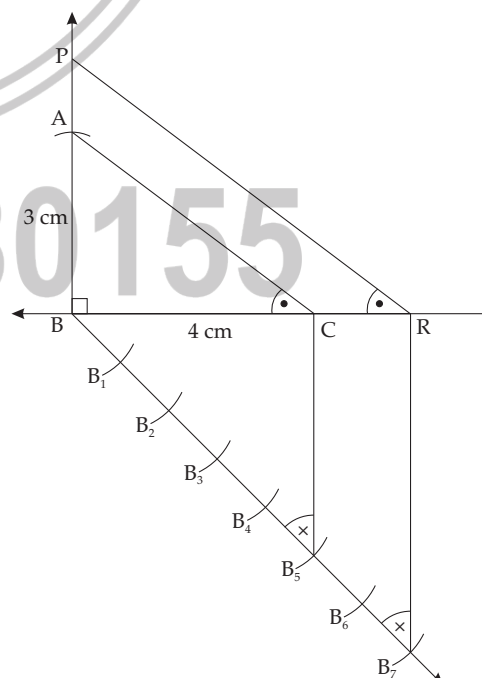


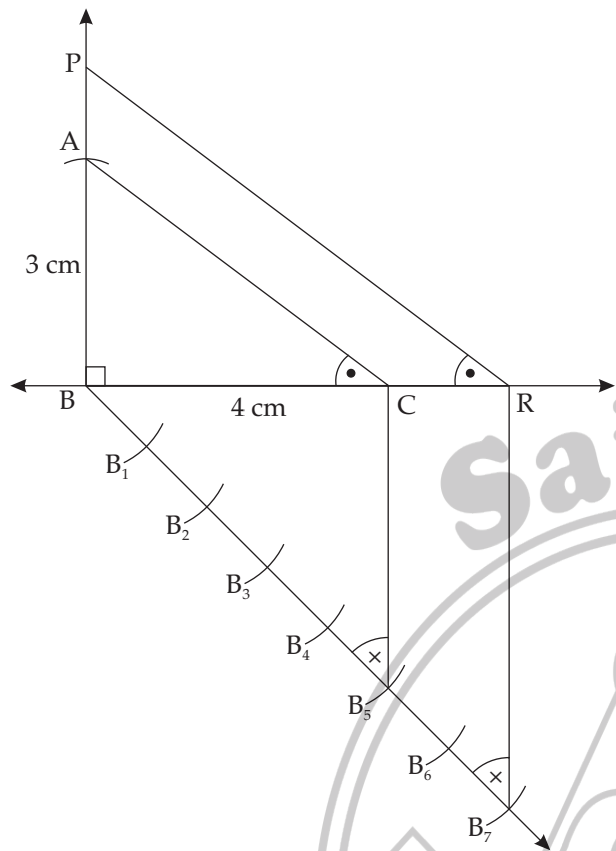
$\triangle PBQ$ is the required triangle similar to the given $\triangle ABC$.

- (2) Draw a $\triangle ABC$, right angled at B such that $AB = 3$ cm and $BC = 4$ cm. Now, construct a triangle similar to $\triangle ABC$, each of whose sides is $\frac{7}{5}$ times the corresponding sides of $\triangle ABC$. (4 marks)

Solution :

(Analytical Figure)

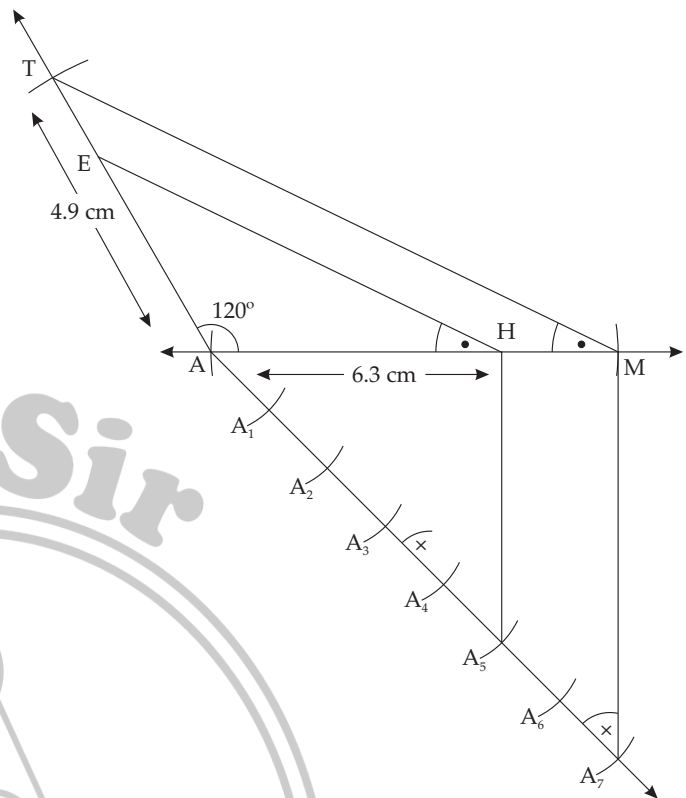
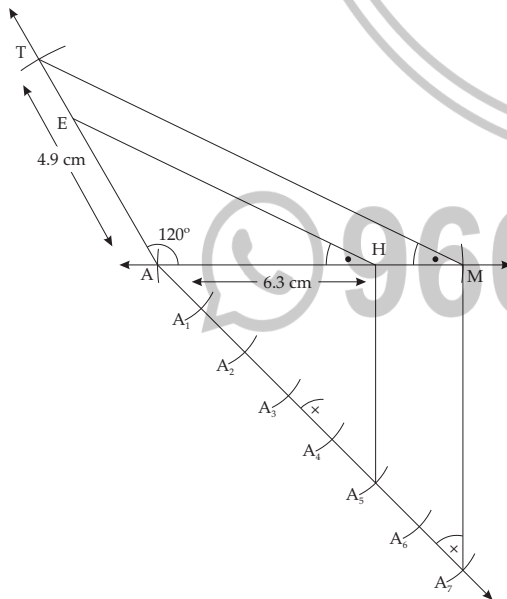




ΔPBR is the required triangle similar to the given ΔABC .

- (3) $\Delta AMT \sim \Delta AHE$, In ΔAMT , $MA = 6.3$ cm, $\angle MAT = 120^\circ$, $AT = 4.9$ cm and $\frac{MA}{HA} = \frac{7}{5}$, construct ΔAHE . (4 marks)

Solution :



- (4) $\Delta LTR \sim \Delta HYD$. Construct ΔHYD , where $HY = 7.2$ cm, $YD = 6$ cm, $\angle Y = 40^\circ$ and $\frac{LR}{HD} = \frac{5}{6}$ and construct ΔLTR . (4 marks)

Solution :

$$\Delta LTR \sim \Delta HYD$$

$$\therefore \frac{LT}{HY} = \frac{TR}{YD} = \frac{LR}{HD} = \frac{5}{6} \quad \dots (i) \text{ (c.s.s.t.)}$$

$$\therefore \angle T = \angle Y = 40^\circ \quad \text{(c.a.s.t.)}$$

$$\frac{LT}{HY} = \frac{5}{6}$$

[From (i)]

$$\therefore \frac{LT}{7.2} = \frac{5}{6}$$

$$\therefore LT = \frac{7.2 \times 5}{6} = \frac{36}{6}$$

$$\therefore LT = 6 \text{ cm}$$

$$\frac{TR}{YD} = \frac{5}{6}$$

[From (i)]

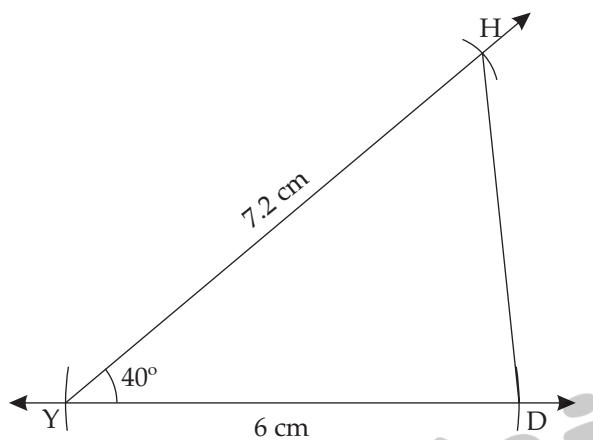
$$\therefore \frac{TR}{6} = \frac{5}{6}$$

$$\therefore TR = \frac{5 \times 6}{6} = \frac{30}{6}$$

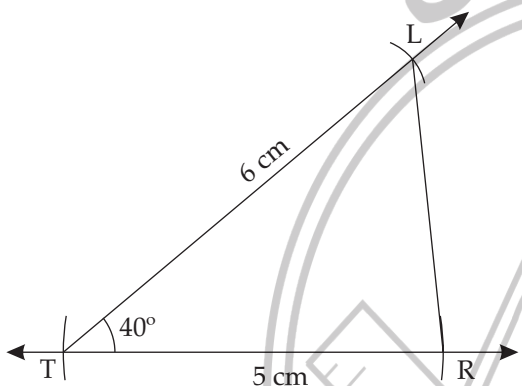
$$\therefore TR = 5 \text{ cm}$$

Information for constructing ΔLTR is complete.

(Given triangle)



(Required triangle)



- (5) Draw a circle with centre O and radius 3.5 cm. Draw tangents PA and PB to the circle, from a point P outside the circle, at points A and B respectively. $\angle APB = 80^\circ$. (4 marks)

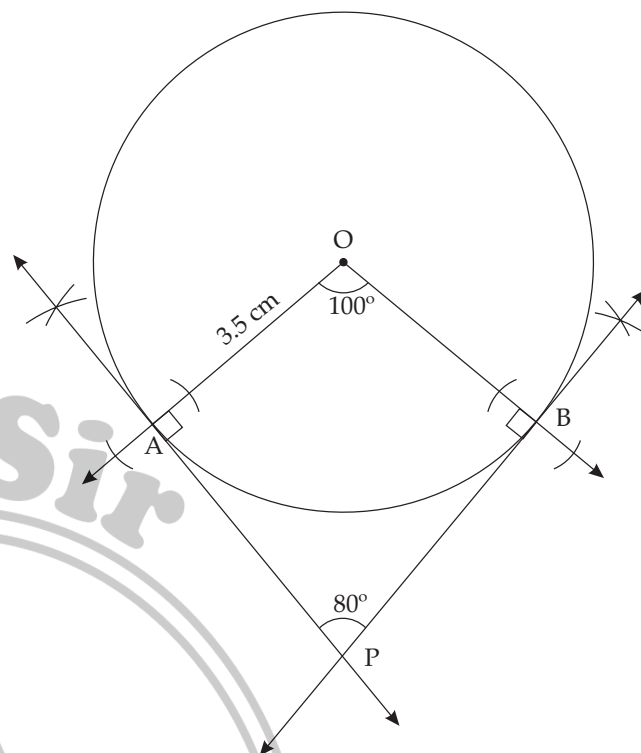
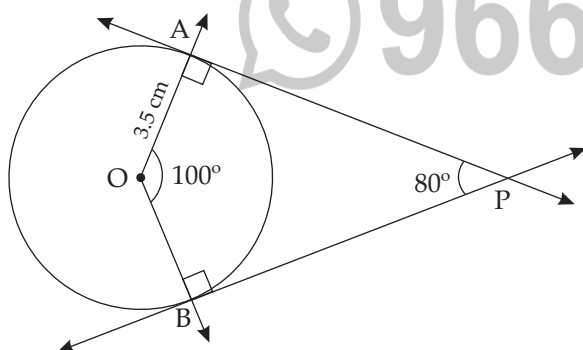
Solution :

In $\square OAPB$, $\angle P = 80^\circ$... [Given]

$\angle OAP = \angle OBP = 90^\circ$ [Tangent perpendicularity theorem]

$\therefore \angle AOB = 100$ [Remaining angle of $\square$]

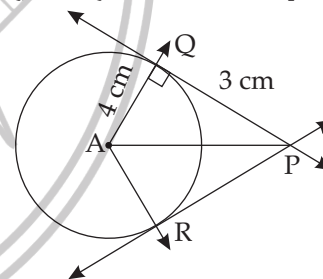
Analytical figure :



- (6) Draw a circle with centre A and radius 4 cm. Draw tangent segments PQ and PR from an external point P such that $PQ = PR = 3$ cm. (4 marks)

Solution :

In $\triangle APQ$, $\angle AQP = 90^\circ$ [Tangent theorem]



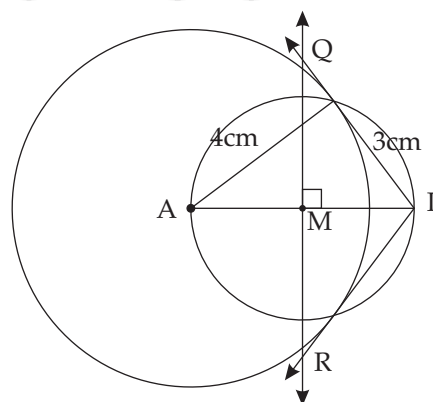
$\therefore AP^2 = AQ^2 + PQ^2$ [Pythagoras theorem]

$\therefore AP^2 = 4^2 + 3^2$

$= 16 + 9$

$\therefore AP^2 = 25$

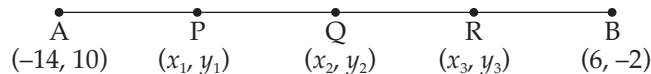
$\therefore AP = 5\text{cm}$



5. Co-ordinate Geometry

- (1) If A(-14, 10) and B(6, -2) find the coordinates of the points which divide seg AB into four equal parts. (4 marks)

Solution :



Let points $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$ be three points which divide seg AB into four equal parts.

$$\therefore AP = PQ = QR = RB \quad \dots (i)$$

$$AQ = AP + PQ = AP + AP = 2AP \quad \dots (ii)$$

[A - P - Q and From (i)]

$$BQ = BR + RQ = AP + AP = 2AP \quad \dots (iii)$$

[B - R - Q and From (i)]

$$\therefore AQ = BQ \quad \text{[From (ii) and (iii)]}$$

$\therefore$ Q is midpoint of seg AB

$$\therefore x_2 = \frac{-14+6}{2}; \quad y_2 = \frac{10+(-2)}{2}$$

[Midpoint formula]

$$\therefore x_2 = \frac{-8}{2}; \quad y_2 = \frac{8}{2}$$

$$\therefore x_2 = -4; \quad y_2 = 4$$

$$\therefore \boxed{Q = (-4, 4)}$$

P is midpoint of seg AQ. [From (i)]

$$\therefore x_1 = \frac{-14-4}{2}; \quad y_1 = \frac{10+4}{2} \quad \text{[Midpoint formula]}$$

$$\therefore x_1 = \frac{-18}{2}; \quad y_1 = \frac{14}{2}$$

$$\therefore x_1 = -9; \quad y_1 = 7$$

$$\therefore \boxed{P = (-9, 7)}$$

R is midpoint of seg BQ. [From (i)]

$$\therefore x_3 = \frac{-4+6}{2}; \quad y_3 = \frac{4+(-2)}{2} \quad \text{[Midpoint formula]}$$

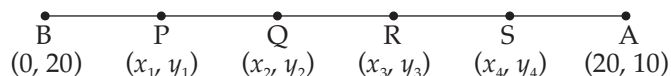
$$\therefore x_3 = \frac{2}{2}; \quad y_3 = \frac{2}{2}$$

$$\therefore x_3 = 1; \quad y_3 = 1$$

$$\therefore \boxed{R = (1, 1)}$$

- (2) If A(20, 10) and B(0, 20) are end points of a seg AB then find the coordinates of points which divide seg AB into 5 congruent parts. (4 marks)

Solution :



Let points $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$ and

$S(x_4, y_4)$ be four points which divide seg AB into five equal parts.

$$\therefore BP = PQ = QR = RS = AS \quad \dots (i)$$

$$BS = BP + PQ + QR + RS \quad [B - P - Q - R - S]$$

$$\therefore BS = BP + BP + BP + BP \quad \dots \text{[From (i)]}$$

$$\therefore BS = 4BP$$

$$\therefore BS = 4AS \quad \text{[From (i)]}$$

$$\therefore \frac{BS}{AS} = \frac{4}{1}$$

$\therefore$ Point S divides seg BA in the ratio 4 : 1.

i.e. $m : n = 4 : 1$

By section formula,

$$\therefore x_4 = \frac{4 \times 20 + 1 \times 0}{4 + 1}; \quad y_4 = \frac{4 \times 10 + 1 \times 20}{4 + 1}$$

$$\therefore x_4 = \frac{80}{5}; \quad y_4 = \frac{40 + 20}{5} = \frac{60}{5}$$

$$\therefore x_4 = 16; \quad y_4 = 12$$

$$\therefore \boxed{S(16, 12)}$$

Q is midpoint of seg BS.

$$\therefore x_2 = \frac{0+16}{2}; \quad y_2 = \frac{20+12}{2} \quad \text{[Midpoint formula]}$$

$$\therefore x_2 = \frac{16}{2}; \quad y_2 = \frac{32}{2}$$

$$\therefore x_2 = 8; \quad y_2 = 16$$

$$\therefore \boxed{Q(8, 16)}$$

P is midpoint of seg BQ.

$$\therefore x_1 = \frac{0+8}{2}; \quad y_1 = \frac{20+16}{2}$$

$$\therefore x_1 = \frac{8}{2}; \quad y_1 = \frac{36}{2}$$

$$\therefore x_1 = 4; \quad y_1 = 18$$

$$\therefore \boxed{P(4, 18)}$$

R is midpoint of seg QS.

$$\therefore x_3 = \frac{8+16}{2}; \quad y_3 = \frac{16+12}{2}$$

$$\therefore x_3 = \frac{24}{2}; \quad y_3 = \frac{28}{2}$$

$$\therefore x_3 = 12; \quad y_3 = 14$$

$$\therefore \boxed{R(12, 14)}$$

$\therefore$ Point P(4, 18), Q(8, 16), R(12, 14) and S(16, 12) divides seg AB into five equal parts.

- (3) Find the coordinates of the circumcentre and the radius of the circumcircle of $\triangle ABC$ if A(2, 3), B(4, -1) and C(5, 2). (4 marks)

Solution :

Let point P(h, k) be the circumcentre of $\triangle ABC$.

$$\therefore PA = PB = PC \quad \dots (i) \text{ [Radii of a circle]}$$

$$\therefore PA = PB \quad \dots \text{ [From (i)]}$$

Using distance formula,

$$\sqrt{(h-2)^2 + (k-3)^2} = \sqrt{(h-4)^2 + (k+1)^2}$$

Squaring both sides

$$\therefore (h-2)^2 + (k-3)^2 = (h-4)^2 + (k+1)^2$$

$$\therefore h^2 - 4h + 4 + k^2 - 6k + 9 = h^2 - 8h + 16 + k^2 + 2k + 1$$

$$\therefore -4h - 6k + 13 = -8h + 2k + 17$$

$$\therefore -4h + 8h - 6k - 2k = 17 - 13$$

$$\therefore 4h - 8k = 4$$

$$\therefore h - 2k = 1 \quad \dots (ii)$$

$$PA = PC \quad \text{[From (i)]}$$

Using distance formula,

$$(h-2)^2 + (k-3)^2 = (h-5)^2 + (k-2)^2$$

$$\therefore h^2 - 4h + 4 + k^2 - 6k + 9 = h^2 - 10h + 25 + k^2 - 4k + 4$$

$$\therefore -4h - 6k + 13 = -10h - 4k + 29$$

$$\therefore -4h + 10h - 6k + 4k = 29 - 13$$

$$\therefore 6h - 2k = 16 \quad \dots (iii)$$

Subtracting equation (ii) from equation (iii)

$$6h - 2k = 16$$

$$- \quad h - 2k = 1$$

$$\begin{array}{r} (-) \quad (+) \quad (-) \\ \hline 5h \quad \quad = 15 \end{array}$$

$$\therefore h = 3$$

Substituting $h = 3$ in equation (ii)

$$3 - 2k = 1$$

$$\therefore -2k = 1 - 3$$

$$\therefore -2k = -2$$

$$\therefore k = 1$$

$$\therefore P(3, 1) \text{ is the centre of the circle.}$$

$$\text{Radius } PA = \sqrt{(2-3)^2 + (3-1)^2} \quad \text{[Distance formula]}$$

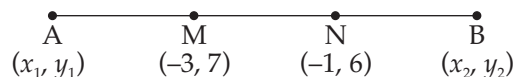
$$= \sqrt{(-1)^2 + (2)^2}$$

$$= \sqrt{1+4}$$

$$\therefore \text{Radius } PA = \sqrt{5} \text{ unit}$$

- (4) Point M(-3, 7) and N(-1, 6) divides segment AB into three equal parts. Find the coordinates of point A and point B. (4 marks)

Solution :



Let A(x_1, y_1), and B(x_2, y_2)

Points M and N divides seg AB into three equal parts.

$$\therefore AM = MN = NB \quad \dots (i)$$

$$\therefore AM = MN$$

$\therefore$ Point M is midpoint of seg AN.

$$\therefore -3 = \frac{x_1 + (-1)}{2} \text{ and } 7 = \frac{y_1 + 6}{2} \quad \text{[Midpoint formula]}$$

$$\therefore -6 = x_1 - 1 \text{ and } 14 = y_1 + 6$$

$$\therefore x_1 = -6 + 1 \text{ and } y_1 = 14 - 6$$

$$\therefore x_1 = -5, y_1 = 8$$

$$\therefore \boxed{A(-5, 8)}$$

$$MN = NB \quad \dots \text{ [From (i)]}$$

$\therefore$ Point N is midpoint of seg MB.

$$\therefore -1 = \frac{-3 + x_2}{2}; 6 = \frac{7 + y_2}{2} \quad \text{[Midpoint formula]}$$

$$\therefore -2 = -3 + x_2; 12 = 7 + y_2$$

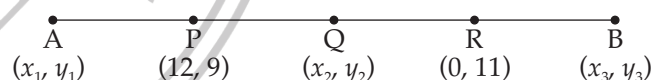
$$\therefore -2 + 3 = x_2; 12 - 7 = y_2$$

$$\therefore x_2 = 1; y_2 = 5$$

$$\therefore \boxed{B(1, 5)}$$

- (5) Segment AB is divided into four equal parts by points P, Q and R such that A - P - Q - R - B. If P(12, 9) and R(0, 11), then find the coordinates of point A, Q and B. (4 marks)

Solution :



$$AP = PQ = QR = RB \quad \dots (i)$$

Let A(x_1, y_1), Q(x_2, y_2) and B(x_3, y_3)

Q is midpoint of seg PR. [From (i)]

$$\therefore x_2 = \frac{12+0}{2}; y_2 = \frac{9+11}{2} \quad \text{[Midpoint formula]}$$

$$\therefore x_2 = \frac{12}{2}; y_2 = \frac{20}{2}$$

$$\therefore x_2 = 6; y_2 = 10$$

$$\therefore \boxed{Q(6, 10)}$$

P is midpoint of seg AQ. [From (i)]

$$\therefore \frac{x_1 + x_2}{2} = 12 \text{ and } \frac{y_1 + y_2}{2} = 9$$

$$\therefore 6 + x_1 = 24 \text{ and } y_1 + 10 = 18$$

$$\therefore x_1 = 24 - 6 \text{ and } y_1 = 18 - 10$$

$$\therefore x_1 = 18 \text{ and } y_1 = 8$$

$$\therefore \boxed{A(18, 8)}$$

R is midpoint of seg BQ.

[From (i)]

$$\therefore \frac{x_2 + x_3}{2} = 0 \text{ and } \frac{y_2 + y_3}{2} = 11$$

$$\therefore x_3 + 6 = 0 \times 2 \text{ and } y_3 + 10 = 11 \times 2$$

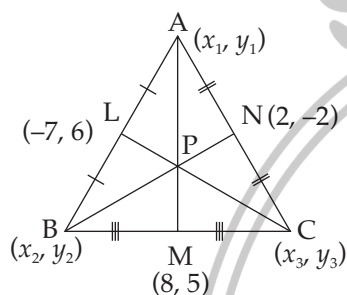
$$\therefore x_3 = 0 - 6 \text{ and } y_3 = 22 - 10$$

$$\therefore x_3 = -6 \text{ and } y_3 = 12$$

$$\therefore \boxed{B(-6, 12)}$$

- (6) If $(-7, 6)$, $(8, 5)$ and $(2, -2)$ are the midpoints of the sides of a triangle. Find the coordinates of its centroid. (4 marks)

Solution :



Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be three vertices of $\triangle ABC$.

L $(-7, 6)$ is midpoint of seg AB.

M $(8, 5)$ is midpoint of seg BC.

N $(2, -2)$ is midpoint of seg AC.

Let $G(x, y)$ be centroid of $\triangle ABC$.

L is midpoint of seg AB.

$$\therefore \frac{x_1 + x_2}{2} = -7 \text{ and } \frac{y_1 + y_2}{2} = 6$$

[Midpoint formula]

$$\therefore x_1 + x_2 = -14 \text{ and } y_1 + y_2 = 12 \quad \dots (i)$$

M is midpoint of seg BC.

$$\therefore \frac{x_2 + x_3}{2} = 8 \text{ and } \frac{y_2 + y_3}{2} = 5$$

[Midpoint formula]

$$\therefore x_2 + x_3 = 16 \text{ and } y_2 + y_3 = 10 \quad \dots (ii)$$

N is midpoint of seg AC.

$$\therefore \frac{x_1 + x_3}{2} = 2 \text{ and } \frac{y_1 + y_3}{2} = -2$$

[Midpoint formula]

$$\therefore x_1 + x_3 = 4 \text{ and } y_1 + y_3 = -4 \quad \dots (iii)$$

Adding (i), (ii) and (iii)

$$2x_1 + 2x_2 + 2x_3 = 6 \text{ and } 2y_1 + 2y_2 + 2y_3 = 18$$

$$\therefore x_1 + x_2 + x_3 = 3 \text{ and } y_1 + y_2 + y_3 = 9 \quad \dots (iv)$$

$\therefore G$ is centroid of $\triangle ABC$

$$\therefore x = \frac{x_1 + x_2 + x_3}{3} \text{ and } y = \frac{y_1 + y_2 + y_3}{3}$$

[Centroid formula]

$$\therefore x = \frac{3}{3} \text{ and } y = \frac{9}{3}$$

$$\therefore x = 1 \text{ and } y = 3$$

$$\therefore \boxed{G = (1, 3)}$$



6. Trigonometry

- (1) If $\sqrt{1+x^2} \sin \theta = x$, prove that

$$\tan^2 \theta + \cot^2 \theta = x^2 + \frac{1}{x^2} \quad (3 \text{ marks})$$

Proof :

$$\sqrt{1+x^2} \sin \theta = x \quad \dots [\text{Given}]$$

$$\therefore \sin \theta = \frac{x}{\sqrt{1+x^2}} \quad \dots (i)$$

$$\therefore \sin^2 \theta + \cos^2 \theta = 1 \quad \dots (\text{Identity})$$

$$\therefore \frac{x^2}{1+x^2} + \cos^2 \theta = 1 \quad \dots [\text{From (i)}]$$

$$\therefore \cos^2 \theta = 1 - \frac{x^2}{1+x^2}$$

$$\therefore \cos^2 \theta = \frac{1+x^2-x^2}{1+x^2}$$

$$\therefore \cos^2 \theta = \frac{1}{1+x^2} \quad \dots (ii)$$

$$\tan^2 \theta = \sin^2 \theta \div \cos^2 \theta$$

$$= \frac{x^2}{(1+x^2)} \div \frac{1}{(1+x^2)} \quad [\text{From (i) and (ii)}]$$

$$= \frac{x^2}{(1+x^2)} \times (1+x^2)$$

$$\tan^2 \theta = x^2 \quad \dots (iii)$$

$$\therefore \cot^2 \theta = \frac{1}{\tan^2 \theta}$$

$$\therefore \cot^2 \theta = \frac{1}{x^2} \quad \dots (iv) [\text{From (iii)}]$$

$$\therefore \tan^2 \theta + \cot^2 \theta = x^2 + \frac{1}{x^2} \quad [\text{Adding (iii) and (iv)}]$$

$$\therefore \boxed{\tan^2 \theta + \cot^2 \theta = x^2 + \frac{1}{x^2}}$$

- (2) Prove : $\frac{\tan \theta}{1 - \cot \theta} + \frac{\cot \theta}{1 - \tan \theta} = 1 + \tan \theta + \cot \theta$

(4 marks)

Proof :

$$\text{L.H.S.} = \frac{\tan \theta}{1 - \cot \theta} + \frac{\cot \theta}{1 - \tan \theta}$$

$$= \left[\frac{\sin \theta}{\cos \theta} \div \left(1 - \frac{\cos \theta}{\sin \theta} \right) \right] + \left[\frac{\cos \theta}{\sin \theta} \div \left(1 - \frac{\sin \theta}{\cos \theta} \right) \right]$$

$$= \left[\frac{\sin \theta}{\cos \theta} \div \left(\frac{\sin \theta - \cos \theta}{\sin \theta} \right) \right]$$

$$+ \left[\frac{\cos \theta}{\sin \theta} \div \left(\frac{\cos \theta - \sin \theta}{\cos \theta} \right) \right]$$

$$\begin{aligned}
 &= \left[\frac{\sin \theta}{\cos \theta} \times \frac{\sin \theta}{(\sin \theta - \cos \theta)} \right] \\
 &\quad + \left[\frac{\cos \theta}{\sin \theta} \times \frac{\cos \theta}{(\cos \theta - \sin \theta)} \right] \\
 &= \frac{\sin^2 \theta}{\cos \theta(\sin \theta - \cos \theta)} + \frac{\cos^2 \theta}{\sin \theta \times (-1)(\sin \theta - \cos \theta)} \\
 &= \frac{\sin^2 \theta}{\cos \theta(\sin \theta - \cos \theta)} - \frac{\cos^2 \theta}{\sin \theta(\sin \theta - \cos \theta)} \\
 &= \frac{\sin^3 \theta - \cos^3 \theta}{\sin \theta \cdot \cos \theta(\sin \theta - \cos \theta)} \\
 &= \frac{(\sin \theta - \cos \theta)(\sin^2 \theta + \sin \theta \cos \theta + \cos^2 \theta)}{(\sin \theta - \cos \theta) \sin \theta \cdot \cos \theta} \\
 &= \frac{\sin^2 \theta + \sin \theta \cos \theta + \cos^2 \theta}{\sin \theta \cdot \cos \theta} \\
 &= \frac{\sin^2 \theta}{\sin \theta \cdot \cos \theta} + \frac{\sin \theta \cos \theta}{\sin \theta \cos \theta} + \frac{\cos^2 \theta}{\sin \theta \cdot \cos \theta} \\
 &= \frac{\sin \theta}{\cos \theta} + 1 + \frac{\cos \theta}{\sin \theta} \\
 \text{L.H.S.} &= \tan \theta + 1 + \cot \theta \\
 \text{R.H.S.} &= 1 + \tan \theta + \cot \theta \\
 \therefore \quad \text{L.H.S.} &= \text{R.H.S.}
 \end{aligned}$$

(3) Prove :

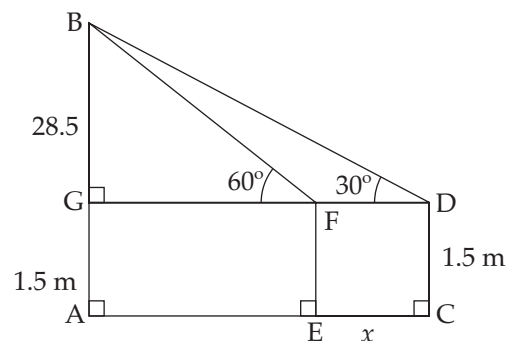
$$\sin^8 \theta - \cos^8 \theta = (\sin^2 \theta - \cos^2 \theta)(1 - 2\sin^2 \theta \cos^2 \theta) \quad (3 \text{ marks})$$

Proof :

$$\begin{aligned}
 \text{L.H.S.} &= \sin^8 \theta - \cos^8 \theta \\
 &= (\sin^4 \theta)^2 - (\cos^4 \theta)^2 \\
 &= (\sin^4 \theta - \cos^4 \theta)(\sin^4 \theta + \cos^4 \theta) \\
 &= [(\sin^2 \theta)^2 - (\cos^2 \theta)^2] [\sin^4 \theta + \cos^4 \theta] \\
 &= (\sin^2 \theta + \cos^2 \theta)(\sin^2 \theta - \cos^2 \theta) [\sin^4 \theta \\
 &\quad + 2\sin^2 \theta \cos^2 \theta + \cos^4 \theta - 2\sin^2 \theta \cos^2 \theta] \\
 &= 1(\sin^2 \theta - \cos^2 \theta) [(\sin^2 \theta + \cos^2 \theta)^2 \\
 &\quad - 2\sin^2 \theta \cos^2 \theta] \\
 &\quad [\because \sin^2 A + \cos^2 A = 1] \\
 &= (\sin^2 \theta - \cos^2 \theta)(1^2 - 2\sin^2 \theta \cos^2 \theta) \\
 \text{L.H.S.} &= (\sin^2 \theta - \cos^2 \theta)(1 - 2\sin^2 \theta \cos^2 \theta) \\
 \text{R.H.S.} &= (\sin^2 \theta - \cos^2 \theta)(1 - 2\sin^2 \theta \cos^2 \theta) \\
 \therefore \quad \text{L.H.S.} &= \text{R.H.S.}
 \end{aligned}$$

- (4) A 1.5 m tall boy is standing at some distance from a 30 m tall building. The angle of elevation from his eyes to the top of the building increases from 30° to 60° as he walks towards the building. Find the distance he walked towards the building. (4 marks)

Solution :



AB $\Rightarrow$ Building of height 30 m

CD $\Rightarrow$ First position of boy of height 1.5 m

EF $\Rightarrow$ Second position of boy,

Let the distance travelled be x m.

$$CD = EF = AG$$

$$\therefore EF = AG = 1.5 \text{ m}$$

$$BG = AB - AG$$

$$\dots [A - G - B]$$

$$\therefore BG = 30 - 1.5 \text{ m} = 28.5 \text{ m}$$

$$\text{In } \triangle BGF, \angle BGF = 90^\circ$$

$$\therefore \tan 60^\circ = \frac{BG}{GF}$$

[Definition]

$$\therefore \sqrt{3} = \frac{28.5}{GF}$$

$$\therefore GF = \frac{28.5}{\sqrt{3}} = \frac{28.5 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}}$$

$$\therefore GF = 9.5\sqrt{3} \text{ m}$$

... (i)

$$\text{In } \triangle BGD, \angle BGD = 90^\circ$$

$$\therefore \tan 30^\circ = \frac{BG}{GD}$$

$$\therefore \frac{1}{\sqrt{3}} = \frac{28.5}{GD}$$

$$\therefore GD = 28.5\sqrt{3} \text{ m}$$

... (ii)

$$AC = GD, GF = AE$$

... (iii)

$$\therefore AC = 28.5\sqrt{3}, AE = 9.5\sqrt{3}$$

[From (i), (ii) and (iii)]

$$AC = AE + CE$$

$$[A - E - C]$$

$$\therefore 28.5\sqrt{3} = x + 9.5\sqrt{3}$$

$$\therefore x = 28.5\sqrt{3} - 9.5\sqrt{3}$$

$$\therefore x = 19\sqrt{3} \text{ m}$$

Distance he walked towards the building is $19\sqrt{3}$ m.

(4) Prove :

$$(\sin A + \operatorname{cosec} A)^2 + (\cos A + \sec A)^2 = 7 + \tan^2 A + \cot^2 A. \quad (4 \text{ marks})$$

Proof :

$$\text{L.H.S.} = (\sin A + \operatorname{cosec} A)^2 + (\cos A + \sec A)^2$$

$$\begin{aligned}
 &= \sin^2 A + 2\sin A \cdot \operatorname{cosec} A + \operatorname{cosec}^2 A \\
 &\quad + \cos^2 A + 2\cos A \sec A + \sec^2 A \\
 &= \sin^2 A + \cos^2 A + 2\sin A \frac{1}{\sin A} + 2\cos A \\
 &\quad \frac{1}{\cos A} + \operatorname{cosec}^2 A + \sec^2 A \\
 &= 1 + 2 + 2 + (1 + \cot^2 A) + (1 + \tan^2 A) \\
 &\quad [\because \sin^2 \theta + \cos^2 \theta = 1, \\
 &\quad \operatorname{cosec}^2 \theta = 1 + \cot^2 \theta, \\
 &\quad \sec^2 \theta = 1 + \tan^2 \theta]
 \end{aligned}$$

$$= 5 + 1 + \cot^2 A + 1 + \tan^2 A$$

$$\text{L.H.S.} = 7 + \cot^2 A + \tan^2 A$$

$$\text{R.H.S.} = 7 + \tan^2 A + \cot^2 A$$

$$\text{L.H.S.} = \text{R.H.S.}$$

$$\begin{aligned}
 \therefore &(\sin A + \operatorname{cosec} A)^2 + (\cos A + \sec A)^2 \\
 &= 7 + \tan^2 A + \cot^2 A.
 \end{aligned}$$

(5) Prove that :

$$\frac{1 - \sin \theta \cos \theta}{\cos \theta (\sec \theta - \operatorname{cosec} \theta)} \times \frac{\sin^2 \theta - \cos^2 \theta}{\sin^3 \theta + \cos^3 \theta} = \sin \theta \quad (4 \text{ marks})$$

Proof :

$$\begin{aligned}
 \text{L.H.S.} &= \frac{1 - \sin \theta \cos \theta}{\cos \theta (\sec \theta - \operatorname{cosec} \theta)} \times \frac{\sin^2 \theta - \cos^2 \theta}{\sin^3 \theta + \cos^3 \theta} \\
 &= \frac{1 - \sin \theta \cos \theta}{\cos \theta \left(\frac{1}{\cos \theta} - \frac{1}{\sin \theta} \right)} \times \frac{(\sin \theta + \cos \theta)(\sin \theta - \cos \theta)}{(\sin \theta + \cos \theta)(\sin^2 \theta - \sin \theta \cos \theta + \cos^2 \theta)} \\
 &= \frac{(1 - \sin \theta \cos \theta)}{\cos \theta \left(\frac{\sin \theta - \cos \theta}{\sin \theta \cos \theta} \right)} \times \frac{(\sin \theta - \cos \theta)}{(1 - \sin \theta \cos \theta)} \\
 &\quad [\because \sin^2 A + \cos^2 A = 1] \\
 &= \frac{1}{(\sin \theta - \cos \theta)} \times (\sin \theta - \cos \theta) \\
 &\quad \sin \theta
 \end{aligned}$$

$$\text{L.H.S.} = \sin \theta$$

$$\text{R.H.S.} = \sin \theta$$

$$\therefore \text{L.H.S.} = \text{R.H.S.}$$

(6) Prove that :

$$\frac{\tan A}{\sec A - 1} + \frac{\tan A}{\sec A + 1} = 2 \operatorname{cosec} A \quad (4 \text{ marks})$$

Proof :

$$\begin{aligned}
 \text{L.H.S.} &= \frac{\tan A}{\sec A - 1} + \frac{\tan A}{\sec A + 1} \\
 &= \tan A \left[\frac{1}{\sec A - 1} + \frac{1}{\sec A + 1} \right] \\
 &= \tan A \left[\frac{\sec A + 1 + \sec A - 1}{(\sec A - 1)(\sec A + 1)} \right]
 \end{aligned}$$

$$= \tan A \times \frac{2 \sec A}{(\sec^2 A - 1)}$$

$$= \tan A \times \frac{2 \sec A}{\tan^2 A} \quad [\because 1 + \tan^2 \theta = \sec^2 \theta, \therefore \tan^2 \theta = \sec^2 \theta - 1]$$

$$= 2 \sec A \div \tan A$$

$$= \frac{2}{\cos A} \div \frac{\sin A}{\cos A}$$

$$= \frac{2}{\cos A} \times \frac{\cos A}{\sin A}$$

$$= \frac{2}{\sin A}$$

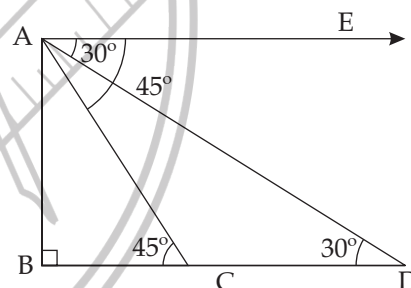
$$\text{L.H.S.} = 2 \operatorname{cosec} A$$

$$\text{R.H.S.} = 2 \operatorname{cosec} A$$

$$\therefore \text{L.H.S.} = \text{R.H.S.}$$

(7) From the top of a light house, 80 metres high, two ships on the same side of light house are observed. The angles of depression of the ships as seen from the light house are found to be of 45° and 30° . Find the distance between the two ships. (Assume that the two ships and the bottom of the lighthouse are in a line) (4 marks)

Solution :



In the above figure, AB represents lighthouse of height 80 m. D and C are positions of two ships.

$\angle EAD$ and $\angle EAC$ are angles of depression.

$$\angle EAD = 30^\circ \text{ and } \angle EAC = 45^\circ \quad [\text{Given}]$$

$$\therefore \angle ADB = 30^\circ \text{ and } \angle ACB = 45^\circ \quad \dots (i) \quad [\text{Alternate angles}]$$

In $\triangle ABD$, $\angle B = 90^\circ$

$$\therefore \tan \angle ADB = \frac{AB}{BD} \quad [\text{Definition}]$$

$$\therefore \tan 30^\circ = \frac{80}{BD} \quad [\text{From (i)}]$$

$$\therefore \frac{1}{\sqrt{3}} = \frac{80}{BD}$$

$$\therefore BD = 80\sqrt{3} \text{ m} \quad \dots (ii)$$

In $\triangle ABC$, $\angle ABC = 90^\circ$

$$\therefore \tan \angle ACB = \frac{AB}{BC} \quad [\text{Definition}]$$

$$\begin{aligned}\therefore \tan 45^\circ &= \frac{80}{BC} \\ \therefore 1 &= \frac{80}{BC} \\ \therefore BC &= 80 \text{ m} \quad \dots \text{ (iii)} \\ \therefore BD &= BC + CD \quad [B - C - D] \\ \therefore 80\sqrt{3} &= 80 + CD \\ \therefore CD &= 80\sqrt{3} - 80 \\ \therefore CD &= 80(\sqrt{3} - 1) \text{ m} \\ \therefore \text{The distance between the two ships is } &80(\sqrt{3} - 1) \text{ m.}\end{aligned}$$

- (8) If a $\cos\theta + b \sin\theta = m$ and a $\sin\theta - b \cos\theta = n$, then prove that $a^2 + b^2 = m^2 + n^2$ (4 marks)

Proof :

$$\begin{aligned}m &= a \cos\theta + b \sin\theta \quad \dots [\text{Given}] \\ \therefore m^2 &= (a \cos\theta + b \sin\theta)^2 \\ \therefore m^2 &= a^2 \cos^2\theta + 2ab \cos\theta \sin\theta + b^2 \sin^2\theta \\ \therefore m^2 &= a^2 \cos^2\theta + 2ab \sin\theta \cos\theta + b^2 \sin^2\theta \quad \dots \text{ (i)} \\ n &= a \sin\theta - b \cos\theta \\ \therefore n^2 &= (a \sin\theta - b \cos\theta)^2 \\ \therefore n^2 &= a^2 \sin^2\theta - 2ab \sin\theta \cos\theta + b^2 \cos^2\theta \quad \dots \text{ (ii)} \\ \therefore m^2 + n^2 &= a^2 \cos^2\theta + 2ab \sin\theta \cos\theta + b^2 \sin^2\theta \\ &\quad + a^2 \sin^2\theta - 2ab \sin\theta \cos\theta + b^2 \cos^2\theta \\ m^2 + n^2 &= a^2 \sin^2\theta + a^2 \cos^2\theta + b^2 \sin^2\theta \\ &\quad + b^2 \cos^2\theta \\ \therefore m^2 + n^2 &= a^2(\sin^2\theta + \cos^2\theta) + b^2(\sin^2\theta + \cos^2\theta) \\ \therefore m^2 + n^2 &= a^2 + b^2 \quad (\because \sin^2 A + \cos^2 A = 1)\end{aligned}$$

- (9) If $\sqrt{3} \tan\theta = 3 \sin\theta$, find the value of $\sin^2\theta - \cos^2\theta$, where $\theta \neq 0$. (4 marks)

Solution :

$$\begin{aligned}\sqrt{3} \tan\theta &= 3 \sin\theta \\ \therefore \frac{\tan\theta}{\sin\theta} &= \frac{3}{\sqrt{3}} \\ \therefore \frac{\sin\theta}{\cos\theta} \div \sin\theta &= \frac{\sqrt{3} \times \sqrt{3}}{\sqrt{3}} \\ \therefore \frac{\sin\theta}{\cos\theta} \times \frac{1}{\sin\theta} &= \sqrt{3} \\ \therefore \frac{1}{\cos\theta} &= \sqrt{3} \\ \therefore \cos\theta &= \frac{1}{\sqrt{3}} \\ \therefore \cos^2\theta &= \frac{1}{3} \quad \dots \text{ (i)} \\ \sin^2\theta + \cos^2\theta &= 1 \quad \dots [\text{From (i)}] \\ \therefore \sin^2\theta + \frac{1}{3} &= 1\end{aligned}$$

$$\begin{aligned}\therefore \sin^2\theta &= 1 - \frac{1}{3} \\ \therefore \sin^2\theta &= \frac{3-1}{3} = \frac{2}{3} \quad \dots \text{ (ii)} \\ \therefore \sin^2\theta - \cos^2\theta &= \frac{2}{3} - \frac{1}{3} \quad [\text{From (i) and (ii)}] \\ \therefore \sin^2\theta - \cos^2\theta &= \frac{1}{3}\end{aligned}$$

(10) Prove that :

$$\left(1 + \frac{1}{\tan^2 A}\right) \left(1 + \frac{1}{\cot^2 A}\right) = \frac{1}{\sin^2 A - \sin^4 A} \quad (4 \text{ marks})$$

Proof :

$$\begin{aligned}\text{L.H.S.} &= \left(1 + \frac{1}{\tan^2 A}\right) \left(1 + \frac{1}{\cot^2 A}\right) \\ &= (1 + \cot^2 A) (1 + \tan^2 A) \\ &= \operatorname{cosec}^2 A \cdot \sec^2 A \\ &= \frac{1}{\sin^2 A \cdot \cos^2 A} \\ &= \frac{1}{\sin^2 A (1 - \sin^2 A)} \quad [\because \sin^2 A + \cos^2 A = 1 \\ &\quad \therefore \cos^2 A = 1 - \sin^2 A] \\ \text{L.H.S.} &= \frac{1}{\sin^2 A - \sin^4 A} \\ \text{R.H.S.} &= \frac{1}{\sin^2 A - \sin^4 A} \\ \therefore \text{L.H.S.} &= \text{R.H.S.}\end{aligned}$$



7. Mensuration

- (1) A tin maker converts a cubical metallic box into 10 cylindrical tins. Side of the cube is 50 cm and radius of the cylinder is 7 cm. Find the height of each cylinder so made if the wastage incurred was 12%. ($\pi = \frac{22}{7}$) (4 marks)

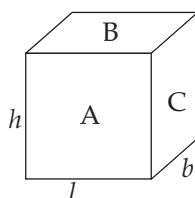
Solution :

$$\begin{aligned}\text{Total surface area of cube} &= 6l^2 \quad (\text{Formula}) \\ &= 6 \times 50 \times 50 \\ &= 15000 \text{ cm}^2 \\ \text{Wastage incurred} &= 12\% \text{ of } 15000 \\ &= \frac{12}{100} \times 15000 \\ &= 1800 \text{ cm}^2 \\ \therefore \text{Area of metal sheet} &= 15000 - 1800 \\ \text{used to make 10 cylindrical tins} &= 13,200 \text{ cm}^2\end{aligned}$$

$$\begin{aligned} \therefore \text{Area of metal sheet used to make 1 cylindrical tin} &= \frac{13200}{10} = 1,320 \text{ cm}^2 \\ \therefore \text{Area of metal sheet required to make 1 cylindrical tin} &= \text{Total surface area of a cylinder} \\ \therefore \text{Total surface area of a cylinder} &= 1,320 \\ \therefore 2\pi r(r+h) &= 1,320 \\ \therefore 2 \times \frac{22}{7} \times 7 \times (7+h) &= 1,320 \\ \therefore 7+h &= \frac{1320}{44} \\ \therefore h &= 30-7 \\ \therefore h &= 23 \text{ cm} \\ \therefore \text{Height of each cylinder} &= 23 \text{ cm} \end{aligned}$$

- (2) The three faces, A, B, C having a common vertex of a cuboid have areas 450 cm^2 , 600 cm^2 and 300 cm^2 respectively. Find the volume of the cuboid. (4 marks)

Solution :



$$\begin{aligned} \text{Area of surface A} &= 450 \text{ cm}^2 \\ \therefore l \times h &= 450 \text{ cm}^2 \quad \dots (1) \\ \text{Area of surface B} &= 600 \text{ cm}^2 \\ \therefore l \times b &= 600 \text{ cm}^2 \quad \dots (2) \\ \text{Area of surface C} &= 300 \text{ cm}^2 \\ \therefore b \times h &= 300 \text{ cm}^2 \quad \dots (3) \end{aligned}$$

Multiplying (1), (2) and (3),

$$\begin{aligned} l^2 \times b^2 \times h^2 &= 600 \times 450 \times 300 \\ \therefore l^2 \times b^2 \times h^2 &= 300 \times 2 \times 450 \times 300 \\ &= 300 \times 900 \times 300 \\ \therefore l^2 \times b^2 \times h^2 &= 300 \times 300 \times 30 \times 30 \\ \therefore l \times b \times h &= 300 \times 30 \quad [\text{Taking square roots}] \\ \therefore l \times b \times h &= 9000 \text{ cm}^3 \quad \dots (4) \\ \text{Volume of cuboid} &= l \times b \times h \end{aligned}$$

$$\therefore \text{Volume of cuboid} = 9000 \text{ cm}^3$$

- (3) Oil tins of cuboidal shape are made from a metallic sheet with length 8 m and breadth 4 m. Each tin has dimensions $60 \times 40 \times 20$ in cm and is open from the top. Find the number of such tins that can be made? (4 marks)

Solution :

$$\begin{aligned} \text{Area of metallic sheet} &= 8 \text{ m} \times 4 \text{ m} \\ \therefore \text{Area of metallic sheet} &= 800 \text{ cm} \times 400 \text{ cm} \dots (1) \\ \text{Total surface area of a tin} &= 2(l+b) \times h + l \times b \end{aligned}$$

$$\begin{aligned} \therefore \text{Total surface area of a tin} &= 2(60+40) \times 20 + 60 \times 40 \\ &= 2 \times 100 \times 20 + 2400 \\ &= 4000 + 2400 \\ \text{Total surface area of a tin} &= 6400 \text{ cm}^2 \quad \dots (2) \\ \text{Number of tins can be made} &= \frac{\text{Area of metallic sheet}}{\text{Total surface area of open tin}} \\ &= \frac{800 \times 400}{6400} \end{aligned}$$

$$\therefore \text{Number of tins can be made} = 50$$

- (4) Plastic drum of cylindrical shape is made by melting spherical solid plastic balls of radius 1 cm. Find the number of balls required to make a drum of thickness 2 cm, height 90 cm and outer radius 30 cm. (4 marks)

Solution :

$$\begin{aligned} \text{For drum, thickness} &= 2 \text{ cm} \\ \text{Inner radius } (r_i) &= \text{outer radius } (r_o) - 2 \\ \therefore r_i &= 30 - 2 = 28 \text{ cm} \\ \text{Inner height } (h_i) &= \text{Outer height } (h_o) - 2 \\ \therefore h_i &= 90 - 2 = 88 \text{ cm} \\ \text{Volume of plastic required for cylindrical drum } (V_1) &= \text{Volume of outer cylinder} - \text{Volume of inner cylinder} \\ &= \pi r_o^2 h_o - \pi r_i^2 h_i \\ &= \pi [30^2 \times 90 - 28^2 \times 88] \\ &= \pi [900 \times 90 - 784 \times 88] \\ &= \pi [81,000 - 68,992] \\ &= 12008\pi \text{ cm}^3 \quad \dots (1) \\ \text{Volume of one plastic ball } (V_2) &= \frac{4}{3} \pi r^3 \\ &= \frac{4}{3} \times \pi \times 1^3 \\ (V_2) &= \frac{4}{3} \pi \quad \dots (2) \end{aligned}$$

$$\begin{aligned} \text{Number of plastic balls required to make the drum} &= \frac{V_1}{V_2} \\ &= \frac{12008\pi}{\frac{4}{3}\pi} \quad [\text{From (1) and (2)}] \\ &= 12008 \times \frac{3}{4} \\ &= 3002 \times 3 \\ &= 9006 \end{aligned}$$

$$\therefore \text{Number of plastic balls required to make the cylindrical drum is 9006.}$$

- (5) Water drips from a tap at the rate of 4 drops in every 3 seconds. Volume of one drop of 0.4 cm^3 . If dripped water is collected in a cylindrical vessel of height 7 cm and diameter is 8 cm. In what time vessel be completely filled? What is the volume of water collected? How many such vessels will be completely filled in 3 hours in 40 minutes?

(4 marks)

Solution :

Volume of water collected

$$\begin{aligned} &= \text{Volume of cylindrical vessel} \\ &= \pi r^2 h \\ &= \frac{22}{7} \times 4 \times 4 \times 7 \end{aligned}$$

$$\text{Volume of water collected } (V_1) = 352 \text{ cm}^3$$

$$\text{Volume of 1 drop of water} = 0.4 \text{ cm}^3$$

$$\text{Volume of 4 drops of water} = 4 \times 0.4 = 1.6 \text{ cm}^3$$

4 drops drips in 3 seconds

$$\therefore \text{Volume of water dripped in 3 seconds} = 1.6 \text{ cm}^3$$

$$\begin{aligned} \therefore \text{Volume of water dripped in 1 seconds } (V_2) \\ &= \frac{1.6}{3} \text{ cm}^3 \end{aligned}$$

$$\begin{aligned} \therefore \text{Volume of time required} \\ \text{to fill the cylindrical vessel} &= \frac{V_1}{V_2} \\ &= 352 \div \frac{1.6}{3} \\ &= 352 \times \frac{3}{1.6} \times 10 \\ &= 660 \text{ seconds} \\ &= 11 \text{ minutes} \end{aligned}$$

$$\begin{aligned} 3 \text{ hours and } 40 \text{ minutes} &= (3 \times 60 + 40) \text{ min} \\ &= 220 \text{ min.} \end{aligned}$$

$$\begin{aligned} \text{Number of vessels} \\ \text{that can be completely} &= \frac{220}{11} \\ \text{filled} &= 20 \end{aligned}$$

20 vessels can be filled in 3 hours and 40 minutes.

- (6) A cone and a hemisphere have equal bases and equal volumes. Find the ratio of their heights.

(3 marks)

Solution :

As cone and hemisphere have equal bases and they have equal radii.

Let the radius of each be r

$$\text{For cone height} = h_1$$

$$\text{For hemisphere height } (h_2) = r \quad \dots (1)$$

$$\begin{aligned} \text{Volume of cone} &= \text{Volume of hemisphere} \\ &\quad \text{(Given)} \end{aligned}$$

$$\therefore \frac{1}{3} \pi r^2 h_1 = \frac{2}{3} \pi r^3$$

$$\therefore \frac{1}{3} \pi r^2 h_1 = \frac{2}{3} \pi r^2 \times r$$

$$\therefore \frac{1}{3} \pi r^2 \times h_1 = \frac{2}{3} \pi r^2 \times h_2 \quad \text{[From (2)]}$$

$$\therefore \frac{h_1}{h_2} = \frac{2}{1} \times \frac{1}{\pi r^2}$$

$$\therefore \frac{h_1}{h_2} = \frac{2}{1}$$

Ratio of heights of cone and hemisphere = 2 : 1.

- (7) A sphere and a cube have the same surface area. Show that the ratio of the volume of the sphere to that of cube is $\sqrt{6} : \sqrt{\pi}$.

(4 marks)

Solution :

$$\text{Surface area of sphere} = 4\pi r^2 \quad \dots (1)$$

$$\text{Surface area of cube} = 6l^2 \quad \dots (2)$$

$$\begin{aligned} \text{Surface area of sphere} &= \text{Surface area of cube} \\ &\quad \text{(Given)} \end{aligned}$$

$$\therefore 4\pi r^2 = 6l^2 \quad \text{[From (1) and (2)]}$$

$$\therefore \frac{r^2}{l^2} = \frac{6}{4\pi}$$

$$\therefore \frac{r}{l} = \frac{\sqrt{6}}{2\sqrt{\pi}} \quad \dots \text{[Taking square roots]}$$

$$\therefore \frac{r^3}{l^3} = \frac{\sqrt{6} \times \sqrt{6} \times \sqrt{6}}{2 \times 2 \times 2 \times \sqrt{\pi} \times \sqrt{\pi} \times \sqrt{\pi}}$$

$$\therefore \frac{r^3}{l^3} = \frac{6\sqrt{6}}{8\pi\sqrt{\pi}} \quad \dots (3)$$

$$\text{Volume of sphere} = \frac{4}{3} \pi r^3$$

$$\text{Volume of cube} = l^3$$

$$\frac{\text{Volume of sphere}}{\text{Volume of cube}} = \frac{4\pi r^3}{3 \times l^3}$$

$$= \frac{4}{3} \times \pi \times \frac{3\sqrt{6}}{4\pi\sqrt{\pi}}$$

[From (3)]

$$= \frac{\sqrt{6}}{\sqrt{\pi}}$$

Ratio of volume of sphere and cube is $\sqrt{6} : \sqrt{\pi}$.

- (8) ₹ 5 coins were made by melting a solid cuboidal block of metal with dimensions $16 \times 11 \times 10$ in cm. How many coins of thickness 2 mm and diameter 2 cm can be made. ($\pi = \frac{22}{7}$) (3 marks)

Solution :

For cylindrical coin,

$$\text{height } (h) = 2 \text{ mm} = \frac{2}{10} \text{ cm}$$

$$\text{Diameter} = 2 \text{ cm} \therefore \text{Radius } (r) = 1 \text{ cm}$$

For cuboidal block,

$$l_1 = 16 \text{ cm}, b_1 = 11 \text{ cm and } h_1 = 10 \text{ cm}$$

Number of coins can be made

$$= \frac{\text{Volume of cuboid}}{\text{Volume of a coin}}$$

$$= \frac{l_1 \times b_1 \times h_1}{\pi r^2 h}$$

$$= \frac{16 \times 11 \times 10}{\frac{22}{7} \times 1^2 \times \frac{2}{10}}$$

$$= \frac{7 \times 16 \times 11 \times 10 \times 10}{22 \times 2}$$

$$\therefore \text{Number of coins made} = 2800$$

- (9) If the radius of a sphere is doubled, what will be the ratio of its surface area and volume as to that of the first sphere? (4 marks)

Solution :

Let r_1 be the radius of first sphere and r_2 be the radius of second sphere.

$$r_2 = 2 \times r_1 \quad \dots (1) \text{ [Given]}$$

Let S_1 and S_2 be the surface areas of first and second sphere.

$$\therefore S_1 = 4\pi r_1^2 \quad \dots (2)$$

$$\therefore S_2 = 4\pi r_2^2$$

$$\therefore S_2 = 4\pi \times (2r_1)^2$$

$$\therefore S_2 = 4\pi \times 4 \times r_1^2$$

$$\therefore S_2 = 4 \times 4\pi r_1^2$$

$$\therefore S_2 = 4 \times S_1$$

[From (2)]

$$\therefore \frac{S_2}{S_1} = 4$$

Let V_1 and V_2 be the volumes of first and second sphere respectively.

$$\therefore V_1 = \frac{4}{3}\pi r_1^3 \quad \dots (3)$$

$$\therefore V_2 = \frac{4}{3}\pi r_2^3$$

$$\therefore V_2 = \frac{4}{3}\pi (2r_1)^3 \quad \text{[From (1)]}$$

$$\therefore V_2 = \frac{4}{3} \times \pi \times 8 \times r_1^3$$

$$\therefore V_2 = 8 \times \frac{4}{3}\pi r_1^3$$

$$\therefore V_2 = 8 \times V_1$$

[From (3)]

$$\therefore \frac{V_2}{V_1} = 8$$

Ratio of surface area is 4 : 1 and the ratio of volume is 8 : 1.

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